CHAPTER-I

INTRODUCTION

To study the effect of age on a unit or on a system of units, various classes of life distributions are introduced and are extensively studied in the univariate case. The classes obtained from the properties of distribution functions such as Increasing failure rate (IFR), Increasing failure rate average (IFRA), New better than used (NBU), New better than used in expectation (NBUE) are welknown among the ones which discribe adverse effect of age on the life-time of a unit, while the classes Decreasing failure rate (DFR), Decreasing failure rate average (DFRA), New worse than used (NWU), New worse than used in expectation, (NWUE) are welknown among the ones which describe benifitial effect of age on life time of units. There are many other classes of univariate life distributions which are not discussed here. For a detailed discussion of the above and others, we refer to Barlow and Proschan (1975).

Many of these classes exhibit quite elegant properties such as closure under various relibility operations and limts in distribution when the underlying random variables are assumed to be independent. But in great many reliability situations, the random variables of interest are not independent. As examples, consider

(a) Minimal path structures of a coherent system having componants in common.

(b) Componants subjected to the same set of stresses.

(c) Structures in which componants share the load, so that failure of one componant results in increased load on each of the remaining componants etc.

Under such circumstances the univariate results will not be applicable and it becomes necessary to extend the univariate concepts of aging to the multivariate case where the componants may or may not be independent.

In this dessertation we study the multivariate extensions of IFR, IFRA and NBU concepts already known in the literature. A very limited literature is available on the multivariate extension of NBUE class and is not studied here due to limitations of time and access to the material.

Below we present a brief summary of each chapter. A detailed introduction of each chapter is given at the begining of each chapter.

In chapter II, we study two multivariate extensions of univariate IFR class and their duel DFR classes.

The first one is due to Brindley and Thompson and the second one is due to Harris. To distinguish these from one another the former is called MIFR class while the later is called MIHR class. Both the classes coincide in the univariate case. The duels are called respectively MDFR and MDHR and these also coincide in the univariate case.

In chapter III, we study several conditions proposed by Esary and Marshall that could be viewed as multivariate extensions of the univariate IFRA concept. These are primarily based on intuitive appeal. Also, we study the multivariate IFRA class proposed by Block and Savits which is a direct multivariate extension of the univariate characterization of IFRA property given in the following lemma.

Lemma 1.1 :

A life distribution F is (univariate) IFRA if and only if for every non-negative nondecreasing borel measurable function h,

$$E\, h(x) \leq E^{1/\alpha}\, h^{\alpha}(x/\alpha) \quad \text{for all } \alpha \in (0,1] \quad .. (1.1)$$

Proof :

Let (1.1) hold. Define $h(x)=0$ if $0 \leq x \leq t$ and 1 if $x > t$.

Then $h^{\alpha}(x/\alpha) = 0$ if $0 \leq x \leq \alpha t$ and 1 if $x > \alpha t$. Then $\bar{F}(t) = E\, h(x) \leq E^{1/\alpha}\, h^{\alpha}(x/\alpha) = \bar{F}^{1/\alpha}(\alpha t)$ and hence F is IFRA. Conversly, let F be IFRA. Let h be any borel measurable non-negative, nondecreasing function. Let us define the sets

$D_{ik} = \{x : h(x) > i.2^{-k}\}$,$i=1,\ldots,k.2^k$; $k= 1,2,\ldots$.

It follows that (1.1) holds for the indicator functions $I_{D_{ik}}(x)$ for all i and k.

Let $h_k(x) = 2^{-k} \sum_{i=1}^{k.2^k} I_{D_{ik}}(x)$. Then

$$E\, h_k(x) = E\, 2^{-k} \sum_{i=1}^{k.2^k} I_{D_{ik}}(x) \leq 2^{-k} \sum_{i=1}^{k.2^k} E^{1/\alpha} I_{\alpha D_{ik}}(x)$$

$$\leq 2^{-k} E^{1/\alpha} \left[\sum_{i=1}^{k.2^k} I_{\alpha D_{ik}}(x) \right]^{\alpha} = E^{1/\alpha} h_k^{\alpha}(x/\alpha) \qquad ..(1.2)$$

Where αD_{ik} is the set obtained by multipling each element of D_{ik} by α. Here the first inequality follows by hypothesis and the second inequality follows by using Minkowski inequality. The last equality follows since $I_{D_{ik}}(x/\alpha) = 1$ if $x \in \alpha D_{ik}$ and zero otherwise. Thus (1.1) holds for every h_k , $k = 1,2,\ldots$. Now since $h_k \uparrow h$, the result follows by using monotone convergence theorem.

In chapter IV, we study two multivariate extensions of univariate NBU concept. The first is a direct extension of an univariate characterization of NBU class while the second is generated on the basis of a shock model arising in physical situations. We show that these classes possess many desirable properties and the later is a subclass of the former.

One of the possible extensions of univariate exponential distribution to multivariate case is the multivariate exponential distribution (reffered to as MVE) of Marshall and Olkin, which has survival function given by

$$\bar{F}(t_1,\ldots,t_n) = \exp\left\{-\sum_{i=1}^{n}\lambda_i t_i - \sum\sum_{i<j=1}^{n}\lambda_{ij}\max(t_i,t_j)-\ldots-\lambda_{12\ldots n}\max(t_1,\ldots t_n)\right\} \quad \ldots (1.3)$$

where $\bar{F}(t_1,\ldots,t_n)=P[T_1>t_1,\ T_2>t_2,\ldots,T_n>t_n]$.

This distribution has been found to play a similar role in multivariate classes of life distributions as univariate exponential plays in the univariate case. It has been found to be a member of almost all the classes of distributions discussed in this dessertation except the MDHR class of chapter II.

Moreover it has been found to form a 'boundary' between the classes MIFR and MDFR of chapter II.

Below we present some terminology used in the sequel.

1.1 Definition :

A function $\phi(\underline{x}) = \phi(x_1,\ldots,x_n)$ where $x_i = 0$ or 1, $i = 1,2,\ldots,n$ and $\phi(\underline{x}) = 0$ or 1 is a coherent structure function of order n if ϕ is nondecreasing in each of it's arguments and $\phi(\underline{0}) = \phi(0,\ldots,0) = 0$, $\phi(\underline{1}) = \phi(1,\ldots,1) = 1$.

1.2 Definition :

Let $T_1,\ldots,T_n$ be nonnegative random veriables and $S_1,\ldots,S_m$ be nonempty subsets of $\{1,\ldots,n\}$. Then

$$\tau(T_1,\ldots,T_n) = \max_{i=1}^{m} \min_{j\in S_i} T_j$$ is called a coherent life function of order n.

Throughout the discussion we consider random variables which are concentrated on the nonnegative orthant, that is the random variables for which $\bar{F}(\underline{0}) = 1$.

Below we present some definitions of classes of life distributions in the univariate case.

1.3 Definition :

A distribution function F is IFR(DFR) if

$$\bar{F}(x/t) = \frac{\bar{F}(t+x)}{\bar{F}(t)} \downarrow (\uparrow) \text{ in } -\infty < t < \infty \text{ for all } x \geq 0.$$

1.4 Definition :

A distribution function F is IFRA(DFRA) if

$$- \frac{1}{t} \log \bar{F}(t) \uparrow (\downarrow) \qquad \text{in } t \geq 0 .$$

1.5 Definition :

A distribution function F is NBU (NWU) if

$$\bar{F}(x+y) \leq (\geq) \bar{F}(x) . \bar{F}(y) \quad \text{for all } x, y \geq 0 .$$

1.6 Definition :

A distribution function F is NBUE (NWUE) if

(a) F has finite (finite or infinite) mean

(b) $\int_t^\infty \bar{F}(x)\, dx \leq (\geq) \mu . \bar{F}(t)$ for all $t \geq 0$ where

$$\mu = \int_0^\infty \bar{F}(x)\, dx .$$

In this dessertation, Definitions, Theorems, Corollaries, Lemmas, Results, Remarks and Examples, when numbered are governed by the chapter and the section in which they appear and are numbered serially within a section. Thus for example in chapter II, 2.2.1 is a definition followed by 2.2.2, which are remarks followed by 2.2.3 which is an example etc. Similarly equations are numbered serially in each section and are governed by the chapter and the section in which they appear.

For example, equation (2.3.1) is the first equation in section 3 of chapter 2.

Recently a new extension of univariate IFR class to multivariate case has been discussed by T.H.Savits (1985). This new class is based on the following equivalent characterization of univariate IFR class given by himself :

' A r.v. T is IFR if and only if $E[h(x,T)]$ is log-concave in x for all functions $h(x,t)$ which are log concave in (x,t) and are nondecreasing in t for each fixed $x \geq 0$'.

Here a function f is logconcave on $A \subset R^n$ means that $f[\lambda \underline{x} + (1-\lambda)\underline{y}] \geq f^{\lambda}(\underline{x})\, f^{1-\lambda}(\underline{y})$. for all $0<\lambda<1$ and $\underline{x},\underline{y} \in A$.

Using this characterization, Savits has proposed the following MIFR class:

Let $\underline{T}$ be a nonnegative random vector. $\underline{T}$ is said to have a multivariate IFR distribution if and only if $E[h(\underline{x},\underline{T})]$ is logconcave in $\underline{x}$ for all functions $h(\underline{x},\underline{t})$ which are logconcave in $(\underline{x},\underline{t})$ and nondecreasing and continuous in $\underline{t} \geq \underline{0}$ for each fixed $\underline{x} \geq \underline{0}$.

This class has been found to possess nice closure properties such as closure under convolution, limits in distribution etc. Also, if $\underline{T}$ has an MIFR distribution(according to above definition), then all marginals of $\underline{T}$ also have the same property. If $\underline{T}$ and $\underline{S}$ are two independent MIFR random vectors then the conjuction $(\underline{T}, \underline{S})$ has MIFR distribution.

We have not discussed in detail this class in this dessertation, as this paper has come to our notice after completion of this work.