CHAPTER-II

MULTIVARIATE IFR/DFR DISTRIBUTIONS

2.1 Introduction :

In this chapter we study the multivariate version of univariate IFR (DFR) class. Several extensions based on intuitive and mathematical appeal are discussed by various authors out of which we study only two, the one due to Brindley and Thompson, in section 2.2. and the other due to Harris in section 2.3. To distinguish these classes we call the class based on Brindley-Thompson's definition as MIFR class while the one based on Harris's definition as MIHR class. Both the classes coincide with IFR (DFR) class in the univariate case.

A parallel development of MDFR class for the former is stright-forward; but for the proper definition of MDHR class needs some further mathematical concepts, and was not discussed by Harris while introducing MIHR class. Later on Brindley and Thompson extended the necessary mathematical concepts and gave a parallel version of MDHR class which we discuss in section 2.4.

In section 2.5 we discuss a subclass of MIHR class which is of some practical importance.

2.2. MIFR Class due to Brindley and Thompson :

In this section we concentrate on MIFR class while it is understood that an analogus development of MDFR class is obtained by reversing the appropriate inequalities and by replacing '$\uparrow$' by '$\downarrow$'.

2.2.1 Definition :

A distribution $F(\underline{x})$ is said to be a multivariate increasing failure rate (MIFR) distribution if it's marginal distribution $F_I[\underline{x}^{(I)}]$ satisfies

$$\frac{\bar{F}_I[\underline{x}^{(I)} + t\underline{1}]}{\bar{F}_I[\underline{x}^{(I)}]} \downarrow \text{ in each } x_i,\ i \in I, \text{ for all } t > 0 \text{ and} \quad \text{----(2.2.1)}$$

for every non empty subset I of $\{1,\ldots,n\}$.

We assume through out that $\bar{F}(\underline{0}) = 1$.

2.2.2 Remarks :

i) The above definition physically demands that for every nonempty subset of the n components, the probability of survival of additional t units of time for these components when they are of age $\underline{x}^{(I)}$ is decreasing in $\underline{x}^{(I)}$[i.e. in each x_i, $i \in I$] for all $t > 0$.

ii) A possible definition could be to demand that

$$\frac{\bar{F}_I[\underline{x}^{(I)} + \underline{t}^{(I)}]}{\bar{F}_I[\underline{x}^{(I)}]} \downarrow \text{ in each } x_i \text{ for all } \underline{t}^{(I)} = (t_{i_1},\ldots,t_{i_k}) \ \& \ \text{for all } I \in \{1,\ldots,n\} .$$

But this implies $\bar{F}(x_1+t_1,\ldots,x_n+t_n) \le \bar{F}(x_1,\ldots,x_n)\ \bar{F}(t_1,\ldots,t_n)$ for all $x_i,\ t_i \ge 0$.

so that

$$\begin{aligned}\bar{F}(x_1,\ldots,x_n) &= \bar{F}(0 + x_1,\ x_2 + 0,\ldots,x_n + 0) \\ &\le \bar{F}(0,x_2,\ldots,x_n).\ \bar{F}_1(x_1) \\ &\le \bar{F}(0,0,x_3,\ldots,x_n)\ \bar{F}_2(x_2).\ \bar{F}_1(x_1) \\ &\le \prod_{i=1}^{n} \bar{F}_i(x_i)\end{aligned}$$

Thus an unde sirable constraint $\bar{F}(x_1,\ldots,x_n) \le \prod_{i=1}^{n} \bar{F}_i(x_i)$ which is a kind of negative independance is imposed on F by this condition. Also this implies that F is both MIFR and MDFR only when the marginals are independent.

iii) From the above definition of MIFR class, it immediately follows that

a) A single MIFR random variable is IFR in the usual sense.

b) Any subset of MIFR random variables is MIFR. Also using factorization of survival function for indepnedent random variables it can be easily proved that

c) Union of two mutually independent sets of MIFR random variables is MIFR.

d) (a) and (c) together imply that if $X_1, \ldots, X_n$ are n independent univariate IFR variables, then $(X_1, \ldots, X_n)$ is MIFR.

e) It is easy to see that if $\underline{X}$ is MIFR, a > 0, $\underline{b} \geq \underline{0}$ then $a\underline{X} + \underline{b}$ is MIFR.

iv) The following example shows that there may exist distributions for which(2.2.1) is monotone in opposite directions for two subsets I_1 and I_2 of $\{1, \ldots, n\}$. Such distributions can not be classified either in MIFR class or in MDFR class.

<u>2.2.3 Example</u> :

The bivariate distribution due to Freund has survival function given by

$$\bar{F}_{(x,y)} = e^{-(\alpha+\beta)x}\left[\frac{\beta-\beta'}{\alpha+\beta-\beta'}e^{-(\alpha+\beta)(y-x)} + \frac{\alpha}{\alpha+\beta-\beta'}e^{-\beta'(y-x)}\right] \quad x \leq y$$

$$= e^{-(\alpha+\beta)y}\left[\frac{\alpha-\alpha'}{\alpha+\beta-\alpha'}e^{-(\alpha+\beta)(x-y)} + \frac{\beta}{\alpha+\beta-\alpha'}e^{-\alpha'(x-y)}\right] \quad y \leq x$$

$$x, y \geq 0.$$

It can be easily checked that $\dfrac{\bar{F}(x+t,\ y+t)}{\bar{F}(x,y)}$ is constant in x and y. Now,

$$\bar{F}_1(x) = \frac{\alpha - \alpha'}{\alpha+\beta - \alpha'}\, e^{-(\alpha+\beta)x} + \frac{\beta}{\alpha+\beta - \alpha'}\, e^{-\alpha' x}, \quad x > 0.$$

$$\bar{F}_2(y) = \frac{\beta - \beta'}{\alpha+\beta - \beta'}\, e^{-(\alpha+\beta)y} + \frac{\alpha}{\alpha+\beta - \beta'}\, e^{-\beta' y}, \quad y > 0.$$

We show that $F_1(x)$ is IFR(DFR) iff $\alpha < \alpha'$ $(\alpha > \alpha')$.

To see this,

Let $A = [(\alpha-\alpha')/(\alpha+\beta-\alpha')]\, e^{-(\alpha+\beta)x}$, $B = [\beta/(\alpha+\beta-\alpha')]e^{-\alpha' x}$.

Then

$$f_1(x) = \frac{-\partial \bar{F}(x)}{\partial x} = (\alpha+\beta)A + \alpha' B, \quad f_1'(x) = -(\alpha+\beta)^2 A - \alpha'^2 B$$

now

$F_1(x)$ is IFR iff $r_1(x) = \dfrac{f_1(x)}{\bar{F}_1(x)} \uparrow$ in x

i.e. iff $\bar{F}_1(x)\, f_1'(x) + f_1^2(x) \geq 0$

i.e. iff $-[(\alpha+\beta)^2 A + \alpha'^2 B][A+B] + [(\alpha+\beta) A + \alpha' B]^2 \geq 0$

i.e. iff $-AB[(\alpha+\beta-\alpha')^2 + \alpha'(\alpha+\beta)] \geq 0$

i.e. iff $AB \leq 0$

i.e. iff $\dfrac{(\alpha-\alpha')\beta}{(\alpha+\beta-\alpha')^2}\, e^{-(\alpha+\beta+\alpha')x} \leq 0$

i.e. iff $\alpha \leq \alpha'$

On the same lines it can be proved that $\bar{F}_2(y)$ is IFR iff $\beta \leq \beta'$.

When the inequalities are strict, corresponding failure rates will be strictly increasing or decreasing. Taking $\alpha < \alpha'$ and $\beta > \beta'$ we see that $F_1(x)$ is IFR while $F_2(y)$ is DFR.

We give some important properties of MIFR class in the next theorem.

2.2.4 Theorem :

i) Minima of subsets of MIFR random Variables are MIFR.

ii) $\underline{X}$ is both MIFR and MDFR iff it has MVE distribution. [we refer to multivariate Marshall-Olkin exponential distribution with distribution function given in (1.3) as MVE].

Proof :

i) Let $\underline{X}$ be MIFR. Let J_i, $i = 1,2,\ldots,m$ be subsets of $\{1,\ldots,n\}$. Let $Y_i = \min\{X_j / j \in J_i\}$ $i = 1,\ldots,m$. For given $y_1,\ldots,y_m \geq 0$ we define $x_j = \max\{y_i / j \in J_i\}$ with the condition that $x_j = 0$ if j belongs to no J_i. Now, the event $\{\underline{Y} > \underline{y}\}$ is equivalent to the event $\{\underline{X} > \underline{x}\}$.

Hence

$$\frac{\bar{F}[\underline{y} + t\underline{1}]}{\bar{F}(\underline{y})} = \frac{P[\underline{Y} > \underline{y} + t\underline{1}]}{P[\underline{Y} > \underline{y}]} \quad \ldots\ldots \qquad (2.2.2)$$

$$= \frac{P[\underline{X} > \underline{x} + t\underline{1}]}{P[\underline{X} > \underline{x}]}$$

$$= \frac{\bar{F}[\underline{x} + t\underline{1}]}{\bar{F}[\underline{x}]}$$

Since $\underline{X}$ is MIFR , this is decreasing in each x_i for all $t > 0$ and since each x_i is increasing in each y_j, it follows that(2.2.2) is decreasing in each y_j. A similar argument holds for every subset of $\underline{Y}$. Hence $\underline{Y}$ is MIFR.

ii) If Part

Let $\underline{X}$ be MVE with survival function given by

$$\bar{F}(\underline{x}) = \exp[-[\sum_{i=1}^{n} \lambda_i x_i + \sum\sum_{i<j} \lambda_{ij} \max(x_i,x_j) + \dots + \lambda_{12,\dots,n} \max(x_1,\dots,x_n)]]$$

Then

$$\frac{\bar{F}(\underline{x} + t\,\underline{1})}{\bar{F}[\underline{x}]} = \frac{\exp[-[\sum_{i=1}^{n} \lambda_i(x_i+t) + \dots + \lambda_{12,\dots,n} \max(x_1+t,\dots,x_n+t)]]}{\exp[-[\sum_{i=1}^{n} \lambda_i x_i + \dots + \lambda_{12,\dots,n} \max(x_1,\dots,x_n)]]}$$

$$= \exp[-[\sum_i \lambda_i + \sum\sum_{i<j} \lambda_{ij} + \dots + \lambda_{12\dots n}]t]$$

This is constant in each x_i for every $t > 0$. Since, every marginal distribution of MVE is again MVE, a similar result holds for every marginal. Hence MVE is both MIFR and MDFR.

Only If part

Let $\underline{X}$ be both MIFR and MDFR. Then by definition (2.2.1) we must have for every k dimentional marginal $\underline{X}^{(I_k)}$ where $I_k = \{i_1,\dots,i_k\} \in \{1,\dots,n\}$,

$$\frac{\bar{F}_{I_k}[\underline{x}^{(I_k)}+t\underline{1}]}{\bar{F}_{I_k}[\underline{x}^{(I_k)}]} = C(t) \text{ where } C(t) \text{ is independent of } \underline{x}^{(I_k)}.$$

Putting $\underline{x}^{(I_k)} = \underline{0}$ in this expression we get $C(t)=\bar{F}_{I_k}[t\,\underline{1}]$.

Hence, $\bar{F}_{I_k}[\underline{x}^{(I_k)}+ t\,\underline{1}] = \bar{F}_{I_k}[t\,\underline{1}].\,\bar{F}_{I_k}[\underline{x}^{(I_k)}]$ .... (2.2.3)

Now by putting

$\underline{x}^{(I_k)}=x\,\underline{1}$ in (2.2.3) we get $\bar{F}_{I_k}[(x+t)\underline{1}]=\bar{F}_{I_k}[x\,\underline{1}].\bar{F}_{I_k}[t\,\underline{1}]$

i.e. $P[\underline{X}^{(I_k)}>(x+t)\underline{1}] = P[\underline{X}^{(I_k)}> x\,\underline{1}].\,P[\underline{X}^{(I_k)}> t\,\underline{1}]$

which is equivalent to

$P[Y > x+t] = P[Y > x].\,P[Y > t]$ where $Y = \min_{i\in I_k}\{X_i\}$.

From property of univariate exponential distributions, this implies that Y must have exponential distribution for some parameter $\theta > 0$. **Therefore**

$$\bar{F}_{I_k}(x\,\underline{1}) = P[Y > x] = e^{-\theta x} \text{ for all } x > 0 \text{ .... (2.2.4)}$$

Now, we complete our proof by induction on k.

For k = 1,(2.2.3) implies that $\bar{F}_1(x) = e^{-\lambda x}$ for some $\lambda > 0$ and thus one dimensional marginals are MVE.

Now let us assume that all k - 1 dimensional marginals are MVE. Without loss of generality we assume that $x_{ik} = \min\{x_{i_1},\dots,x_{i_k}\}$. Now,

$$\begin{aligned}\bar{F}_{I_k}(x_{i_1},\dots,x_{i_k}) &= \bar{F}_{I_k}[(x_{i_1}-x_{i_k})+x_{i_k},\dots,(x_{i_{k-1}}-x_{i_k})+x_{i_k},\ x_{i_k}] \\ &= \bar{F}_{I_k}[(x_{i_1}-x_{i_k}),\dots,(x_{i_{k-1}}-x_{i_k}),0]\bar{F}_{I_k}[x_{i_k}\underline{1}] \quad \text{by(2.2.3)} \\ &= \bar{F}_{I_{k-1}}[(x_{i_1}-x_{i_k}),\dots,(x_{i_{k-1}}-x_{i_k})]e^{-\theta x_{i_k}} \dots \quad \text{by (2.2.4)}\end{aligned}$$

Since by hypothesis, $\bar{F}_{I_{k-1}}$ is k-1 dimensional MVE, we must have

$$\bar{F}_{I_k}(x_{i_1},\ldots,x_{i_k})$$
$$= e^{-\theta x_{i_k}} \exp[-[\sum_{j=1}^{k-1} \lambda_j(x_{i_j}-x_{i_k}) + \ldots + \lambda_{12\ldots k-1} \max[(x_{i_1}-x_{i_k}),\ldots,(x_{i_{k-1}}-x_{i_k})]]]$$

for some $\lambda_i, \lambda_{jl},\ldots, \lambda_{12\ldots n} \geq 0$ for all $i,j,l,\ldots = 1,2,\ldots,k-1$.

$$= \exp[-[\sum_{j=1}^{k} \lambda_j x_{i_j} + \ldots + \lambda_{12\ldots k-1} \max(x_{i_1},\ldots,x_{i_{k-1}})]] \quad \ldots \quad (2\text{-}2\text{-}5)$$

where

$$\lambda_k = [\theta - \sum_{j=1}^{k-1} \lambda_j - \sum\sum_{j<l=1}^{k-1} \lambda_{jl} - \ldots - \lambda_{12\ldots k-1}]$$

If $\lambda_k \geq 0$, (2.2.5) represents a survival function of MVE.

To see that $\lambda_k \geq 0$, we note that $\bar{F}_{I_k}(\underline{x}^{(I_k)})$ should be nonincreasing in x_{i_k}.

$$\frac{\partial}{\partial x_{i_k}} \bar{F}_{I_k}[\underline{x}^{(I_k)}] = C[\underline{x}^{(I_{k-1})}].\ e^{-\lambda_k x_{i_k}} (-\lambda_k) \leq 0$$

where $C[\underline{x}^{(I_{k-1})}] > 0$ depends only cn $x_{i_1},\ldots,x_{i_{k-1}}$.

This implies that $\lambda_k > 0$.

Thus MVE forms the boundary between MIFR and MDFR class.

2.2.5 Remarks :

i) Theorem (2.2.4)(i) says that MIFR class is closed under the formation of series systems.

ii) An equivalent defination of MIFR class :

The hazard function $R(\underline{t})$ is defined as

$$R(\underline{t}) = -\log \bar{F}(\underline{t})$$

Let $r_i(\underline{t}) = \frac{\partial}{\partial t_i} R(\underline{t})$ $i = 1, \ldots, n$. Then the vector $\underline{r}(\underline{t}) = (r_1(\underline{t}), \ldots, r_n(\underline{t}))$ is called the hazard gradient for distribution F.

Block has given an equivalent definition of MIFR class as follows :

' A d.F. F is MIFR iff $\underline{r}(\underline{x} + t\underline{1}) \uparrow$ in t for all $\underline{x} \geq 0$, and a similar condition holds for the, hazard gradient of every marginal distribution.'

Also he shows that the quantity $\underline{r}(\underline{x})$ is stationary (in the sense that $\underline{r}(\underline{x} + t\underline{1}) = \underline{r}(\underline{x})$ for all $t \geq 0$) iff F is MVE, provided that F has absolutely contineous univariate marginals.

2.2.6 A Shock model giving rise to MIFR distribution :

Let us consider shocks to an n-component system. Let H_i, $i = 1, \ldots, n$ be the IFR distributions of occurance times of independent shocks to destroy components $i = 1, \ldots, n$. Also for every subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, the IFR distribution $H_{(I)} = H_{i_1 \ldots, i_k}$ governs the occurance time of shocks which jointly destroy the components in I.

[Here, I is an unordered set.]. All shock occurance times are independently distributed, so that the multivariate distribution of component survival is

$$\bar{F}(t_1,\dots,t_n) = \prod_{i=1}^{n} \bar{H}_i(t_i) \prod_{i<j} \max(t_i,t_j)$$

Then

$$\dots \bar{H}_{12\dots,n}[\max(t_1,\dots,t_n)] .$$

$$\frac{\bar{F}(\underline{t} + x\underline{1})}{\bar{F}(\underline{t})} = \prod_{i=1}^{n} \frac{\bar{H}_i(t_i + x)}{\bar{H}_i(t_i)} \prod_{i<j} \frac{\bar{H}_{ij}(\max(t_i,t_j)+x)}{\bar{H}_{ij}(\max(t_i,t_j))}$$

$$\dots \frac{\bar{H}_{12\dots,n}[\max(t_1,\dots,t_n)+x]}{\bar{H}_{12\dots,n}[\max(t_1,\dots,t_n)]}$$

and since each H is IFR, this is decreasing in $\underline{t}$ for all $x > 0$. [Note that $\max\{t_1,\dots,t_n\}$ is increasing function of each t_i]. By putting $t_i = 0, i \notin I$, it can be seen that similar condition holds for every marginal distribution F_I. Hence F is MIFR.

<u>2.3 MIHR class due to Harris</u> :

<u>2.3.1 Definition</u> :

A distribution function $F(\underline{x})$ on the nonnegative orthant is multivariate IHR if it satisfies the conditions

(i) $\dfrac{\bar{F}(\underline{x} + t\,\underline{1})}{\bar{F}(\underline{x})} \downarrow$ in each x_i for all $t > 0$.

(ii) $P[\underline{X} > \underline{x} \,/\, \underline{X} > \underline{x}'] \uparrow$ in each x_i' for every choice of $\underline{x}$.

We note that condition (i) is equivalent to

$$\frac{\bar{F}(\underline{x} + t\,\underline{1})}{\bar{F}(\underline{x})} \leq \frac{\bar{F}(\underline{x}' + t\,\underline{1})}{\bar{F}(\underline{x}')} \quad \text{for all } x_i \geq x_i' \text{ and } t>0. \qquad \ldots(2.3.1)$$

Condition (ii) is called 'right corner set increasing' or RCSI property, written as RCSI ($\underline{X}$).

2.3.2 Remarks :

i) We note that unlike MIFR class, here the condition (i) neednot be satisfied for all marginals of F, but $\underline{x}$ needs to satisfy a form of positive dependance, namely the RCSI condition. In the later discussion it will be shown that MIHR class is subclass of MIFR class and we also provide an example to show that these two classes are distinct.

ii) We note that a single random variable is always RCSI and hence it immediately follows that a single MIHR r.v. is IFR in the usual sense.

iii) Using the property of factorization of survival function for independent random variables it is easy to see that union of two mutually independent sets of MIHR r.v. is itself a set of MIHR variables.

iv) Remarks (ii) and (iii) imply that if $X_1,\ldots,X_n$ are independent univariate IFR r.v., then $\underline{X}=(X_1,\ldots,X_n) \in$ MIHR.

v) It is easy to see that if $\underline{X} \in$ MIHR, then $\underline{Y} = a\underline{X} + \underline{b}$, $a > 0$, $\underline{b} \geq 0$ is MIHR.

vi) A similar argument as in theorem 2.2.4(a), can be used to see that sets of minimums of MIHR variables satisfy conditions (i) and (ii) of definition 2.3.1 and hence are MIHR. Thus MIHR class is also closed under formation of series systems.

vii) Remarks (iv) and (vi) together imply that MVE is MIHR.

In the next theorem we prove one more property of MIHR. class

2.3.3. Theorem :

Subsets of MIHR random variables are MIHR.

Proof :

Let $\underline{X} \in$ MIHR. Let $(t_i, \underline{x})$ denote the vector $\underline{x}$ with i^{th} componant replaced by t. i.e.

$(t_i, \underline{x}) = (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$ and

$(0_i, \underline{x}) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$.

Now by putting $x_1 = x_1' = 0$ in (2.3.1) we get

$$\frac{\bar{F}[(0_1, \underline{x}) + t\,\underline{1}]}{\bar{F}(0_1, \underline{x})} \leq \frac{\bar{F}[(0_1, \underline{x}') + t\,\underline{1}]}{\bar{F}(0_1, \underline{x}')} \quad \text{for all } x_i \geq x_i' \text{ and } t > 0$$

i.e.

$$\frac{\bar{F}[0_1, \underline{x}']}{\bar{F}[0_1, \underline{x}]} \leq \frac{\bar{F}[(0_1, \underline{x}') + t\,\underline{1}]}{\bar{F}[(0_1, \underline{x}) + t\,\underline{1}]} \quad \ldots\ldots \quad (2.3.2)$$

Also, from RCSI ($\underline{X}$) we get

$$P[\underline{X} > [(0_1,\underline{x}') + t\,\underline{1}] / \underline{X} > [\,0_1,(\underline{x}'+t\,\underline{1})]]$$

$$\leq \quad P[\underline{X} > [(0_1,\underline{x}') + t\,\underline{1}] / \underline{X} > [\,0_1,(\underline{x}+ t\,\underline{1})]]$$

$$= \quad P[\underline{X} > (0_1, \underline{x})+t\,\underline{1} / \underline{X} > [\,0_1,(\underline{x} + t\,\underline{1})]] \text{ Since } x_i \geq x_i' \text{ for all } i.$$

i.e.

$$\frac{\bar{F}[(0_1, \underline{x}')+ t\,\underline{1}]}{\bar{F}[[0_1,(\underline{x}' + t\underline{1})]]} \leq \frac{\bar{F}[(0_1, \underline{x}) + t\,\underline{1}]}{\bar{F}[[0_1,(\underline{x}+ t\,\underline{1})]]} \quad \text{for all } x_i \geq x_i', t > 0.$$

i.e.

$$\frac{\bar{F}[(0_1,\underline{x}')+ t\,\underline{1}]}{\bar{F}[(0_1,\underline{x})+ t\,\underline{1}\,]} \leq \frac{\bar{F}[[0_1,(\underline{x}'+ t\,\underline{1})]]}{\bar{F}[[0_1,(\underline{x} + t\,\underline{1})]]} \quad \text{for all } x_i \geq x_i', t>0 \;. \quad (2.3.3)$$

Combining 2.3.2) and (2.3.3) we get

$$\frac{\bar{F}[0_1, \underline{x}']}{\bar{F}[0_1,\underline{x}]} \leq \frac{\bar{F}[[0_1,(\underline{x}' + t\,\underline{1})]]}{\bar{F}[[0_1,(\underline{x} + t\,\underline{1}\,)]]} \quad \text{for all } x_i \geq x_i',\ t>0.$$

i.e.

$$\frac{\bar{F}_{n-1}(\underline{x}')}{\bar{F}_{n-1}(\underline{x})} \leq \frac{\bar{F}_{n-1}(\underline{x}'+ t\,\underline{1})}{\bar{F}_{n-1}(\underline{x} + t\,\underline{1})} \quad \text{for all } x_i \geq x_i',\ t>0.$$

Where $\bar{F}_{n-1}$ is survival function for the marginal distribution of $(X_2,\ldots,X_n)$. [Here, without ambiguity, we interpret $\bar{F}_{n-1}(\underline{x}') = \bar{F}(x_2', \ldots, x_n')$ etc.]. Thus condition (i) is satisfied for a subset of n-1 dimentions. Putting $x_1 = x_1' = 0$, in condition (ii) of definition (2.3.1) we see that RCSI $(X_2,\ldots,X_n)$. Thus $(X_2,\ldots,X_n) \in$ MIHR. Using a similar argument we see that all subsets of size n-1 are MIHR. By induction on k, the size of the subset, it follows that alll subsets of $\underline{x}$ are MIHR.

2.3.4 Remarks :

The theorem immediately implies that all marginal distributions of $\underline{X}$ satisfy condition (i) of definition 2.3.1 if $\underline{X}$ is MIHR and hence it is also MIFR. Thus MIHR is a subclass of MIFR. After studing some implications of RCSI condition, we present an example to show that MIHR class is a proper subset of MIFR class.

2.3.5 Some Implications of RCSI Condition :

i) Let RCSI($\underline{X}$). Then for every subsets K and M of $\{1,\ldots,n\}$, $P[\underline{X}_K > \underline{x}_K / \underline{X}_M > \underline{x}'_M]$ is increasing in $\underline{x}'_M$ for all $\underline{x}_K$ with the convension that $\underline{X}_K$ is the vector obtained from the set $\{X_i, i \in K\}$ by placing subscripts in assending order.

Proof :

Since RCSI($\underline{X}$), we have $P[\underline{X}_K > \underline{x}_K, \underline{X}_{\bar{K}} > \underline{x}_{\bar{K}} / \underline{X} > \underline{x}']$ is increasing in $\underline{x}'$ for all $\underline{x}_K$, $\underline{x}_{\bar{K}}$ where $\bar{K} = \{1,\ldots,n\} - K$. Taking $x_i = -\infty$ for $i \in \bar{K}$, we see that

$P[\underline{X}_M > \underline{x}_K / \underline{X}_M > \underline{x}'_M, \underline{X}_{\bar{M}} > \underline{x}'_{\bar{M}}]$ is increasing in $\underline{x}'_M$ and $\underline{x}'_{\bar{M}}$ for all $\underline{x}_K$. In particular it is increasing in $\underline{x}'_M$ for all $\underline{x}_K$ and $\underline{x}'_{\bar{M}}$. Now taking $x'_i = -\infty$ for $i \in \bar{M}$ we see that $P[\underline{X}_K > \underline{x}_K / \underline{X}_M > \underline{x}'_M]$ is increasing in $\underline{x}'_M$ for all $\underline{x}_K$.

ii) The RCSI condition implies the series bound, i.e. If RCSI($\underline{x}$), then

$$\bar{F}(\underline{x}) \geq \prod_{i=1}^{n} \bar{F}_i(x_i) \quad \ldots . \qquad (2.3.4)$$

Proof :

Let RCSI $(X_1,\ldots,X_n)$. For n = 1, the result trivially holds as an equality. Let us assume that it is true for $n \leq k-1$. i.e.

$$\bar{F}_{k-1}(x_1,\ldots,x_{k-1}) \geq \prod_{i=1}^{k-1} \bar{F}_i(x_i). \text{ Now,}$$

$$\bar{F}_k(x_1,\ldots,x_k) = P[X_k>x_k].P[X_1>x_1,\ldots,X_{k-1}>x_{k-1}/X_k>x_k]$$

$$\geq P[X_k>x_k].P[X_1>x_1,\ldots,X_{k-1}>x_{k-1}/X_k>-\infty]$$

(using implication I)

$$= P[X_k>x_k].\ P[X_1>x_1,\ldots,X_{k-1} > x_{k-1}]$$

$$= \bar{F}_k(x_k).\ \bar{F}_{k-1}(x_1,\ldots,x_{k-1})$$

$$\geq \prod_{i=1}^{k} \bar{F}_i(x_i) \qquad \text{by hypothesis.}$$

The result now follows by induction on k.

Now we present an example to show that MIHR class is proper subset of MIFR class.

2.3.6 Example :

Gumbel's bivariate distribution has survival function $\bar{G}(x,y) = e^{-x-y-\sigma xy}$ x, y $\geq$ 0, $\sigma > 0$. Now,

$$\frac{\bar{G}(x + t, y + t)}{\bar{G}(x, y)} = e^{-2t-\sigma t^2-\sigma t(x+y)}$$

This is strictly decreasing in both x and y. Now,

$\bar{G}_1(x) = e^{-x}$, $\bar{G}_2(y) = e^{-y}$ and thus one dimentional marginals also satisfy (2.2.1) and hence, (X, Y) is MIFR.

But

$$\begin{aligned}\bar{G}(x,y) &= e^{-x}.\ e^{-y}.\ e^{-\sigma xy}\\ &= \bar{G}_1(x).\ \bar{G}_2(y).\ e^{-\sigma xy}\\ &< \bar{G}_1(x).\ \bar{G}_2(y).\end{aligned}$$

Hence, implication (ii) of subsection 2.3.5 implies that (X, Y) are not RCSI. Hence (X, Y) is not MIHR.

2.3.7 Some system relibility bounds :

We can combine implication (ii) of subsection 2.3.5 with some of the inequalities for univariate IFR distributions to obtain system bounds.

Barlow and Proschan (1975) give the following inequalities for univariate IFR distributions :

(i) If F is IFR with mean μ_1, then

$$\bar{F}(t) \geq \begin{cases} e^{-t/\mu_1} & \text{if} \quad t < \mu_1 \\ 0 & \text{if} \quad t \geq \mu_1. \end{cases}$$

Using (1.3.4) we modify this for MIHR case as follows :

If $F(\underline{x})$ is IHR and if $E(X_i) = \mu_i \quad i = 1,\ldots,n$, then

$$\bar{F}(\underline{x}) \geq \prod_{i=1}^{n} \bar{F}_i(x_i)$$

$$\geq \begin{cases} \exp\left[-\sum_{i=1}^{n} x_i/\mu_i\right] & \text{if } x_i < \mu_i \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If F is IFR and $F(\xi_p) = p$, i.e. ξ_p is a p^{th} percentile for F,

$$\text{then } \bar{F}(t) \geq e^{-\alpha t} \quad \text{if } t \leq \xi_p$$

$$\leq e^{-\alpha t} \quad \text{if } t \geq \xi_p.$$

where $\alpha = [-\log(1-p)]/\xi_p$.

We modify this for MIHR case as follows :

If $F(\underline{x})$ is MIHR and $F_i(\xi_{p_i}) = p_i$, $i = 1,\ldots,n$, i.e. ξ_{p_i} is p_i^{th} percentile for F_i, then

$$\bar{F}(\underline{x}) \geq \begin{cases} \exp\left[-\sum_{i=1}^{n} \alpha_i x_i\right] & \text{if } x_i \leq \xi_{p_i} \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha_i = \dfrac{-\log(1-p_i)}{\xi_{p_i}} \qquad i = 1,\ldots,n.$

In the next section we discuss the analogus MDHR class developed by Brindely and Thompson.

2.4 The MDHR Class :

For developing MDHR class analogus to MIHR class, a negative dependence concept parallel to RCSI dependence is needed.

A possible definition of such a concept would be to demand that $P[\underline{X} > \underline{x} / \underline{X} > \underline{x}']$ to be decreasing in each x_i' for every choice of $x_1, \ldots, x_n$. But unfortunately this probability is always increasing in $\{x_i' : x_i' < x_i\}$ for every choice of $\underline{x}$.

However, Brindley and Thompson have tried to partially reverse the concept of RCSI as follows :

2.4.1 Definition :

Random variables $\underline{X}$ are right corner set decreasing, written as RCSD ($\underline{X}$) if $P[\underline{X} > \underline{x} / \underline{X} > \underline{x}']$ is decreasing in $\{x_i' : x_i' \geq x_i\}$ for every choice of $\underline{x}$.

Using this dependence concept, Brindley and Thompson have defined the class of MDHR distributions as follows :

2.4.2 Definition :

A d.F. $F(\underline{x})$ on the nonnegative orthant is decreasing hazard rate (MDHR) if it satisfies conditions

(i) $\dfrac{\bar{F}(\underline{x} + t\,\underline{1})}{\bar{F}(\underline{x})} \geq \dfrac{\bar{F}(\underline{x}' + t\,\underline{1})}{\bar{F}(\underline{x}')}$ for all $x_i \geq x_i'$ $i=1,\ldots,n$.

(ii) RCSD($\underline{X}$).

2.4.3 Remarks :

i) The following properties of MDHR class can be easily varified.

(a) A single multivariate DHR survival time is DFR in the usual sense.

(b) The union of two mutually independent sets of DHR survival times is DHR.

(c) Any subset of DHR survival times is DHR.

ii) (c) implies that MDHR class is a subclass of MDFR class discussed in the previous section.

iii) RCSD condition implies $\bar{F}(x_1,\ldots,x_n) \leq \prod_{i=1}^{n} \bar{F}_i(x_i)$ [Proof similar to that for (2.3.4)].

iv) Bivariate exponential distributions of Marshall and Olkin with $\lambda_{12} > 0$ are not DHR, since in this case,

$$\begin{aligned} \bar{F}(x,y) &= e^{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)} \qquad \lambda_1>0,\ \lambda_2>0, \lambda_{12}>0. \\ &= \bar{F}_1(x).\bar{F}_2(y).\ e^{\lambda_{12}\min(x,y)} \\ &> \bar{F}_1(x).\bar{F}_2(y) \qquad \text{if} \quad \lambda_{12} > 0. \end{aligned}$$

Thus (X,Y) is not RCSD and hence not MDHR.

v) We have proved that all MVEs are MDFR. Thus above example shows that MDHR class is a proper subclass of MDFR class.

vi) Since the two variables of a BVE arise as minimums of subsets of independent exponential variables, namely

$X = \min(U_1, U_{12})$ and $Y = \min(U_2, U_{12})$ where U_1, U_2, U_{12} are independent exponential variables, the example in remark (iv) above also shows that minimums of subsets of MDHR survival times need not be MDHR.

vii) A d.F. $F(\underline{x})$ is both MIHR and MDHR iff it is product of independent exponentials.

Since if $F(\underline{x})$ is both MIHR and MDHR, it must also be both MIFR and MDFR, and hence, F must be a MVE. Also $\underline{X}$ must be both RCSI and RCSD, ahd hence, F must satisfy the inequalities (i) $\bar{F}(\underline{x}) \geq \prod_{i=1}^{n} \bar{F}_i(x_i)$ and (ii) $\bar{F}(\underline{x}) \leq \prod_{i=1}^{n} \bar{F}_i(x_i)$ respectively which implies $\bar{F}(\underline{x}) = \prod_{i=1}^{n} \bar{F}_i(x_i)$. Thus F must be MVE with independent marginals. i.e. it is composed of independent exponentials.

2.5 A subclass of bivariate IHR distributions :

Harris discusses a more restricted class of IHR distributions generated from an independent basis with appropriate marginals, which satisfies all the properties of MIHR class. He takes $X = \min(U,W)$ and $Y = \min(V, W+a)$ where U,V and W are arbitrary independent IHR random variables and $a \geq 0$ is an arbitrary constant.

The following theorem and it's corollary reveal the importance of such a class.

2.5.1 Theorem :

If $Y = \phi(X)$, where ϕ is a strictly increasing non-negative function which is not identically zero or infinity and X has a marginal exponential distribution, then the pair (X,Y) has a bivariate IHR distribution iff

$$\phi(x) = x + a \quad a \geq 0.$$

Proof :

Let ψ be the inverse function of ϕ defined by

$$\psi(y) = \inf_x \{\phi(x) > y\} .$$

Since X is non-negative r.v., the function ψ will be non-negative, moreover since ϕ is strictly increasing in $(0, \infty)$, we must have $\psi(0)=0$ [o.w. if possible, let $\psi(0) = C > 0$. i.e. $\inf_x \{\phi(x) > 0\} = C$ which implies that $\phi(x) = 0$ for all $x > C$, contradicting the strictly increasing property of ϕ]. Then we have

$$P[X>x, \phi(X)>y]=P[X> \max(x,\psi(y)]=\exp[-\lambda \max(x,\psi(y)]$$

for all $x, y \geq 0$ and for some $\lambda > 0$.

For $\bar{F}$ to be IHR, it must satisfy condition (i) and (ii) of definition 2.3.1,

Now, condition (i) will be satisfied by $\bar{F}$ iff

$$\frac{\bar{F}(x+t,\ y+t)}{\bar{F}(\ x,\ y\ )} \leq \frac{\bar{F}(x'+t,\ y'+t)}{\bar{F}(x',\ y')} \quad \text{for all } x \geq x' \geq 0 \text{ and } y \geq y' \geq 0,\ t>0.$$

i.e. iff $\exp[-\lambda[\max(x+t,\psi(y+t)) - \max(x,\psi(y))]]$

$\leq \exp[-\lambda[\max(x'+t,\psi(y'+t)) - \max(x',\psi(y'))]]$

for all $x \geq x'$, $y \geq y'$, $t>0$.

i.e. iff $\max(x+t,\ \psi(y+t)) - \max(x,\ \psi(y))$

$\geq \max[x'+t,\ \psi(y'+t)] - \max(x',+\psi(y'))$

for all $x \geq x' \geq 0$, $y \geq y' \geq 0$, $t>0$.

.... (2.5.1)

Further proof depends on the following lemmas :

Lemma 1 :

If (2.5.1) holds, then $\psi(y+t) \leq \psi(y)+t$

for all $y \geq 0$ and $t > 0$.

Proof :

Suppose, if possible, there exists $a \geq 0$ and $t > 0$ such that

$\psi(a+t) > \psi(a)+t$ .... (i)

let $x' = \psi(a)$ and $x = \psi(a+t)-t$, then $x > x' \geq 0$.

let $y = y' = a$. Putting these values in (2.5.1) we get $\psi(a+t) - [\psi(a+t)-t] \geq \psi(a+t) - \psi(a)$. i.e. $t \geq \psi(a+t)-\psi(a)$ which is contradiction to (i), hence the claim.

Lemma 2 :

Condition (2.5.1) is equivalent to

$\max(x+t,\ \psi(y+t)) - \max(x,\ \psi(y)) = t$ ... (2.5.2)

for all $x \geq 0$, $y \geq 0$, $t>0$.

Proof :

Let (2.5.1) holds. Then from Lemma 1 putting $y=0$, we get $\psi(t)\leq t$ for all $t>0$. Therefore by putting $x' = y' = 0$ in (2.5.1) we get

$$\max(x+t, \psi(y+t)) - \max(x,\psi(y)) \geq \max(t,\psi(t)) = t \quad \ldots\ldots(ii)$$

Again from Lemma 1 we get

$$\max(x+t, \psi(y+t)) - \max(x, \psi(y))$$
$$\leq \max(x+t, \psi(y)+t) - \max(x, \psi(y)) = t. \quad \ldots\ldots(iii)$$

from (ii) and (iii) we conclude that

$$\max(x+t,\psi(y+t)) - \max(x, \psi(y)) = t.$$

on the other hand, if (2.5.2) holds, condition (2.5.1) reduces to $t \geq t$ which is trivially true. Hence the Claim.

Lemma 3 :

Condition (2.5.2) is equivalent to

$$\psi(y+t)=\psi(y)+t \text{ for all } t>0 \text{ whenever } \psi(y)>0. \quad .. (2.5.3)$$

Proof :

Let (2.5.2) holds. For y such that $\psi(y)>0$, take x such that $0 \leq x \leq \psi(y)$. Then (2.5.2) becomes

$$\max(x+t, \psi(y+t)) - \max(x, \psi(y))$$
$$= \max (x+t, \psi(y+t)) - \psi(y) = t.$$

i.e. $\max (x+t, \psi(y+t)) = \psi(y) + t$

Since $x < \psi(y)$, $x+t \neq \psi(y)+t$. Therefore $\psi(y+t)=\psi(y)+t$, thus (2.5.3) holds. On the other hand if (2.5.3) holds, we have $\max (x+t, \psi(y+t)) - \max(x, \psi(y)) = t$ for all y such that $\psi(y) > 0$. Hence the claim.

Now we continue with the proof of the theorem.

It follows now that it is enough to prove that (2.5.3) holds iff $\phi(x) = x+a$ for some $a \geq 0$.

we note that (2.5.3) implies

$\psi(y+t) = \psi(y)+t = \psi(t)+y$ for all y, t such that $\psi(y)>0$, $\psi(t)>0$. Thus $\psi(y)-\psi(t) = y-t$ for all y, t such that $\psi(y)$, $\psi(t)>0$. Let us fix t= a for some a such that $\psi(a)>0$. Then we have $\psi(y)=y+\psi(a)-a = y-C$ where $C = a-\psi(a)$. note that by Lemma 1, $C \geq 0$. Thus we must have $\psi(y)$ of the form $\psi(y) = y-C$, $C \geq 0$ for all $y>C$ if (2.5.3) holds. ...(2.5.4)

Also since ϕ is increasing, so is ψ and hence we must have $\psi(C) = 0$ [o.w. if possible, let $\psi(C) = h > 0$. Let $0 < x < h$. Then from (2.5.4) $\psi(x+C) = x < h$ contradicting increasing nature of ψ] and $\psi(y) = 0$ for $y < C$.

Thus we have

$$\begin{aligned} \psi(y) &= y - C & & y > C \\ &= 0 & & y \leq 0,\ C \geq 0. \end{aligned}$$

which gives $\phi(x) = x - C$. $x \geq 0$ $C \geq 0$

RCSI $(X, \phi(X))$ follows from

$$P[X > x, \phi(X) > y \,/\, X > x', \phi(X) > y']$$

$$= \quad P[X > \max(x, \psi(y)) / X > \max(x', \psi(y'))].$$

and since $\max(x', \psi(y'))$ is increasing function of

x' and y', this probability is increasing in both x' and y'. Hence the theorem.

Corollary : If $X = \min(U,W)$, $Y = \min(V,\phi(W))$ where U,V and W are independent exponential random variables and ϕ is strictly increasing function then the pair (X,Y) has a bivariate IHR distribution iff $\phi(x)=x+a$, $a\geq 0$.

Proof :

Let $U\sim\exp(\lambda_1)$, $V\sim\exp(\lambda_2)$, $W\sim\exp(\lambda_{12})$.

Then we have $\bar{F}(x,y) = e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}\max(x,\psi(y))}$.

$\bar{F}$ will satisfy condition (i) of definition (3.1) iff

$$\frac{\bar{F}(x+t,\ y+t)}{\bar{F}(x,y)} \leq \frac{\bar{F}(x'+t,\ y'+t)}{\bar{F}(x',\ y')} \text{ for all } x\geq x'\geq 0,\ y\geq y'\geq 0,\ t > 0.$$

i.e. iff $e^{-(\lambda_1+\lambda_2)t}.\ e^{-\lambda_{12}[\max(x+t,\psi(y+t))]-\max(x,\psi(y))]}$

$\leq\ e^{-(\lambda_1+\lambda_2)t}.\ e^{-\lambda_{12}[\max(x'+t,\ \psi(y'+t)]-\max(x,\ \psi(y)]}$

i.e. iff $\max(x+t,\ \psi(y+t))-\max(x,\ \psi(y))$

$$\geq \max(x'+t,\ \psi(y'+t))-\max(x',\ \psi(y'))$$

for all $x\geq x'\geq 0$, $y\geq y'\geq 0$, $t > 0$.

Which is exactly same as condition (2.5.1) and hence from Theorem 2.5.1, we must have $\phi(x) = x+a \quad a \geq 0$.

To see that RCSI(X,Y) we observe that

$P[X > x, Y > y / X > x', Y > y']$

$= P[U>x, V>y, W>\max(x,\psi(y)) / U>x', V>y', W>\max(x',\psi(y'))]$

since (U,V,W) is MIHR and $\max(x',\psi(y'))$ is increasing function of x' and y'. it follows that the above probability is increasing in x' and y'. Hence RCSI (X,Y).

Thus (X,Y) is MIFR iff $\phi(x) = x + a$.

A practical situation where such type of model is appropriate, is described below :

Suppose, three independent sources of shocks are present in the environment. A shock from source 1 destroyes componant 1 at a random time U. A shock from source 2 destroyes componant 2 at random time V. A shock from source 3 destroyes componant 1 at a random time W and the componant 2 at a random time, which is known to be an increasing function $\phi(W)$, of W. Then the life time X of componant 1 is given by $X = \min(U,W)$, $Y = \min(V,\phi(W))$.

Suppose U,V and W are exponentially distributed. Then corollary of theorem 2.5.1 says that life times X and Y have bivariate IHR distribution iff $\phi(W)=W+a$ for some $a \geq 0$. Hence if it is known that say $\phi(W)=aX$ for some $a>0$, then (X,Y) will not have an IHR distribution. Note however that in this case, the marginals X,Y will have univariate IHR distribution. Thus the marginal distributions are IFR need not mean that the joint distribution is IHR.