CHAPTER-III

THE MIFRA DISTRIBUTIONS

3.1 Introduction :

In this chapter we study several conditions that extend the univariate IFRA property to the multivariate case.

In section 3.2 we present six multivariate IFRA conditions proposed by Esary and Marshall in 1979. In section 3.3 we study the interrelationships among these conditions and in section 3.4 we give the counter examples to show that no other relationships hold among the conditions. In section 3.5 we discuss some properties of the conditions and their relation with association, absolute continuity and independence. In section 3.6 we present another multivariate extension of univariate IFRA class based on it's recent characterization proposed by Block and Savits (1981) namely, a r.v. T is (univariate) IFRA iff for every nonnegative nondecreasing function h, $E\ h(X) \leq E^{1/\alpha}\ h^{\alpha}(X/\alpha)$. Also we discuss the properties of this class in the same section. In section 3.7 we present some families of distributions which belong to this class. In section 3.8 we show that the definition of MIFRA class presented in section 3.6 can be slightly modified which leads to an easy characterization of this MIFRA class.

3.2 Multivariate IFRA conditions due to Esary and Marshall :

Below we present the six MIFRA conditions proposed by Esary and Marshall.

Condition A : This condition can be stated through three equivalent statements :

(a) $\frac{R(\alpha \underline{t})}{\alpha}$ is increasing in $\alpha>0$ whenever each $t_i \geq 0$.

(b) $R(\alpha\, \underline{t}) \leq \alpha\, R(\underline{t})$ for all $\alpha \in [0,1]$ whenever each $t_i \geq 0$.

(c) $R(\underline{t}) \leq \underline{t} . \underline{r}\ (\underline{t})$ whenever each $t_i \geq 0$ provided $R(\underline{t})$ is differentiable.

Where $R(\underline{t}) = -\log \bar{F}(\underline{t})$ and $\underline{r}\ (\underline{t}) = (r_1(\underline{t}),\ r_2(\underline{t}),..,r_n(\underline{t}))$,

$$r_i(\underline{t}) = \frac{\partial}{\partial t_i} R(\underline{t}).$$

The equivalance of (a) and (b) is quite easy to demonstrate while equivalence between (a) and (c) follows by observing that $\frac{\partial}{\partial \alpha} \left[\frac{R(\alpha \underline{t})}{\alpha} \right]$ is nonnegative.

Condition B : The random variables $T_1,...,T_n$ have joint distribution such as $\tau(T_1,...,T_n)$ has an IHRA distribution for all coherent life functions τ.

Condition C : $T_1,...,T_n$ have a representation as $T_i = \tau_i(X_1...,X_k)$ where $X_1,...,X_k$ are independent IFRA random variables and $\tau_1,..., \tau_n$ are coherent life functions of order k.

Condition D : For some independent IHRA random variables $X_1,\ldots,X_k$ and nonempty subsets S_i of $\{1,\ldots,k\}$, T_i have a representation $T_i = \min_{j \in S_i} X_j$, $i = 1,\ldots,n$.

Condition E : $\min_{i \in S} T_i$ is IHRA for all nonempty subsets S of $\{1,\ldots,n\}$.

Condition F : $T_1,\ldots,T_n$ have a joint distribution such that $\min_i a_i T_i$ is IHRA whenever each $a_i \geq 0$.

Some comments on the MIFRA conditions :

We observe that condition A is the only condition which is direct analog of univariate IFRA property. Other conditions have arisen as models appropriate for practical situations.

In dealing with large systems, it is common practice to determine the life distribution of various subsystems and then to combine such partial results succesively as larger subsystems are studied. Condition B has a direct bearing on such a procedure.

When making a system analysis by combining information from subsystems as discribed above, the subsystem life lengths $T_1,..,T_n$ are often dependent as a result of the subsystems having componants in common. Condition C is an appropriate model in such circumstances.

Further we note that if each of the coherent systems in condition C is a series system, and if each X_j is exponentially distributed, then each T_i can be viewed as minimum over subsets of independent exponential variables and thus will have MVE distribution. Thus it is of some interest to modify condition C by admitting only series systems. This case arises very often in practice especially when $T_1, \ldots, T_n$ are the minimal path life lengths for a coherent system with independent componants.
This modification leads to condition D.

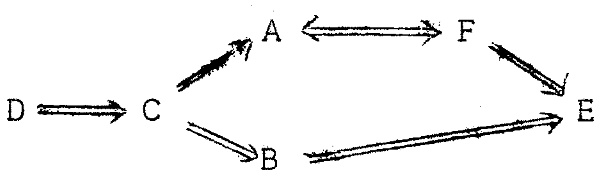
A similar modification of condition B gives rise to condition E.

The condition F is shown to be equivalent to condition A.

3.3 The interrelationships among the conditions :

3.3.1 Theorem :

The following diagram summarizes the relationships between the conditions A to F.



Proof :

The proofs $D \Rightarrow C$, $B \Rightarrow E$ are trivial. $C \Rightarrow B$ follows from the fact that compositions of coherent life functions are again coherent life functions. $F \Rightarrow E$ follows by taking $a_i = 1$ for $i \in S$ and zero otherwise in condition F. The remaining implications follow as given below :

$A \Leftrightarrow F$: Let A holds. Therefore $R(\alpha \underline{t}) \leq \alpha R(\underline{t})$ for all $\alpha \in [0,1]$ whenever each $t_i \geq 0$. Let $z > 0$ and $a_i = \frac{z}{t_i}$. We get,

$$R(\alpha \frac{z}{a_1}, \ldots, \alpha \frac{z}{a_n}) \leq \alpha R(\frac{z}{a_1}, \ldots, \frac{z}{a_n})$$

i.e. $-\log \bar{F}(\alpha \frac{z}{a_1}, \ldots, \alpha \frac{z}{a_n}) \leq -\alpha \log \bar{F}(\frac{z}{a_1}, \ldots, \frac{z}{a_n})$.

But $\bar{F}(\frac{\alpha z}{a_1}, \ldots, \frac{\alpha z}{a_n}) = P[a_1 T_1 > \alpha z, \ldots, a_n T_n > \alpha z]$

$$= P[\min_i a_i T_i > \alpha Z] = \bar{G}(\alpha z)$$

where $\bar{G}$ is survival function for $Y = \min_i a_i T_i$. Thus we have $-\log \bar{G}(\alpha z) \leq -\alpha \log \bar{G}(z)$ and hence Y is IHRA for every $a_i \geq 0$.

$F \Rightarrow A$ follows by exactly reversing the steps.

$C \Rightarrow A$: Let $T_i = \tau_i(X_1, \ldots, X_k)$ $i=1,\ldots,n$ where X_j^s are independent.

$$\text{Let } X_j(t) = \left.\begin{array}{ll} 1 & \text{if } t < X_j \\ 0 & \text{if } t \geq X_j \end{array}\right\} j = 1,\ldots,k .$$

and $\underline{X}(t) = (X_1(t), \ldots, X_k(t))$. Let $\phi_1, \ldots, \phi_n$ be coherent structure functions of order K corresponding to the life-functions $\tau_1,\ldots,\tau_n$. Let $P_{i1},\ldots,P_{ik_i}$ be minimal path sets for ϕ_i, $i=1,\ldots,n$. In case $\phi_i(\underline{X})\equiv 0$, $\tau_i(\underline{X})\equiv 0$ and in case $\phi_i(\underline{X})\equiv 1$, $\tau_i(\underline{X})\equiv\infty$. The joint survival function $\bar{H}$ of $\tau_1,\ldots,\tau_n$ is given by

$$\begin{aligned}\bar{H}(t_1,\ldots,t_n) &= P[\phi_i(\underline{X}(t_i))= 1 \text{ for all } i=1,\ldots,n] \\ &= P[\prod_{i=1}^{n} \phi_i(\underline{X}(t_i))= 1] \\ &= E[\prod_{i=1}^{n} \phi_i(\underline{X}(t_i))] \\ &= E[\prod_{i=1}^{n} [\coprod_{j=1}^{k_i} \prod_{\ell\in P_{ij}} X_\ell(t_i)]] \text{ where } \coprod_{i=1}^{k} y_i = 1-\prod_{i=1}^{k}(1-y_i). \\ &= E[g(\underline{X}(\underline{t})].\end{aligned}$$

where $\underline{X}(\underline{t})= (X_i(t_j))$ is k x n matrix. We note that $g(\underline{X}(\underline{t}))$ contains terms of the form $\prod_i \prod_j X_i(t_j)$ and since $X_i(t_j)$ is binary and decreasing in t_j, we can replace $\prod_j X_i(t_j)$ by $X_i(\max_j t_j)$. Moreover since X_i are statistically independent, so are the rows of the matrix $\underline{X}(\underline{t})$. Hence $\bar{H}(t_1,\ldots,t_n)$ depends only on the k x n matrix. $\mathbb{P} = E\,\underline{X}(\underline{t}) = (\bar{F}_i(t_j)) = (p_{ij})$. Thus $\bar{H}(t_1,\ldots,t_n)= h(\mathbb{P})$ where h depends only on $\mathbb{P}$. Let $\mathbb{P}^\alpha$ denotes the matrix $(\bar{F}_i(\alpha t_j))$. We need to prove that

$R(\alpha \underline{t}) = -\log \bar{H}(\alpha \underline{t}) \leq -\alpha \log \bar{H}(\underline{t}) = \alpha R(\underline{t})$ i.e.

$-\log h(\mathbb{P}^{\alpha}) \leq -\alpha \log h(\mathbb{P})$ .. (3.3.1)

We first prove three lemmas, using which (3.3.1) follows. Suppose for convenience that $0 = t_o \leq t_1 \leq \ldots \leq t_n < t_{n+1} = \infty$. Let $d_\ell = (1,\ldots,1,0,\ldots,0)$ be vector with first ℓ componants equal to 1 and remaining componants equal to zero. Let $\mathbb{P}_{ij}$ be obtained from $\mathbb{P}$ after replacing i^{th} row by d_j .

Lemma 3.3.2 :

$$\frac{\partial h(\mathbb{P})}{\partial p_{ij}} = h(\mathbb{P}_{ij}) - h(\mathbb{P}_{i,j-1})$$

Proof :

Since $t_1 \leq \ldots. \leq t_n$, it follows that $X_i(t_j)$ is increasing in j and hence we can write for given i,

$$\prod_{j=1}^{n} \phi_j[\underline{X}(t_j)] = \sum_{\ell=0}^{n} [X_i(t_\ell) - X_i(t_{\ell+1})] \prod_{j=1}^{\ell} \phi_j(\underline{X}(t_j),1_i) \times \prod_{j=\ell+1}^{n} \phi_j(\underline{X}(t_j), 0_i)$$

where $(z,1_i)$ is the vector z with i^{th} componant replaced by 1 and $(z, 0_i)$ is the vector z with ith componant replaced by zero. [we note that this follows since $X_i(t_\ell) - X_i(t_{\ell+1}) = 1$ only if $X_i(t_\ell) = 1$ and $X_i=(t_{\ell+1})=0$ and zero otherwise].

Thus $h(\mathbb{P}) = E \prod_{j=1}^{n} \phi(\underline{X}(t_j))$

$$= \sum_{\ell=0}^{n} E[X_i(t_\ell) - X_i(t_{\ell+1})].E \prod_{j=1}^{\ell} \phi_j(\underline{X}(t_j),1_i) \times \prod_{j=\ell+1}^{n} \phi_j(\underline{X}(t_j),0_i)$$

$$= \sum_{\ell=0}^{n} (p_{i\ell} - p_{i,\ell+1})\, h(\mathbb{P}_{i\ell}) \quad .. \quad (3.3.2)$$

With the convension that $p_{i0} = 1$ and $p_{i,n+1} = 0$ $i = 1,2,..,k$.
Hence,

$$\frac{\partial h(\mathbb{P})}{\partial p_{ij}} = \frac{\partial}{\partial p_{ij}}[(p_{i,j-1}-p_{ij})h(\mathbb{P}_{i,j-1})+(p_{ij}-p_{i,j+1})h(\mathbb{P}_{ij})]$$

$$= h(\mathbb{P}_{ij}) - h(\mathbb{P}_{i,j-1})$$

Lemma 3.3.3. :

With $\psi(x) = -x \log x$, $x \in (0, 1]$

$$\sum_i\sum_j \psi(p_{ij}) \frac{\partial h}{\partial p_{ij}} \geq \psi[h(\mathbb{P})].$$

Proof :

We prove this lemma by induction on r, the order of the coherent structure ϕ_i. If $k = 1$, then either $\phi_i(X) \equiv 0$ or $\phi_i(X) \equiv 1$ or $\phi_i(X) \equiv x$. Consequently either $h(\mathbb{P}) = h(p_1,\ldots,p_n) \equiv 0$ [This happens if $\phi_i(X) \equiv 0$ for at least one i] or $h(\mathbb{P}) = 1$ [This happens if $\phi_i(X) \equiv 1$ for all i] or $h(\mathbb{P}) = p_{ij}$ for some i [This happens if $\phi_j(X) \equiv 1$ or X for $j = 1,\ldots,i-1$ and $\phi_i(X) = X$ with $\phi_j(X) = 1$ for $j = i+1, \ldots, n$].

If $h(\mathbb{P}) = p_{1i}$, then $\frac{\partial h}{\partial p_{1j}} = 1$ for $j = i$ and zero otherwise.

Consequently $\sum_j \psi(p_{1j}) \frac{\partial h}{\partial p_{1j}} = \psi(p_{1i}) = \psi(h)$. The equality

$\sum_j \psi(p_{1j}) \frac{\partial h}{\partial p_{1j}} = \psi(h)$ is trivial if $h \equiv 0$ or $h \equiv 1$. Thus

lemma holds for $k = 1$.

Mow, suppose the lemma holds for all semicoherent structure functions of order k-1. Then for structure functions of order k we have

$$\sum_{ij}\sum \psi(p_{ij}) \frac{\partial h}{\partial p_{ij}} = \sum_{ij}\sum \psi(p_{ij}) \frac{\partial}{\partial p_{ij}} \left[\sum_{\ell=0}^{n} (p_{1\ell} - p_{1,\ell+1}) h(\mathbb{P}_{1,\ell}) \right]$$

..(using 3.3.2)

$$= \sum_{i=2}^{k} \sum_{j=1}^{n} \psi(p_{ij}) \sum_{\ell=0}^{n} (p_{1\ell} - p_{1,\ell+1}) \frac{\partial}{\partial p_{ij}} h(\mathbb{P}_{1,\ell}) +$$

$$\sum_{j=1}^{n} \psi(p_{1j}) \left[\frac{\partial}{\partial p_{1j}} (p_{1,j-1} - p_{1j}) \, h(\mathbb{P}_{1,j-1}) + \right.$$

$$\left. \frac{\partial}{\partial p_{1j}} (p_{1j} - p_{1,j+1}) \, h(\mathbb{P}_{1j}) \right]$$

$$= \sum_{\ell=0}^{n} (p_{1\ell} - p_{1,\ell+1}) \sum_{i=2}^{k} \sum_{j=1}^{n} \psi(p_{ij}) \frac{\partial}{\partial p_{ij}} h(\mathbb{P}_{1,\ell}) +$$

$$\sum_{j=1}^{n} \psi(p_{1j}) [h(\mathbb{P}_{1j}) - h(\mathbb{P}_{1,j-1})]$$

$$\geq \sum_{\ell=0}^{n} (p_{1\ell} - p_{1,\ell+1}) \psi[h(\mathbb{P}_{1,\ell})] +$$

$$\sum_{j=1}^{n} \psi(p_{1j}) [h(\mathbb{P}_{1,j}) - h(\mathbb{P}_{1,j-1})]$$

[using the hypothesis and the fact that $p_{1\ell} \geq p_{1,\ell+1}$]

After rearranging the terms we get

$$= \psi[h(\mathbb{P}_{10})] + \sum_{j=1}^{n} [[\psi(p_{1j})h(\mathbb{P}_{1j}) - p_{1j}\psi[h(\mathbb{P}_{1,j-1})]] +$$

$$+[p_{1j}\psi[h(\mathbb{P}_{1j})] - \psi(p_{1j})h(\mathbb{P}_{1,j-1})]]$$

To simplify the notation we write $h_j = h(\mathbb{P}_{1j})$ and $p_{1j} = p_j$. Then the inequality becomes

$$\sum_{ij}\sum \psi(p_{ij})\frac{\partial h}{\partial p_{ij}} \geq \psi(h_o) + \sum_{j=1}^{n} [[\psi(p_j)h_j - p_j\, \psi(h_{j-1})] -$$

$$[\psi(p_j)h_{j-1} - \psi(h_j)p_j]]$$

$$= \psi(h_o) + \sum_{j=1}^{n} [\psi(p_j h_j) - \psi(p_j h_{j-1})] \quad .. \qquad (3.3.3)$$

We observe that $\frac{\partial^2 \psi(x)}{\partial x^2} = -\frac{1}{x}$ and hence ψ is concave on $(0,\infty)$. Therefore we have

$$\psi(x+\theta) - \psi(x) \geq \psi(y+\theta) - \psi(y) \quad \text{for all } 0 \leq x \leq y,\ \theta > 0.$$

Taking

$\theta = (h_j - h_{j-1})p_j$, $x = p_j h_{j-1}$, $y = h_o + \sum_{i=1}^{j-1} (h_i - h_{i-1})p_i$ it follows that

$$\psi(p_j h_j) - \psi(p_j h_{j-1}) \geq \psi[h_o + \sum_{i=1}^{j} (h_i - h_{i-1})p_i] - \psi[h_o + \sum_{i=1}^{j-1} (h_i - h_{i-1})p_i]$$

summing both sides over j gives

$$\sum_{j=1}^{n} [\psi(p_j h_j) - \psi(p_j h_{j-1})] \geq \psi[h_o + \sum_{i=1}^{n} (h_i - h_{i-1}) p_i] - \psi(h_o)$$

which together with (3.3.3) gives

$$\begin{aligned} \sum_i \sum_j \psi(p_{ij}) \frac{\partial h}{\partial p_{ij}} &\geq \psi[h_o + \sum_{i=1}^{n} (h_i - h_{i-1}) p_i] \\ &= \psi[\sum_{j=0}^{n} (p_j - p_{j+1}) h_j] \\ &= \psi[h(\mathbb{P})] \quad \text{.. .. by(3.3.2)} \end{aligned}$$

Hence the lemma.

Lemma 3.3.4 :

Let η be the real valued function of k x n matrices $U = (u_{ij})$ defined by $\eta(U) = \log h[(e^{-u_{ij}})]$. Then $\eta(\alpha U) \leq \alpha \eta(U)$ whenever $\alpha \in [0, 1]$.

Proof :

It can be easily verified that this inequality is equivalent to the statement that $\frac{\eta(\alpha U)}{\alpha}$ is increasing in $\alpha > 0$. Hence it is enough to prove that

$$\frac{\partial}{\partial \alpha} [\frac{\eta(\alpha U)}{\alpha}] \geq 0 \quad \text{for all } \alpha > 0. \text{ With}$$

$V_{ij} = e^{-\alpha u_{ij}}$ this implies,

$$- \alpha \frac{\partial}{\partial \alpha} [\log h(V_{ij})] \geq - \log h(V_{ij})$$

$$\Longrightarrow \frac{-\alpha}{h(V_{ij})} \left[\sum_i \sum_j \frac{\partial h(V_{ij})}{\partial V_{ij}} \quad \frac{\partial V_{ij}}{\partial \alpha} \right] \geq - \log h(V_{ij})$$

$$\Longrightarrow -\alpha \left[\sum_i \sum_j \frac{\partial h(V_{ij})}{\partial V_{ij}} \cdot V_{ij}(-u_{ij}) \right] \geq - \qquad h(V_{ij}) \log h(V_{ij})$$

$$\Longrightarrow -\alpha \left[\sum_i \sum_j \frac{\partial h(V_{ij}) V_{ij}}{\partial V_{ij}} \left[\frac{\log V_{ij}}{\alpha} \right] \right] \geq - h(V_{ij}) \log h(V_{ij})$$

$$\Longrightarrow \qquad \sum_i \sum_j \psi(V_{ij}) \frac{\partial h}{\partial V_{ij}} \geq \psi[h(V_{ij})]$$

and by lemma 3.3.3 this holds. Hence the lemma.

Next, in order to prove $C \Longrightarrow A$, we go back to (3.3.1). Let R_i, be the hazard function of X_i, $i=1,2,..,k$. Then

$$
\begin{aligned}
R(\alpha t_1,\ldots,\alpha t_n) &= - \log h(\mathbb{P}^{\alpha}) \\
&= - \log h[(\bar{F}_i(\alpha t_j))] \\
&= - \log h[(e^{-\log}(\bar{F}_i(\alpha t_j))] \\
&= \eta[(R_i(\alpha t_j)] \\
&\leq \eta[\alpha R_i(t_j)] \quad \text{[since each } X_i \text{ is IHRA and } \eta \text{ is increasing in each argument]} \\
&\leq \alpha\, \eta[R_i(t_j)] \quad \text{[by lemma 3.3.4]} \\
&= \alpha R (t_1,\ldots,t_n).
\end{aligned}
$$

Thus 3.3.1 and hence the result follows.

3.3.5 Remark :

We note that the MVE distribution satisfies condition D and hence all other conditions.

Our next section presents counter examples to show that no other relationships hold among these conditions.

3.4 Counter Example :

(1) $C \not\Rightarrow D$. Let (T_1, T_2) satisfy condition D. Therefore we have $T_1 = \min_{i \in S_1} X_i$ and $T_2 = \min_{i \in S_2} X_i$ where S_1 and S_2 are subsets of $\{1,\ldots,k\}$ and $X_1 \ldots, X_k$ are mutually indeppendent. We can write $T_1 = \min(X,Z)$ and $T_2 = \min(Y,Z)$ where

$X = \min_{i \in S_1 - S_2 \cap S_1} X_i$, $Y = \min_{i \in S_2 - S_1 \cap S_2} X_i$ and $Z = \min_{i \in S_1 \cap S_2} X_i$

[minimum over an empty set is to be interpreted as ∞]

We note that X, Y and Z are mutually independent r.v.s. Consequently the joint survival function of (T_1, T_2) has the form

$P[T_1 > t_1,\ T_2 > t_2] = \bar{F}_X(t_1)\bar{F}_Y(t_2)\bar{F}_Z(\max(t_1, t_2))$ so that..(3.4.1)

$$P[T_1 \leq t_1,\ T_2 \leq t_2] = 1 - \bar{F}_{T_1}(t_1) - \bar{F}_{T_2}(t_2) + \bar{F}_{T_1 T_2}(t_1, t_2)$$
$$= 1 - \bar{F}_X(t_1)\bar{F}_Z(t_1) - \bar{F}_Y(t_2)\bar{F}_Z(t_2) +$$
$$\bar{F}_X(t_1)\bar{F}_Y(t_2)\bar{F}_Z[\max(t_1, t_2)] \quad ..(3.4.2)$$

Next, consider the random variables T_1' and T_2' of the form $T_1' = \max(U,W)$, $T_2' = \max(V,W)$ where U,V and W are independent and uniformaly distributed over (0,1). Then (T_1', T_2') satisfy condition C and $P[T_1 \leq t_1, T_2 \leq t_2] =$ $t_1 . t_2 \min(t_1, t_2)$ for $0 \leq t_1,\ t_2 \leq 1$. .. (3.4.3)

If (T_1', T_2') also satisfy condition D, then (3.4.3) must be of the form (3.4.2) for some independent random variables X,Y,Z. Putting $t_2 = 0$ in (3.4.1) we get $\bar{F}_{T_1}(t_1) = \bar{F}_X(t_1)\,\bar{F}_Z(t_1)$. Similarly we get $\bar{F}_{T_2}(t_2) = \bar{F}_Y(t_2)\,\bar{F}_Z(t_2)$. This together with (3.4.3) gives $\bar{F}_X(t_1)\,\bar{F}_Z(t_1) = 1 - t_1^2$ and $\bar{F}_Y(t_2)\bar{F}_Z(t_2) = 1 - t_2^2$. Putting this in (3.4.2) and again using (3.4.3) with the convention that $t_1 < t_2$ we get

$$t_1^2\, t_2 = 1 - (1 - t_1^2) - (1 - t_2^2) + \bar{F}_X(t_1)(1 - t_2^2)$$

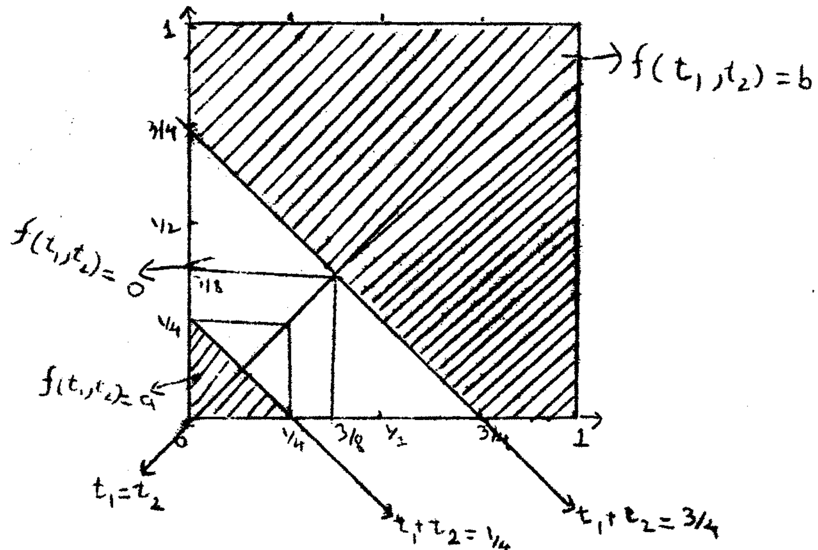
$$= t_1^2 - (1 - t_2^2)\, F_X(t_1)$$

After simplification, this gives $F_X(t_1) = \frac{t_1^2}{1+t_2}$ which implies that $F_X(t_1)$ depends upon t_2, which is a contradiction. Thus (T_1', T_2') can not satisfy condition D which implies that condition $C \not\Rightarrow D$.

(2) $A, F \not\Rightarrow B$ [consequently $A, F \not\Rightarrow C$ and $E \not\Rightarrow B$]: Suppose that (T_1, T_2) has density

$$f(t_1, t_2) = \begin{cases} a & \text{if } t_1 \geq 0,\ t_2 \geq 0 \text{ and } t_1 + t_2 \leq \frac{1}{4}. \\ b & \text{if } 0 \leq t_1 \leq 1,\ 0 \leq t_2 \leq 1,\ t_1 + t_2 \geq \frac{3}{4}. \\ 0 & \text{elsewhere.} \end{cases}$$

Where $a = \frac{32}{47}$, $b = 2a = \frac{64}{47}$. This is illustrated in the figure below :



From the figure it is clear that $\max(T_1,T_2)$ has no density in the region (1/4,3/8) and hence it can not be IFRA.

[Since for $\frac{1}{4}\leq t_1<t_2\leq\frac{3}{8}$, $R(t_1) = R(t_2)$ and $\frac{R(t_1)}{t_1} > \frac{R(t_2)}{t_2}$].

Thus condition B fails. [Here we have shown that life time of parallel system of componants is not IFRA]. Next, we will show that (T_1,T_2) satisfy condition F (and hence A). Let $Y = \min(\alpha_1 T_1,\alpha_2 T_2)$. Let $\alpha_1 > \alpha_2$. We have

$\bar{F}_Y(t) = P[T_1> \frac{t}{\alpha_1}, T_2> \frac{t}{\alpha_2}]= P[T_1>x, T_2>m x]$ where $x=\frac{t}{\alpha_1}, m=\frac{\alpha_1}{\alpha_2}$.

Thus to find survival function of Y, it is enough to find

$\bar{F}_m(x) = P[T_1>x, T_2>m x]$ for all $x \geq 0$, $m \in [0,1]$ .. (3.4.4)

From the joint distribution of (T_1,T_2) this probability is obtained as follows :

$$\bar{F}_m(x) = \frac{a}{4}\, x[3(m+1)+2(1-m)^2 x] \qquad 0 \le x \le \frac{1}{4(m+1)}$$

$$= \frac{b}{4}\, x[1+m+2(1+m^2)x] \qquad \frac{1}{4(m+1)} \le x \le \frac{3}{4(m+1)}$$

$$= bx\,[1+m-mx] \qquad \frac{3}{4(m+1)} \le x \le 1.$$

$$= 0 \qquad \text{elsewhere.}$$

Differentiation of $\bar{F}_m(x)$ with a negative sign gives it's density as

$$f_m(x) = \frac{a}{4}\,[3(m+1)+4x(1-m)^2] \qquad 0 \le x \le \frac{1}{4(m+1)}$$

$$= \frac{b}{4}\,[1+m+4(1+m^2)x] \qquad \frac{1}{4(m+1)} \le x \le \frac{3}{4(m+1)}$$

$$= b\,(1+m-2mx) \qquad \frac{3}{4(m+1)} \le x \le 1.$$

$$= 0 \qquad \text{elsewhere.}$$

From the above expression it can be easily seen that $\frac{\partial^2 \log f_m(x)}{\partial x^2}$ is of the form $\frac{-C_2^2}{(C_1+C_2x)^2}$ where $C_1, C_2 > 0$ are constants, which implies that $\log f_m$ is concave, so that f_m is a PF_2 density. Thus Y is IFR. The case $\alpha_1 > \alpha_2$ follows by symmetry. Thus min $(\alpha_1 T_1, \alpha_2 T_2)$ is IFR for all $\alpha_1, \alpha_2 > 0$, and hence (T_1, T_2) satisfy conditions A and F. Thus A, F $\not\Rightarrow$ B.

<u>B $\not\Rightarrow$ A, F</u> (consequently B $\not\Rightarrow$ C and C $\not\Rightarrow$ A,F). Let T_1 be uniformly distributed on [0,1] and let T_2 be equal to $T_1 + \frac{1}{2}$ if $0 \le T_1 \le \frac{1}{2}$ and $T_1 - \frac{1}{2}$ if $\frac{1}{2} < T_1 \le 1$.

The joint distribution function of (T_1, T_2) can be easily computed as

$$
\begin{aligned}
F(t_1,t_2) &= 0 && \text{if (a) } 0 \le t_1, t_2 \le 1/2 \\
&= t_1+t_2-1 && \text{if (b) } 1/2 \le t_1, t_2 \le 1. \\
&= t_1 - \tfrac{1}{2} && \text{if (c) } t_2 \le \tfrac{1}{2} \le t_1 \text{ and } \\
& && t_1 - t_2 - \tfrac{1}{2} \le 0 . \\
&= t_2 && \text{if (d) } t_2 \le \tfrac{1}{2} \le t_1 \text{ and } t_1 - t_2 - \tfrac{1}{2} \ge 0. \\
&= t_2 - \tfrac{1}{2} && \text{(e) } t_1 \le \tfrac{1}{2} \le t_2 \text{ and } t_2 - t_1 - \tfrac{1}{2} \le 0. \\
&= t_1 && \text{(f) } t_1 \le \tfrac{1}{2} \le t_2 \text{ and } t_2 - t_1 - \tfrac{1}{2} \ge 0.
\end{aligned}
$$

From (b),(d) and (f) it follows that marginal distribution of both T_1 and T_2 is $U(0,1)$. Also

$$
\begin{aligned}
P[\min(T_1,T_2)>t] = P[T_1>t, T_2>t] &= 1 - F_1(t) - F_2(t) + F(t, t) \\
&= 1 - 2t \qquad 0 \le t \le \tfrac{1}{2} \\
&= 0 \qquad \text{otherwise}
\end{aligned}
\quad \Bigg\} \text{ using (a) and (b)}
$$

Which has density $f(t) = 2,\ 0 \le t \le \frac{1}{2}$. Hence $\min(T_1,T_2)$ is $U(0, \frac{1}{2})$. Also

$$
\begin{aligned}
P[\max(T_1,T_2)<t] = P[T_1 \le t, T_2 \le t] &= 2t - 1, \quad \tfrac{1}{2} \le t \le 1 \\
&= 0 \quad \text{otherwise.}
\end{aligned}
$$

Which has density $f(t) = 2, \frac{1}{2} \le t \le 1$. Hence $\max(T_1,T_2)$ is $U(\frac{1}{2},1)$. Thus all possible coherent systems which can be formed out of the two componants have uniform and hence IHRA distribution. Thus condition B holds. Next we show that condition A is violated by (T_1, T_2).

Consider $Y = \min(T_1, aT_2)$ where $a > 2$

$$
\begin{aligned}
P[Y \leq Z] &= 1 - P[\,Y > z\,] \\
&= 1 - P[T_1 > z,\ T_2 > \tfrac{z}{a}] \\
&= 1 - [1 - F_1(z) - F_2(z/a) + F(z, \tfrac{z}{a})] \\
&= F_1(z) + F_2(z/a) - F(z, z/a) \\
&= z + \tfrac{z}{a} - F(z, z/a) \qquad [\text{since } T_1, T_2 \text{ are } U(0,1)] \\
&= z + \tfrac{z}{a} \qquad \text{for } z \leq 1/2 \\
&= z + \tfrac{z}{a} - (z - \tfrac{1}{2}) = \tfrac{z}{a} + \tfrac{1}{2} \qquad 1/2 \leq z \leq a/2(a-1) \\
&= z + \tfrac{z}{a} - \tfrac{z}{a} = z \qquad \text{for } a/2(a-1) \leq z \leq 1.
\end{aligned}
$$

For Y to have IFRA distribution $-z^{-1}\log\bar{F}_Y(z)$ must be increasing on $(0,\infty)$. The derivative of $z^{-1} \log P\,[\min(T_1, aT_2) \leq z]$ for $\frac{1}{2} \leq z \leq \frac{a}{a-1}$ is non-negative iff $z \cdot \frac{2}{az + a} \leq \log\left(\frac{z}{a} + \frac{1}{2}\right)$ which is violated for $z < 1$, hence condition F (and A) does not hold. Thus condition B does not imply conditions A and F.

3.5 Some Properties of the Conditions :

3.5.1 It is easily varifiable that all the conditions A to F satisfy (P1) :$(T_1, \ldots, T_n)$ satisfy condition(*)$\Longrightarrow$ each nonempty subset of$(T_1, \ldots, T_n)$ satisfy condition (*).

Condition A and C to F also satisfy

(P2) :$(S_1,\ldots,S_n)$ satisfy condition (*), $(T_1,\ldots,T_n)$ satisfy condition (*) and $(S_1,\ldots,S_n),(T_1,\ldots,T_m)$ are independent $\Longrightarrow (S_1,\ldots,S_n, T_1,\ldots,T_n)$ satisfy condition(*). Whether or not condition B satisfies P_2 is unknown.

<u>3.5.2 Relation with Assosiation</u> :

The random variables $T_1,\ldots,T_n$ of conditions c And D are generated as increasing functions of independent random variables, and as such they are associated. On the other hand, let U be uniformly distributed over [0,1] and V = 1-U then $\bar{F}_{(u,v)} = P[u<U<1-v] = 1-v-u$ if $u+v<1$

$= 0$ otherwise.

Hence

$$\bar{F}\left(\frac{u}{a_1}, \frac{u}{a_2}\right) = 1 - u\left[\frac{a_1+a_2}{a_1 a_2}\right] \quad \text{if } u < \frac{a_1 a_2}{a_1+a_2}$$

$$= 0 \quad \text{otherwise.}$$

Which implies that min (a_1U_1, a_2V_2) is $U[0,\frac{a_1a_2}{a_1+a_2}]$ and hence it is IFRA. Thus (U,V) satisfies conditions A and F.

Also,

$$F(u,u) = 2u - 1 \qquad \tfrac{1}{2} \leq u \leq 1$$

$$= 0 \qquad \text{otherwise.}$$

which implies that max(U,V) is $U(0, \frac{1}{2})$ and is IHRA. Thus U,V, min(U,V) and max(U,V) are all IHRA and thus (u,v) satisfies conditions B and E.

Since U and V have correlation -1, the conditions A,B,E and F do not imply association or any other notion of positive dependance.

3.5.3 Absolute Continuity and Independence :

(a) If $T_1,\ldots,T_n$ satisfy condition D and are jointly absolutely contineous, then they are independent.

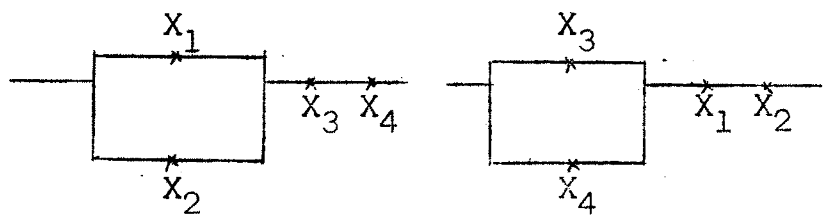
Proof :

Suppose that $T_1 = \min_{j \in S_1} X_j$ and $T_2 = \min_{j \in S_2} X_j$ where X_j are independent.

Let $S_1' = S_1 \cap S_2^c$, $S_2' = S_2 \cap S_1^c$, $S_{12}' = S_1 \cap S_2$. Let $Y = \min_{j \in S_1'} X_j$, $Z = \min_{j \in S_2'} X_j$, $W = \min_{j \in S_{12}'} X_j$. Then

$T_1 = \min(Y,W)$, $T_2 = \min(Z,W)$ and Y,Z,W are independent. Since T_1,T_2 are jointly absolutely contineous, $P[T_1=T_2=W]=0$. Hence S_1 and S_2 are disjoint w.p.1. Hence T_1,T_2 are independent. Under condition D, pairwise independance implies mutual independance (since then all S_i^s can be taken to be disjoint). Hence the result follows.

(b) There do exist absolutely contineous distributions satisfing condition C where the random variables are dependent. For example, Suppose X_i has the absolutely contineous distribution F_i $i=1,2,3,4$, Let T_1 and T_2 be the life length of the coherent systems given below :



T_1 T_2

For $t_1 \leq t_2$ Let

A be the event that $X_1 > t_2$

B " " " " $X_2 > t_2$

C " " " " $(X_3 > t_1,\ X_4 > t_2) \cup (X_3 > t_2,\ X_4 > t_1)$

we note that the events A,B and C are independent,and $P[A] = \bar{F}_1(t_2)$, $P[B] = \bar{F}_2(t_2)$, $P[C] = [\bar{F}_3(t_1)\bar{F}_4(t_2) + \bar{F}_3(t_2)\bar{F}_4(t_1) - \bar{F}_3(t_2)\bar{F}_4(t_2)]$. It follows that

$$\bar{F}(t_1, t_2) = P[T_1 > t_1,\ T_2 > t_2] = P[A \cap B \cap C] = P[A]\ P[B]\ P[C]$$

$$= \bar{F}_1(t_2)\bar{F}_2(t_2)[\bar{F}_3(t_1)\bar{F}_4(t_2) + \bar{F}_3(t_2)\bar{F}_4(t_1) - \bar{F}_3(t_2)\bar{F}_4(t_2)]$$

From this expression, the absolute continuity of joint distribution of T_1, T_2 is evident. But since $T_1 = \max[\min(X_1, X_3, X_4),\ \min(X_2, X_3, X_4)]$ and $T_2 = \max[\min(X_1, X_2, X_3),\ \min(X_1, X_2, X_4)]$ it follows that T_1 and T_2 are dependent.

In the next section we discuss the MIFRA class putforth by H.Block and T.H.Savits [1981] .

3.6 The MIFRA class of Block and Savits :

In 1976 a new charactorization of univariate IFRA distribution was obtained by Block and Savits in terms of an integral inequality : a life distribution F is IFRA iff for every nonnegative nondecreasing function h,

$$\int h(x)dF(x) \leq [\int h^{\alpha}(x/\alpha)dF(x)]^{1/\alpha} \quad 0 < \alpha \leq 1. \quad .. (3.6.1)$$

In this section we present the natural multivariate extension of (3.6.1) and investigate the properties of the class of distributions satisfing this extension.

3.6.1 Definition :

Let $\underline{T} = (T_1,..,T_n)$ be a nonnegative random vector with distribution function F. Then $\underline{T}$ is said to have a multivariate IFRA distribution iff

$$E[h(\underline{T})] \leq E^{1/\alpha}[h^{\alpha}(\frac{1}{\alpha}\underline{T})], \quad 0 < \alpha \leq 1 \quad ... \quad (3.6.2)$$

for all contineous, nonnegative, nondecreasing function h.

3.6.2 Remark :

The continuity assumption on h is a technical simplification. In section 3.8 we show that this assumption on h can be relaxed.

3.6.3 Theorem :

The class π of multivariate IFRA distributions satisfing (3.6.2) possesses the following properties :

(P1) π is closed under the formation of coherent systems.

(P2) π is closed under limits in distributions.

(P3) If $\underline{T} \in \pi$, any joint marginal belongs to π.

(P4) If $\underline{T} = (T_1,\ldots,T_n)$, $\underline{S} = (S_1,\ldots, S_m) \in \pi$ and are independent, then $(\underline{T}, \underline{S}) \in \pi$.

(P5) π is closed under nonnegative scaling.

(P6) π is closed under convolution (when the operation makes sense).

(P7) If $\underline{T} \in \pi$ and $\tau_1,\ldots, \tau_m$ are any coherent life function of order n, then $(\tau_1(\underline{T}),\ldots,\tau_m(\underline{T})) \in \pi$.

Before we prove this theorem, we establish the following lemma :

3.6.4 Lemma :

Let $\underline{T} \in \pi$ and $\psi_1,\ldots,\psi_m$ be any functions of n variables which are contineous, nondecreasing and satisfy the inequality $\psi_i(\underline{x}/\alpha) \leq \frac{1}{\alpha}\, \psi_i(\underline{x})$ for all $\underline{x} \in R^n$ and $0<\alpha\leq 1$. Then setting $S_i = \psi_i(\underline{T})$ for $i = 1,\ldots,m$ it follows that $\underline{S} = (S_1,\ldots,S_m) \in \pi$.

Proof :

Let h be any contineous, nonnegative, nondecreasing function of m variables. Then for $0 < \alpha \leq 1$,

$E[\,h(\underline{S})]$

$$= E[\,h(\psi_1(\underline{T}),\ldots,\psi_m(\underline{T})]$$

$$\leq E^{1/\alpha}[\,h^{\alpha}(\psi(\underline{T}/\alpha),\ldots,\psi_m(\underline{T}/\alpha)] \qquad \text{since } \underline{T} \in \pi$$

$$\leq E^{1/\alpha}[\,h^{\alpha}[\,\tfrac{1}{\alpha}\psi_1(\underline{T}),\ldots,\,\tfrac{1}{\alpha}\psi_m(\underline{T})]] \quad \text{by hypothesis.}$$

$$= E^{1/\alpha}[\,h^{\alpha}(\underline{S}/\alpha)].$$

Hence the lemma.

The proof of the main theorem easily follows using this lemma, which is presented below :

Proof of theorem 3.6.3 :

(P1) and (P7): Since (P7) reduces to (P1) when $m = 1$, we only need to prove (P7). Let $\tau_1,\ldots,\tau_m$ be coherent life functions of order n, corrosponding to the coherent structure functions $\phi_1,\ldots,\phi_m$ of order n respectively. Let $P_{i1},\ldots,P_{ip_i}$ be the minimal path sets for ϕ_i, $i = 1,\ldots,m$. Therefore we have $\tau_i(\underline{X}) = \max\limits_{1\leq k\leq p_i} \min\limits_{j\in P_{ik}} X_j$. Since $\tau_i(\frac{1}{\alpha}\underline{X}) = \max\limits_{1\leq k\leq p_i} \min\limits_{j\in P_{ik}} X_j/\alpha = \frac{1}{\alpha}\,\tau_i(\underline{X})$, the result follows by lemma 3.6.4 .

(P2) : Suppose that for every k, $\underline{T}_k = (T_{1k},\ldots,T_{nk}) \in \pi$ and converges weakly to $\underline{T} = (T_1,\ldots,T_n)$ as $k\to\infty$. Let h be any contineous, nonnegative, nondecreasing function. Let $0 < \alpha \leq 1$ and N be any nonnegative real number. We also let N denote the constant function whose value is N. Then by definition of weak convergence , we have that,

$E[h \wedge N(\underline{T}_k) \longrightarrow E[h \wedge N(\underline{T})]$ and

$E[(h \wedge N)^{\alpha}(\frac{1}{\alpha}\underline{T}_k)] \longrightarrow E[(h \wedge N)^{\alpha}(\frac{1}{\alpha}\underline{T})]$ as $k \longrightarrow \infty$, where $h \wedge N = \min(h,N)$. Since

$E[h \wedge N(\underline{T}_k)] \leq E^{1/\alpha}[(h \wedge N)^{\alpha}(\frac{1}{\alpha}\underline{T}_k)]$ for all k, by letting $k \longrightarrow \infty$ on both sides we get

$E[h \wedge N(\underline{T})] \leq E^{1/\alpha}[(h \wedge N)^{\alpha}(\frac{1}{\alpha}\underline{T})]$ for all N.

We note that $h \wedge N \uparrow h$ and $(h \wedge N)^{\alpha} \uparrow h^{\alpha}$ as $N \to \infty$. The result now follows by letting $N \longrightarrow \infty$ and using monotone convergence theorem.

(P3): By taking $\tau_j(\underline{T}) = T_{i_j}$ $j = 1,\ldots,m$ in (P7) it follows that $(T_{i_1},\ldots,T_{i_m}) \in \pi$ for all nonempty subsets $\{i_1,\ldots,i_m\}$ of $\{1,2,\ldots,n\}$.

(P4): Let $\underline{T}$ and $\underline{S}$ have joint distribution function F and G respectively. Let $h(\underline{x}, \underline{y})$ where $\underline{x} \in R^n$ and $\underline{y} \in R^m$ be contineous, bounded, nonnegative and nondecreasing. Then

$$
\begin{aligned}
E[h(\underline{T}, \underline{S})] &= \iint h(\underline{x},\underline{y})\, dF(\underline{x})dG(\underline{y}) \\
&\leq \int [\int h^{\alpha}(\tfrac{1}{\alpha}\underline{x}, \underline{y})\, dF(\underline{x})]^{1/\alpha}\, dG(\underline{y}) \qquad \text{since } \underline{T} \in \pi \\
&\leq \{\int [[\int h^{\alpha}(\tfrac{1}{\alpha}\underline{x}, \tfrac{1}{\alpha}\underline{y})\, dF(\underline{x})]^{1/\alpha}]^{\alpha}\, dG(\underline{y})\}^{1/\alpha} \qquad \text{since } \underline{S} \in \pi \\
&= [\iint h^{\alpha}(\tfrac{1}{\alpha}\underline{x}, \tfrac{1}{\alpha}\underline{y})\, dF(\underline{x})\, dG(\underline{y})]^{1/\alpha} \\
&= E^{1/\alpha}[h^{\alpha}[\tfrac{1}{\alpha}\underline{T}, \tfrac{1}{\alpha}\underline{S}]]
\end{aligned}
$$

If h is not bounded, then we consider $h \wedge N$ and let $N \longrightarrow \infty$. The result follows by monotone convergence theorem.

(P5): Let $a_1, a_2, \ldots, a_n \geq 0$ and set $\psi_i(x_1,\ldots,x_n) = a_i x_i$ ($1 \leq i \leq n$). Then the result follows by lemma 3.6.4.

(P6): If $\underline{T} = (T_1,\ldots,T_n)$ and $\underline{S} = (S_1,\ldots,S_n) \in \pi$ and are independent, the convolution corrousponds to $(T_1+S_1,\ldots,T_n+S_n)$. By (P4) $(\underline{T},\underline{S}) \in \pi$. We set $\psi_i((\underline{x},\underline{y}) = x_i+y_i \quad 1 \leq i \leq n$. The result now follows by lemma 3.6.4.

3.6.5 Remark :

(i) If $\underline{T} \in \pi$ and $\underline{b} \geq 0$ is vector of constants, then $(\underline{T} + \underline{b}) \in \pi$, since let $\psi_i(\underline{x}) = x_i+b_i$ and $\psi_i(\frac{1}{\alpha}\underline{x}) = \frac{1}{\alpha}x_i+b_i \leq \frac{1}{\alpha}(x_i+b_i) = \frac{1}{\alpha}\psi_i(\underline{x})$ for $0 < \alpha \leq 1$. Again the result follows by lemma 3.6.4.

(ii) Using lemma 3.6.4 it is easy to show that a generalised version of (P6) holds. i.e. $\underline{T} \in \pi$ and $S_1,\ldots,S_m$ are nonempty subsets of $\{1,2,\ldots,n\}$ implies that $(\sum_{i \in S_1} T_i, \ldots, \sum_{i \in S_m} T_i) \in \pi$. To see this, we take $\psi_i(\underline{x}) = \sum_{j \in S_i} X_j$, $i = 1,\ldots,m$ and the result follows.

3.7 Examples of Multivariate IFRA Distributions :

(A) Generated from univariate independent IFRA distributions :

The following theorem gives functions of independent

IFRA distributions which are multivariate IFRA:

3.7.1 Theorem :

Let $X_1, \ldots, X_n$ be independent IFRA random variables and let $\emptyset \neq S_i \subset \{1,2,\ldots,n\}$ for $i = 1,\ldots,m$.

(i) $(X_1,\ldots,X_n) \in \pi$

(ii) If $T_i = \min_{j \in S_i} X_j$ $i = 1,\ldots,m$. Then $(T_1,\ldots,T_m) \in \pi$

(iii)If $\tau_1,\ldots,\tau_m$ are coherent life functions of order n, then $(\tau_1(X_1,\ldots,X_n), \ldots, \tau_m(X_1,\ldots,X_n)) \in \pi$.

(iv) If $T_i = \sum_{j \in S_i} X_j$ $i = 1,\ldots,m$ then $(T_1,\ldots,T_m) \in \pi$.

Proof :

All of these easily follow from theorem 3.6.3 and lemma 3.6.4.

3.7.2 Corollary :

The multivariate exponential distribution of Marshall and Olkin is MIFRA.

Proof :

Let $\underline{T} = (T_1,\ldots,T_n)$ has MVE distribution. Then it has a representation $T_i = \min [X_J ; i \in J]$ $i=1,2,\ldots,n$ where the sets J are the elements of a class $\mathcal{J}$ of nonempty subsets of $\{1,\ldots,n\}$ and random variables X_J, $J \in \mathcal{J}$ are independent exponentially distributed. The result now follows by using theorem 3.7.1 (ii).

3.7.3 Corollary :

Let $X_1,\ldots,X_n$ be independent identically distributed IFRA random variables and $Y_1,\ldots,Y_n$ be the corrousponding order statistics. Then $(Y_1,\ldots,Y_n) \in \pi$.

Proof :

Let τ_k be the life function corrousponding to a (n-k+1) out of n system. Then $Y_k = \tau_k(X_1,\ldots,X_n)$. Since $(X_1,\ldots,X_n)$ is multivariate IFRA, it follows from theorem 3.7.1 (ii) that $(Y_1,\ldots,Y_n) \in \pi$.

3.7.4 Remark :

It is clear in the previous corollary that the hypothesis can be weakened to $(X_1,\ldots,X_n) \in \pi$.

(B) Multivariate Weibull distributions :

(i) Marshall and Olkin (1967) introduced a multivariate weibull distribution which has the form $(T_1',\ldots,T_n') = (T_1^{1/\alpha},\ldots,T_n^{1/\alpha})$ where $\alpha_i > 0$ $i = 1,\ldots,m$ and $(T_1\ldots,T_n)$ has MVE distribution.

Define $\psi_i(X_1,\ldots,X_n) = x_i^{1/\alpha_i}$ $i = 1,\ldots,n$. Then for $\alpha_i \geq 1$, $i=1,\ldots,n$, ψ_i^s satisfy the hypothesis of lemma 3.6.4. Also by corollary 3.7.2 $(T_1,\ldots,T_n) \in \pi$. Hence by lemma 3.6.4 it follows that $(T_1',\ldots,T_n') \in \pi$, for $\alpha_i \geq 1$ $i = 1,\ldots,n$.

(ii) A second type of multivariate weibull distribution was introduced by David and by Lee and Thompson (1974) and has the form $(T_1,\ldots,T_n)$ where $T_i = \min(U_J: i \in J)$, $\emptyset \neq J \subset \{1,\ldots,n\}$, $P[U_J > x] = e^{-\lambda_j x^{\alpha_j}}$ $x \geq 0$ and the U_J are independent.

We note that for $\alpha_j \geq 1$, U_J^s are univariate IFRA and hence by theorem 3.7.1 (ii) it follows that $(T_1,\ldots,T_n) \in \pi$.

(C) Multivariate gamma distribution :

Let $X_o, X_1, \ldots, X_n$ are independent random variables, X_j having standard gamma distribution with parameter θ_j $j = 0,1,\ldots,m$. We define $Y_j = X_o + X_j$ $j = 1,\ldots,m$. Johnson and Kotz (1977) have shown that $(Y_1,\ldots,Y_m)$ has multivariate gamma distribution given by

$$F_{Y_1,\ldots,Y_m}(y_1,\ldots,y_m) = \int_0^{\bar{y}} x_o^{\theta_o - 1}\left[\prod_{j=1}^{m}(y_j - x_o)^{\theta_j - 1}\right] e^{(m-1)x_o} dx_o$$

where $\bar{y} = \min(y_1,\ldots,y_m)$. Since for $\theta_j \geq 1$, X_j has univariate IFRA distribution, it follows by theorem 3.7.1 (iv) that $(Y_1,\ldots,Y_m) \in \pi$.

(D) A Bivariate exponential distribution :

Johnson and Kotz (1977) give the following bivariate exponential distribution,

Let U_o, U_1, U_2 be independent standard normal variates.

Define $X_j = U_o^2 + U_j^2 \quad j = 1,2.$ Then it has been claimed that (X_1, X_2) has bivariate IFRA distribution. Also it has been claimed that this follows by a similar argument as for the multivariate gamma distribution given in (C). But we find that this argument is not correct, and the problem of determining whether (X_1, X_2) is bivariate IFRA or not remains unsolved.

(E) Construction of MIFRA distributions :

Suppose that $(X_1, \ldots, X_n)$ has a MIFRA distribution and let Y be any nonnegative random variable on the same probability space. In this section we investigate conditions under which $(X_1, \ldots, X_n, Y)$ has also multivariate IFRA distribution.

Let $\bar{G}(y/x_1, \ldots, x_n) = P[Y > y / X_1 = x_1, \ldots, X_n = x_n]$ for $x_i \geq 0$, $i = 1, \ldots, n$ and $y > 0$. The random variable Y is said to be stochastically increasing in $(X_1, \ldots, X_n)$ if $\bar{G}(y/x_1, \ldots, x_n)$ is nondecreasing in $x_1, \ldots, x_n$. If $\bar{G}(y/x_1, \ldots, x_n)$ is contineous in $x_1, \ldots, x_n$ we say Y is stochastically contineous in $X_1, \ldots, X_n$.

First we prove the following two lemmas.

3.7.5 Lemma :

Assume Y is stochastically increasing and contineous in $X_1,\dots,X_n$. Then $E[\phi(Y)/X_1=x_1,\dots,X_n=x_n]$ is contineous nonnegative and nondecreasing in $x_1,\dots,x_n$ for every contineous, nonnegative, nondecreasing and bounded function ϕ.

Proof :

We note that $E[\phi(y)/\underline{X}=\underline{x}] = \int \phi(y)\ dG(y/\underline{x})$ and is nonnegative for every $\underline{x} \in R^n$ since $\phi(y)$ is nonnegative. Next, for $\phi(y) = C.I_{(t,\infty)}(y)$ we have $E[\phi(Y)/\underline{x}] = C.\bar{G}(t/\underline{x})$ where $C > 0$ is constant and by hypothesis, this is contineous and nondecreasing in $x_1,\dots,x_n$ for all $t \geq 0$. (3.7.1)

Now let ϕ be any contineous, nondecreasing, nonnegative and bounded function of y. Let

$D_{ik} = \{y : \phi(y) > i2^{-k}\}$ $i = 1,2,\dots,k.2^k$; $k = 1,2,\dots$

We define $\phi_k(y) = 2^{-k} \sum_{i=1}^{k.2^k} I_{D_{ik}}(y)$. It follows from (3.7.1) that for all k, $E[\phi_k(Y)/\underline{X}=\underline{x}]$ is contineous and nondecreasing in $x_1,\dots,x_n$. (3.7.2) We observe that $\phi_k(y)=2^{-k}\ i$ for $2^{-k}\ i < \phi(y) \leq 2^{-k}(i+1)$ and hence as $k \longrightarrow \infty$ $\phi_k(y) \longrightarrow \phi(y)$ for all y. It is easy to see that $\phi_k(y)$ is increasing in k and hence by monotone convergence theorem it follows that

$$\int \phi_k(y)\ dG(y/\underline{x}) \longrightarrow \int \phi(y)\ dG(y/\underline{x}) \quad \text{for all } \underline{x} \in R^n.$$

i.e. $E[\phi_k(Y)/\underline{X}=\underline{x}] \longrightarrow E[\phi(Y)/\underline{X}=\underline{x}]$ for all $\underline{x} \in R^n$. By (3.7.2) now it follows that $E[\phi(Y)/\underline{X}=\underline{x}]$ is nondecreasing in $x_1,\dots,x_n$.

To prove the continuity, we proceed as follows : Consider a sequence of points in R^n, $\underline{x}_k=(x_{1k},\ldots,x_{nk})$ which converges in $\underline{x}$. Since y is stochastically contineous in $\underline{x}$, it follows that $G(y/\underline{x}_k)\rightarrow G(y/\underline{x})$ for every y. Now by using Helly-Bray theorem it follows that $\int\phi(y)\,dG(y/\underline{x}_k)\rightarrow\int\phi(y)\,dG(y/\underline{x})$. i.e. $E[\phi(y)/\underline{X}=\underline{x}_k]\rightarrow E[\phi(y)/\underline{X}=\underline{x}]$ for every sequence $\underline{x}_k \in R^n$ converging to $\underline{x}$. Hence $E[\phi(y)/\underline{X}=\underline{x}]$ is contineous in $\underline{x}$.

3.7.6 Lemma :

If $\bar{G}(y/\frac{1}{\alpha}\underline{x}) \leq \bar{G}^{1/\alpha}(\alpha y/\underline{x})$ $0 < \alpha \leq 1$ then for every nonnegative and nondecreasing ϕ,

$$E[\phi(y)/\tfrac{1}{\alpha}\underline{x}] \leq E^{1/\alpha}[\phi^{\alpha}(y/\alpha)/\underline{x}]$$

Proof :

For $\phi(y) = C.\,I_{(t,\infty)}(y)$ $t \geq 0$,

$$\begin{aligned} E[\phi(y)/\tfrac{1}{\alpha}\underline{x}] &= C.\,\bar{G}(t/\tfrac{1}{\alpha}\underline{x}) \\ &\leq C.\,\bar{G}^{1/\alpha}[\alpha t/\underline{x}] \quad \text{by hypothesis} \\ &= E^{1/\alpha}[\phi^{\alpha}(y/\alpha)/\underline{x}] \quad \text{since } \phi(\tfrac{y}{\alpha})=C.I_{(\alpha t,\infty)}(y) \end{aligned} \quad \ldots (3.7.3)$$

Now let ϕ be any nonnegative, nondecreasing function of Y. As in the proof of previous lemma, we construct the sequence $\phi_k = 2^{-k}\sum_{i=1}^{k.2^k} I_{D_{ik}}$ of simple functions which increases to ϕ.

Let us write $\phi_k = \sum_{i=1}^{k.2^k} \phi_{ik}$ where $\phi_{ik} = 2^{-k} I_{D_{ik}}$. We note that for each i, $\phi_{ik}(y) \geq 0$, $\int|\phi_{ik}(y)|dG(y/\underline{x}) < \infty$ and $\int |\phi_k(y)|^\alpha dG(y|\underline{x}) < \infty$ for $\alpha \in (0,1)$. Hence minkowski, inequality is applicable which gives

$$[\int|\phi_k(y)|^\alpha dG(y|\underline{x})]^{1/\alpha} \geq \sum_{i=1}^{k.2^k} [\int|\phi_{ik}(y)|^\alpha dG(y|\underline{x})]^{1/\alpha}$$

i.e. $E^{1/\alpha}[\phi_k^\alpha(Y)/\underline{x}] \geq \sum_{i=1}^{k.2^k} [E^{1/\alpha}[\phi_{ik}^\alpha(Y)/\underline{x}]]$ for all $\alpha \in (0,1)$ and all $\underline{x} \geq 0$. ... (3.7.4)

Also application of (3.7.3) on each ϕ_{ik} gives

$$E[\phi_k(Y)| \frac{1}{\alpha} \underline{x}] = \sum_{i=1}^{k.2^k} E[\phi_{ik}(Y) | \frac{1}{\alpha} \underline{x}] \leq \sum_{i=1}^{k.2^k} E^{1/\alpha}[\phi_{ik}^\alpha(Y/\alpha)|\underline{x}]$$

...(3.7.5)

combining (3.7.5.) & (3.7.4) with $\phi_k(y)$ replaced by $\phi_k(y/\alpha)$ we get $E[\phi_k(Y)| \frac{1}{\alpha} \underline{x}] \leq E^{1/\alpha}[\phi_k^\alpha(\frac{Y}{\alpha})|\underline{x}]$. Thus the lemma holds for every ϕ_k. Now letting $k \longrightarrow \infty$ on both sides and using monotone convergence theorem the result follows for the required function ϕ.

Making use of these two lemmas we prove the following theorem.

3.7.7 Theorem :

Let $\underline{X} = (X_1,\ldots,X_n) \in \pi$ and let Y be stochastically increasing and contineous in $X_1,\ldots,X_n$ and satisfy the inequality in the hypothesis of lemma 3.7.6. Then $(\underline{X},Y)$ is MIFRA.

Proof :

Let $h(\underline{X},Y)$ be any contineous, nonnegative, nondecreasing and bounded function. Then $E[\, h(\underline{X},Y)] = \int\int h(\underline{x},y)\, dG(y/\underline{x})\, dF(\underline{x})$. Since $h(\underline{x},y)$ is nonnegative and nondecreasing in y, it follows from lemma 3.7.6 that

$$\int h(\underline{x},y)\, dG(y/\underline{x}) \leq [\int h^{\alpha}(\underline{x}, \tfrac{y}{\alpha})\, dG(y/\alpha\underline{x})]^{1/\alpha} \text{ for all } \alpha \in (0,1] \quad \ldots \quad (3.7.6)$$

Now since $h^{\alpha}(\underline{x}, \frac{y}{\alpha})$ is contineous, nondecreasing, nonnegative and bounded function of y, by appling lemma 3.7.5 it follows that

$$h^*(\underline{x}) = [\int h^{\alpha}(\underline{x}, \tfrac{y}{\alpha})\, dG(y/\alpha\underline{x})\,]^{1/\alpha}$$ is contineous, nonnegative and nondecreasing in $x_1,\ldots,x_n$. Consequently since $\underline{X} \in \pi$, taking expectation w.r.t. $\underline{X}$ on both sides of (3.7.6) we get

$$\begin{aligned} E\, h(\underline{X},Y) &\leq E\, h^*(\underline{X}) \\ &\leq E^{1/\alpha}[h^{*\alpha}(\tfrac{1}{\alpha}\underline{X})] \qquad \text{since } \underline{X} \in \pi \\ &= [\int [h^{\alpha}(\tfrac{1}{\alpha}\underline{x}, \tfrac{y}{\alpha})\, dG(y/\underline{x})]\, dF(\underline{x})\,]^{1/\alpha} \\ &= E^{1/\alpha}\, h^{\alpha}\, (\tfrac{1}{\alpha}\underline{X}, \tfrac{Y}{\alpha}) \end{aligned}$$

If h is not bounded, we consider $h_N = h \wedge N$. For every N, h_N is bounded and satisfies inequality (3.6.2).

Also $h_N \uparrow h$. Now using monotone convergence theorem it follows that (3.6.2) holds for h. Hence $(\underset{\sim}{X},Y) \in \pi$.

The following corollary is immidiate consequence of the above theorem.

3.7.8 Corollary :

Let $\underset{\sim}{T} = (T_1,\ldots,T_n)$ be a nonnegative random vector such that (i) T_1 is (univariate) IFRA and (ii) for $K = 1,\ldots,n-1$. T_k is stochastically increasing and contineous in $T_1,\ldots,T_{k-1}$ and satisfies the inequality of lemma 3.7.6. Then $\underset{\sim}{T} \in \pi$.

Using above corollary we construct the following IFRA random variable.

Let $\underset{\sim}{X}$ be exponential with parameter $\lambda_1 > 0$ and set

$$\bar{G}(y/x) = \exp(-\lambda_2 y) \qquad y < x$$
$$= \exp[-(\lambda_2 + \lambda_{12})y + \lambda_{12} x] \quad y \geq x$$

where $\lambda_{12}, \lambda_2 \geq 0$. Then (X,Y) is MIFRA with joint distribution

$$\bar{F}_{X,Y}(x,y) = \exp(-\lambda_1 x - \lambda_2 y) \qquad y < x$$
$$= \frac{\lambda_{12}}{\lambda_{12} - \lambda_1} \exp[-(\lambda_2 + \lambda_1) y] - \frac{\lambda_1}{\lambda_{12} - \lambda_1} \exp[-(\lambda_1 - \lambda_{12})x - (\lambda_2 + \lambda_{12})y]$$

for $y \geq x$.

In the next section we try to modify the definition 3.6.1 of MIFRA class.

3.8 A Modification of definition of MIFRA class :

In this section we will remove the continuity assumption on h in the definition of MIFRA class. Other derived results lead to alternative charactorizations of MIFRA class.

First we introduce a few concepts necessary in the discussion.

A subset $D \subseteq R^n$ is said to be an upper set if whenever $\underline{x} \in D$ and $\underline{y} \geq \underline{x}$, then $\underline{y} \in D$. When D is open, it is called as an upper domain. A set of the form $\{\underline{y} : \underline{y} > \underline{x}\}$ is called as upper quadrant domain. A finite union, of upper quadrant demains is a fundamental upper domain.

3.8.1 Remark :

It is easy to observe that $f = I_D$ where D is an upper set iff f is binary increasing function. Furthermore f is a left (right) contineous binary increasing function if and only if $f = I_D$ where D is an upper domain (closed upper set).

The following results allow us to remove continuity assumption on h.

3.8.2 Lemma :

Let D be either an upper closed set or an upper domain in R^n. For $\underline{T} = (T_1,\ldots,T_n)$ multivariate IFRA,(3.6.2) holds for $h = I_D$ i.e. $P[T \in D] \leq P^{1/\alpha}[T \in \alpha D)$ $0 < \alpha \leq 1$.

Proof :

Let D be the closed upper set. For $k = 2,3,\ldots$ consider the sets $D_k = (1-\frac{1}{k})D$. We observe that $D_k \downarrow D$. For every point $\underline{x} = (x_1,\ldots,x_n) \in D - D^o$ (D^o is interior of D) we define the set $D_{\underline{x}} = \left\{\underline{y} : \frac{y_1}{x_1} = \frac{y_2}{x_2} = \ldots = \frac{y_n}{x_n}\right\}$.

We note that every point in $D_k - D$ belongs to one and only one $D_{\underline{x}}$ for every k. We define

$$\begin{aligned} h_k(\underline{y}) &= 1 \quad \text{if } \underline{y} \in D . \\ &= 0 \quad \text{if } \underline{y} \in D_k^C \\ &= \frac{k}{x_n}\left[y_n - (1-\frac{1}{k})\, x_n\right] \quad \text{if } \underline{y} \in D_{\underline{x}} \cap (D_k - D). \end{aligned}$$

We observe that $h_k \downarrow I_D$.

When D is upper domain, we define $D_k = \frac{k}{k-1}D$ $k=2,3,\ldots$ and note that $D_k \uparrow D$. Now for every point $\underline{x}=(x_1\ldots,x_n)$ $\in \bar{D} - D$ we define the set $D_{\underline{x}}$ as above. Where $\bar{D}$ is the closure of set D.

Every point in $D - D_k$ belongs to one and only one $D_{\underline{x}}$ for every k.

We define

$$h_k(\underline{y}) = 1 \quad \text{if } \underline{y} \in D_k$$
$$= 0 \quad \text{if } \underline{y} \in D^C$$
$$= \frac{k-1}{x_n}[y_n - x_n] \text{ if } \underline{y} \in D_{\underline{x}} \cap (D - D_k) \text{ for some } \underline{x} \in \bar{D}-D.$$

we observe that $h_k \uparrow I_D$. Since in both the cases h_k^s are bounded contineous functions and since the inequality (3.6.2) holds for every h_k, the result follows by monotone convergence theorem.

3.8.3 Lemma :

Let D be any Borel measurable upper set in R^n. Then if $\underline{T} = (T_1, \ldots, T_n)$ is multivariate IFRA,(3.6.2) holds for $h = I_D$ i.e. $P[T \in D] \leq P^{1/\alpha}[T \in \alpha D]$ $0 < \alpha \leq 1$.

Proof :

Let D be any borel measurable upper set in R^n. For $k = 2, \ldots$ and positive integers $i_1, \ldots, i_{n-1}$ let

$a_k(i_1, \ldots, i_{n-1}) = \inf\{t : (i_1 2^{-k}, \ldots, i_{n-1} 2^{-k}, t) \in D\}$.

If $\{\ \} = \emptyset$, set $a_k(i_1, \ldots, i_{n-1}) = \infty$. We set $D_k(i_1, \ldots, i_{n-1})$
$= \{\underline{x} : X \geq (i_1 2^{-k}, \ldots, i_{n-1} 2^{-k}, a_k(i_1, \ldots, i_{n-1}))\}$ if
$(i_1 2^{-k}, \ldots, i_{n-1} 2^{-k}, a_k(i_1, \ldots, i_{n-1})) \in D$ and $D_k(i_1, \ldots, i_{n-1})$
$= \{\underline{x} : X \geq \frac{k-1}{k}(i_1 2^{-k}, \ldots, i_{n-1} 2^{-k}, a_k(i_1, \ldots, i_{n-1}))\}$ if
$(i_1 2^{-k}, \ldots, i_{n-1} 2^{-k}, a_k(i_1, \ldots, i_{n-1})) \notin D$.

Let

$$D_k = \bigcup_{1 \le i_1, \ldots, i_{n-1} \le k \cdot 2^k} D_k(i_1, \ldots, i_{n-1}).$$ Clearly

each D_k is closed upper domain and $D_k \uparrow D$. Hence for given $\epsilon > 0$, there exists a k such that

$P[T \in D] - \epsilon \le P[T \in D_k] \le P^{1/\alpha}(T \in \alpha D_k) \le P^{1/\alpha}(T \in \alpha D)$.

Here the second inequality follows by lemma 3.8.2 and the third inequality follows since $\alpha D_k \subset \alpha D$. Now by letting $\epsilon \downarrow 0$ we get the desired result.

<u>3.8.4 Remark</u> :

It should be noted that lemma 3.8.2 gives that if (3.6.2) holds for contineous nonnegative nondecreasing functions, then it also holds for binary nondecreasing right and left contineous functions. Similarly the proof of lemma 3.8.3 shows that if (3.6.2) holds for nondecreasing binary right contineous (or left contineous) then it holds for binary nondecreasing borel measurable functions. It only remains to show that if(3.6.2) holds for binary nondecreasing Borel measurable functions, then it holds for arbitrary nondecreasing Borel measurable functions. This is contained in our next result.

3.8.5 Theorem :

The random vector $\underline{T} = (T_1, \ldots, T_n)$ is MIFRA (i.e. belongs to π) iff (3.6.2) is valid for all Borel measurable nonnegative nondecreasing functions h.

Proof :

The if part of the theorem is trivial. To prove the 'only if' part we proceed as follows :

Let h be any Borel measurable nonnegative nondecreasing function. As in the proof of lemma 3.7.5 we construct a sequence of functions

$$h_k = 2^{-k} \sum_{i=1}^{k.2^k} I_{D_{ik}} \quad i=1,\ldots,k.2^k, \quad k = 1,2,\ldots, \text{ where}$$

$D_{ik} = \{\underline{x} : h(\underline{x}) > i2^{-k}\}$. We observe that $h_k \uparrow h$. Further since h is an increasing function, it follows that for every i and k, D_{ik} is upper set and hence by lemma 3.8.3 every $I_{D_{ik}}$ satisfies (3.6.2). Hence

$$Eh_k(\underline{X}) = E \sum_{i=1}^{k.2^k} 2^{-k} I_{D_{ik}}(\underline{X}) \leq 2^{-k} \sum_{i=1}^{k.2^k} B^{1/\alpha} I_{\alpha D_{ik}} \quad (\underline{X})$$

$$\leq 2^{-k} E^{1/\alpha} [\sum_{i=1}^{k.2^k} I_{\alpha D_{ik}}(\underline{X})]^{\alpha} \leq E^{1/\alpha}[h_k^{\alpha} (\underline{X}/\alpha)] \quad .$$

Here second inequality follows by using minkovaskey inequality.

Thus h_k satisfies (3.6.2) for every k. Now since $h_k \uparrow h$, by using monotone convergence theorem it follows that h satisfies (3.6.2). Hence the theorem.

The next theorem gives a characterization of MIFRA in terms of idicator functions of fundamental upper domains. The condition $\bar{F}_{(\underset{\sim}{0})} = 1$ is assumed here only because we restrict ourselves to fundamental upper domains of $R^{n*} = \{\underset{\sim}{x} : x_i > 0 \text{ for all } i\}$. If we use fundamental upper domains of R^n instead, no such condition is necessary.

3.8.6 Theorem :

Assume $\bar{F}_{(\underset{\sim}{0})} = 1$. Then $\underset{\sim}{T} = (T_1, \ldots, T_n) \in \pi$ if and only if inequality (3.6.2) is valid for the indicator function of every fundamental upper domain in R^{n*}.

Proof :

The necessity of the condition follows from lemma 3.8.2. Thus we need only prove the sufficiency. Let $D \subset R^{n*}$ be any upper domain. For k = 1,2,... and any positive integers $i_1, \ldots, i_{n-1}$ let $a_k(i_1, \ldots, i_{n-1}) = \inf \{t : (i_1 2^{-k}, \ldots, i_{n-1} 2^{-k}, t) \in D\}$. If $\{ \} = \emptyset$ we set $a_k(i_1, \ldots, i_{n-1}) = \infty$. We define $D_k(i_1, \ldots, i_{n-1}) = \{\underset{\sim}{x} : \underset{\sim}{x} > (i_1 2^{-k}, \ldots, i_{n-1} 2^{-k}, a_k(i_1, \ldots, i_{n-1}))\}$

and put $D_k = \bigcup_{i \leq i_1,\ldots,i_{n-1} \leq k.2^k} D_k(i_1,\ldots,i_{n-1})$.

Clearly D_k is a fundamental upper domain being a finite union of upper quadrant domains. By hypothesis, (3.6.2) is true for I_{D_k} and for every k. Also it is easy to varify that $I_{D_k} \uparrow I_D$ and thus from monotone convergence theorem it follows that $P^{1/\alpha}(T \in \alpha D) \geq P[T \in D]$ for $0<\alpha\leq 1$. From remark 3.8.4 it follows that the result follows for all Borel measurable nonnegative nondecreasing functions restricted to R^{n+}. But since $\bar{F}_{(0)} = 1$ we may remove this condition so that T is MIFRA.

Our next theorem 3.8.7 is an application of theorem 3.8.6.

3.8.7 Theorem :

Let $(T_1,\ldots,T_n)$ be a nonnegative random vector such that for every choice of nonnegative a_i we have $\min_i (a_i T_i)$ is exponential. Then $(T_1,\ldots,T_n)$ is multivariate IFRA.

Proof :

Let a_{ij} be nonnegative constants and let us define $T_{ij} = a_{ij}T_j$ $(1 \leq i \leq k,\ 1 \leq j \leq n)$. Let $\emptyset \neq S \subset \{(i,j) : 1 \leq i \leq k,\ 1 \leq j \leq n\}$, letting

$S_j = \{i : (i,j) \in S\}$ we can see that

$$\min_{(i,j)\in S} T_{ij} = \min_j \left(\min_{i \in S_j} a_{ij} \right) T_j = \min_j a'_j T_j \quad \text{where}$$

$a'_j = \min_{i \in S_j} a_{ij}$ and by assumption, this is exponentially distributed, and hence the collection of random variables $\{T_{ij} : 1 \leq i \leq k,\ i \leq j \leq n\}$ has exponential minimums. Using the application 5.3 of corollary 4.3 from Esary and Marshall (1974) it follows that any coherent life function of these random variables has a univariate IFRA distribution .. (3.8.1)

Now in order to prove $\underline{T} \in \pi$ using theorem 3.8.6 it is enough to prove that inequality (3.6.2) is valid for indicator functions of every fundamental upper domain in R^{n+} i.e. for every fundamental upper domain D,
$P[T \in D] \leq P^{1/\alpha}[T \in \alpha D]$.

Let D be any fundamental upper domain in R^{n+} . Let D be the union of k upper quadrant domains D_i i=1,...,k where no D_i is subset of the other. It can be easily seen that D_i has the form $D_i = \{\underline{x} : \underline{x} \geq \underline{y}_i\}$ where $\underline{Y}_i =$ $(a_{i1}\, x, \ldots, a_{in}\, x)$ for some constants a_{ij} i=1,...,k ; j = 1,...,n with $a_{i1} = 1$ and $x > 0$. Now $I_D(\underline{t}) = 1$ if and if $\underline{t} \geq \underline{y}_1$, or $\underline{t} \geq \underline{y}_2$ or ... or $\underline{t} \geq \underline{y}_k$, and zero otherwise.

That is, $I_D(\underline{t}) = 1$ if and only if $\frac{t_1}{a_{11}} > x, \ldots, \frac{t_n}{a_{1n}} > x$, or $\frac{t_1}{a_{21}} > x, \ldots, \frac{t_n}{a_{2n}} > x$ or ... or $\frac{t_1}{a_{k1}} > x, \ldots, \frac{t_n}{a_{kn}} > x$ and zero otherwise. That is

$I_D(\underline{t}) = 1$ if an only if $\max_{i=1}^{k} \min_{j=1}^{n} \frac{t_j}{a_{ij}} > x$. Thus

$P[T \in D] = P[I_D=1] = P[X > x] = \bar{F}_X(x)$ where

$X = \max_{i=1}^{k} \min_{j=1}^{n} \frac{T_j}{a_{ij}}$. Similarly it is easy to see that

$P[T \in \alpha D] = P[I_{\alpha D}=1] = \bar{F}_X(\alpha x)$. Now since by (3.8.1) X is (univariate) IFRA, it follows that

$$\bar{F}_X(x) \le \bar{F}_X^{1/\alpha}(\alpha x) \text{ for all } \alpha \in (0,1].$$ Hence the theorem.

3.8.8 Remark :

From the arguments in theorem 3.8.7 it becomes clear that $\underline{T} = (T_1, \ldots, T_n)$ is MIFRA if the functions $g(\underline{T})$ of the form

$$g(\underline{T}) = \max_{i=1}^{k} \min_{j=1}^{n} \frac{T_j}{a_{ij}} \quad 0 \le a_{ij} \le \infty \quad i=1,\ldots,k;\ j=1,\ldots,n \qquad \ldots (3.8.2)$$

have univariate IFRA distribution[[since this implies that indicator functions of every fundamental upper domain satisfy (3.6.2)]conversly, let $(T_1, \ldots, T_n) \in \pi$, and $f(\underline{T}) = h(g(\underline{T}))$ be any nonnegative, nondecreasing function of $g(\underline{T})$. Note that $f = h \circ g$ is a nonnegative, nondecreasing function of $\underline{T}$ and hence we have by (3.6.2)

$$E\, h\,[g(\underline{T})] = E\, f(\underline{T}) \le E^{1/\alpha}\, f^{\alpha}(\tfrac{1}{\alpha}\,\underline{T})$$

$$= E^{1/\alpha}[\, h^{\alpha}[g(\tfrac{1}{\alpha}\,\underline{T})]\,] = E^{1/\alpha}[\, h^{\alpha}[\tfrac{1}{\alpha}\, g(\underline{T})]]$$

and hence by (3.6.1) $g(\underline{T})$ has univariate IFRA distribution. Thus we can state the following remark :

' $\underline{T} = (T_1,\ldots,T_n) \in \pi$ if and only if every function $g(\underline{T})$ of the form (3.8.2) has an univariate IFRA distribution'.