CHAPTER IV

MULTIVARIATE NBU DISTRIBUTIONS

4.1 Introduction :

In the last chapter we studied multivariate extensions of univariate IFRA class of distributions. In this chapter we try to extend the well known univariate NBU concept to multivariate case where the componants of a system are interrelated.

In section 4.2 we introduce a MNBU class of distributions proposed by A. W. Marshall and M. Shaked (1982) and study some contidions that are equivalent to the definition of this class. We call this class of distributions as MNBU [1] class. In section 4.3 we study closure properties of this class and in section 4.4 we present some examples of distributions belonging to this class. In section 4.5 we present another class of multivariate NBU distributions proposed by F. Prochan and J. Sethuraman (1983). We call this as MNBU [2] class. Also we give some immidiate implications of the definition of this class in the some section. In section 4.6 we discuss the properties of this class and give some necessary and sufficient conditions for an MNBU [2] random vector to be MVE. In section 4.7 we discuss the relation between MNBU [1] and MNBU [2] classes. Also we introduce some other MNBU classes and discuss their relation with MNBU [2].

4.2 The MNBU [1] Class :

It can be observed that the definition 1.5 of univariate NBU class presented in chapter I can be equivalently expressed as follows :

' A r.v. T is univariate NBU if

$P(T \in (\alpha+\beta)A) \leq P(T \in \alpha A)\ P(T \in \beta A)$ for every $\alpha, \beta \geq 0$ and every set $A = (s, \infty)$ where $s \geq 0$ .. (4.2.1)

This follows since if (4.2.1) holds, for given $t_1, t_2 \geq 0$ by taking $\alpha = t_1/s$, $\beta = t_2/s$, $A = (s, \infty)$ for $s > 0$ we get

$P[T > t_1+t_2] = P[T \in (\alpha+\beta)A] \leq P[T \in \alpha A]\, P[T \in \beta A] =$ $P[T > t_1]\, P[T > t_2]$ and thus T is NBU. On the other hand if T is NBU, then for given α, β, $s > 0$

$$P[T \in (\alpha+\beta)] = P[T > \alpha s+\beta S] \leq P[T > \alpha S]\ P[T > \beta S]$$
$$= P[T \in \alpha A]\ P[T \in \beta A].$$

and hence (4.2.1) holds.

We try to extend condition (4.2.1) to multivariate case. Before doing so, we observe that sets A of the form (s,∞) in the condition (4.2.1) are open and have increasing indicator functions. They have natural multidimentional analogs, namely the upper sets defined in section 3.8. of chapter III. Making use of these observations we define our MNBU [1] class as follows :

<u>4.2.1 Definition</u> :

A random vector $\underline{T} = (T_1,\ldots,T_n)$ with joint d. f. F is said to be multivariate new better than used [MNBU[1]] if $\bar{F}(\underline{0}) = 1$ and $P[T \in (\alpha+\beta)A] \leq P[T \in \alpha A]\, P[T \in \beta A]$ for every α, $\beta \geq 0$ and for every open upper set A .

Our next theorem gives a number of conditions equivalent to definition 4.2.1, before presenting which we introduce some terminology useful for it's statement.

A real function g defined on $[0,\infty)^n$ is said to be subhomogenous if $\alpha\, g(\underline{t}) \leq g(\alpha\, \underline{t})$ for every $\alpha \in [0,1]$ and every $\underline{t} \geq \underline{0}$ . .. (4.2.2)

Or equivalently, if $\alpha\, g(\underline{t}) \geq g(\alpha\, \underline{t})$ for every $\alpha > 1$ and $\underline{t} \geq \underline{0}$ . .. (4.2.3)

If equality holds in (4.2.2) for every $\alpha \in [0,1]$ and every $\underline{t} \geq \underline{0}$

or If equality holds in (4.2.3) for every $\alpha \gtrless 1$ then it is said to be homogenous.

4.2.2 Theorem :

For a random vector $\underline{T} = (T_1,\ldots,T_n)$ such that $\bar{F}(\underline{0}) = 1$. The following conditions are equivalent.

(i) $\underline{T}$ is MNBU [1].

(ii) For every $\alpha > 0$, $\beta > 0$ and every increasing binary (i.e. indicator) function ϕ,

$$E\,\phi\left(\frac{1}{\alpha+\beta}\,\underline{T}\right) \leq E\,\phi\left(\frac{1}{\alpha}\,\underline{T}\right) E\,\phi\left(\frac{1}{\beta}\,\underline{T}\right)$$

(iii) For every $\alpha > 0$, $\beta > 0$, $\gamma \in (0,1)$ and every non-negative increasing function h defined on $[0,\infty)^n$,

$$E\,h\left(\frac{1}{\alpha+\beta}\,\underline{T}\right) \leq E\,h^{\gamma}\left(\frac{1}{\alpha}\,\underline{T}\right) E\,h^{1-\gamma}\left(\frac{1}{\beta}\,\underline{T}\right) \qquad \ldots (4.2.4)$$

(iv) for every nonnegative increasing sub homogenous function g, $g(\underline{T})$ has an NBU distribution.

(v) for every nonnegative increasing homogenous function g, $g(\underline{T})$ has an NBU distribution.

Proof :

The equivalence of these conditions is established by showing that (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(ii)$\Longrightarrow$(iv)$\Longrightarrow$(v)$\Longrightarrow$(i). The proof of (iii)$\Longrightarrow$(ii) and (iv)$\Longrightarrow$(v) is trivial. Other proofs are given below :

(i) $\Longrightarrow$ (iii) : We note that $\emptyset$ is an increasing binary function if and only if it is indicator function of an upper set. Therefore let $\emptyset$ be the indicator function of the upper set A. Fix α, $\beta > 0$. Let A^o be the interior of A. Let $A_k = (1 - \frac{1}{k})A^o$. We note that A_k are open and $A_k \downarrow A^o$. Also $\alpha A_k \downarrow \alpha A^o$ and $\beta A_k \downarrow \beta A^o$. Hence for given $\epsilon > 0$ we can find k such that

$P[\underline{T} \in \alpha A_k] \leq P[\underline{T} \in \alpha A^o] + \epsilon \leq P[\underline{T} \in \alpha A] + \epsilon$ and

$P[\underline{T} \in \beta A_k] \leq P[\underline{T} \in \beta A^o] + \epsilon \leq P[\underline{T} \in \beta A] + \epsilon$

Thus noting that $\emptyset(\frac{1}{\alpha+\beta}\underline{T})$ is indicator function of $(\alpha + \beta)A$, we get

$$E\,\emptyset(\frac{1}{\alpha+\beta}\underline{T}) = P[\underline{T} \in (\alpha+\beta)A] \leq P[\underline{T} \in (\alpha+\beta)A_k] \leq P[\underline{T} \in \alpha A_k]\,P[\underline{T} \in \beta A_k]$$

$$\leq [P[\underline{T} \in \alpha A] + \epsilon]\,[P[\underline{T} \in \beta A] + \epsilon]$$

$$= [E\,\emptyset(\frac{1}{\alpha}\underline{T}) + \epsilon]\,[E\,\emptyset(\frac{1}{\beta}\underline{T}) + \epsilon].$$

Here the 2nd inequality follows from definition 4.2.1. Now the result follows by letting $\epsilon \to 0$.

(ii) $\Longrightarrow$ (iii) : Let h be nonnegative increasing function defined on R_n^+. Let us define the function h_k, $k = 1, 2, \ldots$ as follows

$$h_k(\underline{t}) = \frac{i-1}{2^k} \quad \text{if} \quad \frac{i-1}{2^k} \leq h(\underline{t}) < \frac{i}{2^k} \quad i = 1,2,\ldots,k.2^k$$

$$= k \quad \text{if} \quad h(\underline{t}) \geq k.$$

Let A_{ik} $i = 1,\ldots,k.2^k$, $k = 1,2,\ldots$ be the sets defined by $A_{ik} = \left\{ \underline{t} : h(\underline{t}) \geq \frac{i}{2^k} \right\}$. We note that A_{ik} are upper sets and $A_{1k} > A_{2k} \ldots > A_{k.2^k,k}$. Thus

$$h_k(\underline{t}) = \sum_{i=1}^{k.2^k} \frac{1}{2^k} I_{A_{ik}}(\underline{t}) \quad \text{and} \quad h_k(\underline{t}) \uparrow h(\underline{t}).$$

Because of monotone convergence theorem now it is enough to prove the result for h_k i.e. for functions of the form $f(\underline{t}) = \sum_{i=1}^{m} a_i I_{A_i}(\underline{t})$ where $a_i \geq 0$ $i = 2,\ldots,m$ and $A_1 \supset \ldots \supset A_m$ are upper sets. For notational convenience, let $A_{m+1} = \emptyset$. Then

$$E f\left(\frac{1}{\alpha+\beta} \underline{T}\right) = \sum_{i=1}^{m} a_i P[\underline{T} \in (\alpha+\beta)A_i]$$

$$\leq \sum_{i=1}^{m} a_i P[\underline{T} \in \alpha A_i]\, P[\underline{T} \in \beta A_i]$$

$$= \sum_{i=1}^{m} a_i \left[\sum_{j=i}^{m} P[\underline{T} \in \alpha(A_j - A_{j+1})]\right]$$

$$\left[\sum_{j=i}^{m} P[\underline{T} \in \beta(A_j - A_{j+1})]\right]$$

$$= \sum_{i=1}^{m} \sum_{j=i}^{m} \sum_{\ell=i}^{m} a_i \, P[\underline{I} \in \alpha(A_j - A_{j+1})] P[\underline{I} \in \beta(A_\ell - A_{\ell+1})]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} (a_1 + a_2 + \dots + a_{\min(i,j)}) P[\underline{I} \in \alpha(A_i - A_{i+1})] \, P[\underline{I} \in \beta(A_j - A_{j+1})]$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{m} (a_1 + \dots + a_i)^{\gamma} (a_1 + \dots + a_j)^{1-\gamma} P[\underline{I} \in \alpha(A_i - A_{i+1})] \, P[\underline{I} \in \beta(A_j - A_{j+1})]$$

$$\left[\sum_{i=1}^{m} (a_1 + \dots + a_i)^{\gamma} P[\underline{I} \in \alpha(A_i - A_{i+1})]\right] \times \left[\sum_{j=1}^{m} (a_1 + \dots + a_j)^{1-\gamma} P[\underline{I} \in \beta(A_j - A_{j+1})]\right]$$

$$= E[f^{\gamma}(\tfrac{1}{\alpha} \underline{I})] E[f^{1-\gamma}(\tfrac{1}{\beta} \underline{I})]$$

We note that the last equality follows since $f^{\gamma}(\frac{1}{\alpha}\underline{I})$ takes value $(\sum_{j=1}^{i} a_i)^{\gamma}$ on $\alpha(A_i - A_{i+1})$.

<u>(ii) ⇒ (iv)</u> : Let g be a nonnegative subhomogeneous increasing function. We fix a > 0 and set $\phi(\underline{t}) =$ $\phi(\underline{t}) = I_{\{S\,:\,g(S) > a\}}(\underline{t})$.

we note that ϕ is increasing binary function Now, for $\alpha \in (0, 1)$,

$$P[g(\underline{T})>\alpha a]\ P[g(\underline{T})>(1-\alpha)a] \geq P[\alpha g(\tfrac{1}{\alpha}\underline{T})>\alpha a]\ P[(1-\alpha)g(\tfrac{1}{1-\alpha}\underline{T}) > (1-\alpha)a]$$

$$= P[g(\tfrac{1}{\alpha}\underline{T})>a]\ P[g(\tfrac{1}{1-\alpha}\underline{T}) > a]$$

$$= E\,\phi\,(\tfrac{1}{\alpha}\underline{T})\ E\,\phi\,[\tfrac{1}{1-\alpha}\underline{T}]$$

$$\geq E\,\phi\,(\underline{T})$$

$$= P[\,g(\underline{T}) > a\,]\ .$$

Here the first inequlaity follows by (4.2.3) and the second by hypothesis. Since α is **arbitrary,** it follows that $g(\underline{T})$ has NBU distribution.

(v) $\Rightarrow$ (i) : Let $A \subset R^{n+}$ be an open upper set, Let us define the function g on R^{n+} by

$$g(\underline{t}) = \{\sup\ \theta>0 : \tfrac{1}{\theta}\underline{t} \in A\} \quad \text{if} \{\theta>0 : \tfrac{1}{\theta}\,\underline{t} \in A\} \neq \phi$$

$$= 0 \qquad\qquad = 0 \qquad\qquad \text{otherwise}$$

We note that $g(\underline{t})$ is nonnegative. Further for $\underline{t}_1 > \underline{t}_2$ let $\theta^* = g(\underline{t}_2) = \sup\{\theta>0 : \frac{1}{\theta}\,\underline{t}_2 \in A\}$. Now

$$\frac{1}{\theta^*}\underline{t}_2 < \frac{1}{\theta^*}\underline{t}_1 \Rightarrow \frac{1}{\theta^*}\,\underline{t}_1 \in A \Rightarrow \theta^* \leq \sup\{\theta>0,\ \frac{1}{\theta}\,\underline{t}_1 \in A\} = g(\underline{t}_1)$$

i.e. $g(\underline{t}_2) \leq g(\underline{t}_1)$ and thus $g(\underline{t})$ is nondecreasing.

Also for $\alpha > 0$,

$$g(\alpha\underline{t}) = \sup\{\theta>0 : \frac{1}{\theta}\,\alpha\,\underline{t} \in A\}$$

$$= \alpha\ \sup\{\theta>0 : \frac{1}{\theta}\,\underline{t} \in A\}$$

i.e. $g(\underline{t})$ is homogenous function. Also for every $\sigma \geq 0$,

$$P[g(\underline{T}) > \sigma] = P[\sup\{\theta > 0 : \frac{1}{\theta}\underline{T} \in A\} > \sigma] = P[\frac{1}{\sigma}\underline{T} \in A] = P[\underline{T} \in \sigma A].$$

Since $g(\underline{T})$ is NBU,

$$P\{\underline{T} \in (\alpha+\beta)A\} = P\{g(\underline{T}) > \alpha+\beta\} \leq P\{g(\underline{T}) > \alpha\} \; P\{g(\underline{T}) > \beta\}$$
$$= P[T \in \alpha A] \; P[T \in \beta A].$$

and hence $\underline{T}$ is MNBU [1].

4.2.3 Remark :

Various modifications of the conditions given in theorem 4.2.1 are possible which are listed below :

(a) In (iii) the nonnegative increasing functions can be replaced by the nonnegative increasing contineous functions, since if (iii) holds for nonnegative increasing contineous functions, then first using a similar argument as in the proof of lemma 3.8.2 of chapter III, and noting that $\emptyset^{\gamma} \equiv \emptyset$ for all γ, it can be proved that (ii) holds for nonnegative increasing right contineous binary functions, and then by using a similar argument as in the proof of lemma 3.8.3 of chapter III it can be shown that (ii) holds for all borel measurable nonnegative nondecreasing binary functions, thus (iii)$\Longrightarrow$(ii) follows. Other implications do not pose any problem with this change in (iii).

(b) In (iii) it is sufficient to require that (4.2.4) holds for some $\gamma \in (0,1)$. This can be observed easily.

(c) In condition (v) the nonnegative increasing homogenous functions can be replaced by the functions $g(\underline{T})$ of the form $g(\underline{T}) = \max_{i=1}^{m} \min_{j=1}^{n} a_{ij} T_j$ ...(4.2.5)

Since first we observe that as indicated in the proof of theorem 3.8.7 of chapter III, for any fundamental upper domain A, $P[T \in \sigma A]$ can be expressed as $P[g(\underline{T}) > \sigma x]$ for some $x > 0$ and for every $\sigma > 0$ where $g(T)$ is of the form (4.2.5). Since $g(\underline{T})$ is univariate NBU, it follows that (4.2.1)holds for every fundamental upper domain A. Now for any upper domain D, a sequence of fundamental upper domains D_k can be constructed as shown in the proof of theorem 3.8.6 of chapter III, such that $D_k \uparrow D$ or $I_{D_k} \uparrow I_D$. Then $I_{\sigma D_k} \uparrow I_{\sigma D}$ for every $\sigma > 0$. By using monotone convergence theorem it then follows that (4.2.1) holds for every upper domian D. Thus this modified form of (v)$\Longrightarrow$(i). The other implications of the theorem do not pose any problem with this change in (v).

4.2.4 Remark :

In remark 3.8.8 of chapter III, we have seen that $\underline{T}$ is MIFRA according to Block and Savits (1981) if and only if every function $g(\underline{T})$ of the form (4.2.5) has univariate IFRA distribution. Since univariate IFRA $\Longrightarrow$ univariate NBU, from remark 4.2.3(c) above it follows that MIFRA $\Longrightarrow$ MNBU [1].

4.3 Closure properties of the class MNBU [1] :

4.3.1 Theorem :

The class MNBU [1] of multivariate NBU distributions possesses the following prpperties :

(P1) If $\underline{T}$ is MNBU [1] and g_j is a nonnegative subhomogeous increasing function defined on $[0,\infty)^n, j=1,\ldots,m$ then $(g_1(\underline{T}),\ldots,g_m(\underline{T}))$ is MNBU [1].

(P2) If $\underline{T}$ is MNBU [1], then any joint marginal is MNBU[1].

(P3) If $\underline{T}$ is MNBU[1] and is the life function of a coherent system, then $\tau(\underline{T})$ is NBU.

(P4) If $\underline{T}$ is MNBU[1] and $a_i \geq 0$ $i=1,\ldots,n$ then $\Sigma\, a_i T_i$ is NBU.

(P5) If $\underline{T}$ is MNBU[1] and $a_i \geq 0$ $i=1,\ldots,n$ then $(a_1T_1,\ldots,a_nT_n)$ is MNBU [1].

(P6) If $\underline{S} = (S_1,\ldots,S_m)$ and $\underline{T} = (T_1,\ldots,T_n)$ are MNBU and if $\underline{S}$ and $\underline{T}$ are independent, then $(\underline{S},\underline{T})$ is MNBU.

(P7) If T_ℓ , $\ell=1,2,\ldots$ is a sequence of MNBU[2] random vectors that converges in distribution to $\underline{T}$ then $\underline{T}$ is MNBU[1].

Proof:

(P1): Let g be a nonnegative subhomogenous increasing function defined on $[0,\infty)^m$. Then the composition $g[g_1(\underline{t}),\ldots,g_m(\underline{t})]$ is a nonnegative subhomogenous increasing function defined on $[0,\infty)^n$. Consequently the result follows from (iv) of theorem 4.2.2.

(P2): By taking $g_i(\underline{T}) = T_{j_i}$, i=1,...,m it follows that $(T_{j_1},\ldots,T_{j_m})$ is MNBU[1] for every subset $\{j_1,\ldots,j_m\} \subset \{1,\ldots,n\}$.

(P3): We observe that a coherent life function τ has the form (4.2.5) and hence it is nonnegative, increasing subhomogeneous function of $\underline{T}$. The result now follows from (v) of theorem 4.2.2.

(P4): Again we observe that $g(\underline{T}) = \Sigma a_i T_i$ is nonnegative increasing subhomogenous function of $\underline{T}$. The result follows by (v) of theorem 4.2.2.

(P5): Since $g_i(\underline{T}) = a_i T_i$; i=1,...,n are nonnegative increasing homogenous functions of $\underline{T}$, the result follows from from (P1).

(P6): We prove the result by showing that $(\underline{S},\underline{T})$ satisfies (ii) of theorem 4.2.2. Let α, $\beta \geq 0$ and let $\emptyset$ be an increasing binary function defined on R^{m+n}. Let us denote the distribution function of $\underline{S}$ by F and the distribution function of $\underline{T}$ by G.

Now,

$$E\,[\phi(\tfrac{1}{\alpha+\beta}\,\underline{S},\ \tfrac{1}{\alpha+\beta}\,\underline{T})] = \int_{\underline{t}}\int_{\underline{s}} \phi\,(\tfrac{1}{\alpha+\beta}\,\underline{s},\ \tfrac{1}{\alpha+\beta}\,\underline{t})\,dF(\underline{s})\,dG(\underline{t})$$

$$\leq \int_{\underline{t}}[\int_{\underline{s}}\phi(\tfrac{1}{\alpha}\,\underline{s},\ \tfrac{1}{\alpha+\beta}\,\underline{t})\,dF(\underline{s})][\int_{\underline{s}'}\phi(\tfrac{1}{\beta}\underline{s}',\ \tfrac{1}{\alpha+\beta}\underline{t})\,dF(\underline{s}')]\,dG(\underline{t})$$

[since S satisfies (ii)]

$$= \int_{\underline{s}}\int_{\underline{s}'}[\int_{\underline{t}}\phi(\tfrac{1}{\alpha}\,\underline{s},\ \tfrac{1}{\alpha+\beta}\,\underline{t})\phi(\tfrac{1}{\beta}\underline{s}',\tfrac{1}{\alpha+\beta}\underline{t})\,dG(\underline{t})]\,dF(\underline{s})\,dF(\underline{s}')$$

$$\leq \int_{\underline{s}}\int_{\underline{s}'}[\int_{\underline{t}}(\tfrac{1}{\alpha}\underline{s},\ \tfrac{1}{\alpha}\underline{t})\ \phi(\tfrac{1}{\beta}\underline{s}',\ \tfrac{1}{\alpha}\underline{t})\ dG(\underline{t})\ ]\ \times$$

$$[\int_{\underline{t}'}\phi\ (\tfrac{1}{\alpha}\ \underline{s},\ \tfrac{1}{\beta}\underline{t}')\ \phi(\tfrac{1}{\beta}\underline{s}',\ \tfrac{1}{\beta}\underline{t}')\,dG(\underline{t}')]\,dF(\underline{s})\,dF(\underline{s}')$$

[since $\underline{T}$ satisfies (ii) and product of increasing binary function is increasing binary function].

$$\leq \int_{\underline{s}}\int_{\underline{s}'}\int_{\underline{t}}\int_{\underline{t}'}\ \phi(\tfrac{1}{\alpha}\ \underline{s},\ \tfrac{1}{\alpha}\ \underline{t})\ \phi(\tfrac{1}{\beta}\ \underline{s}',\tfrac{1}{\beta}\ \underline{t}')\ dG(\underline{t}')\,dG(\underline{t})\,dF(\underline{s}')\,dF(\underline{s})$$

[since $\phi \leq 1$].

$$= E\ \phi\ (\tfrac{1}{\alpha}\ \underline{S},\ \tfrac{1}{\alpha}\ \underline{T})\ E\ \phi(\tfrac{1}{\beta}\ \underline{S}\ ,\ \tfrac{1}{\beta}\ \underline{T})$$

Thus $(\underline{S},\ \underline{T}) \in$ MNBU [1].

(P7): Let h be any bounded, contineous, nonnegative increasing function. Then by the definition of weak convergence,

$E\ h(\frac{1}{\alpha+\beta}\ \underline{T}_\ell) \longrightarrow E\ h(\frac{1}{\alpha+\beta}\ \underline{T})$, $E\ h^\gamma(\frac{1}{\alpha}\ \underline{T}_\ell) \longrightarrow E\ h^\gamma(\frac{1}{\alpha}\ \underline{T})$ and $E\ h^{1-\gamma}(\frac{1}{\beta}\ \underline{T}_\ell)$ $\quad E\ h^{1-\gamma}(\frac{1}{\beta}\ \underline{T})$. Further since each $\underline{T}_\ell \in$ MNBU[1],

$E\ h\ (\frac{1}{\alpha+\beta}\ \underline{T}\ ) \leq E\ h^{\gamma}\ (\frac{1}{\alpha}\ \underline{T}\ )\ E\ h^{1-\gamma}(\frac{1}{\beta}\ \underline{T}\ )$. Taking limit as ∞ on both sides we get

$E\ h(\frac{1}{\alpha+\beta}\ \underline{T}) \leq E\ h^{\gamma}(\frac{1}{\alpha}\ \underline{T})\ E\ h^{1-\gamma}(\frac{1}{\beta}\ \underline{T})$. If h is not bounded, we consider the functions $h_N = \min(h,N), N=1,2,\ldots$ $h_N \uparrow h$ and the inequality holds using monotone convergence theorem. Now our result follows using remark (4.2.3) (a).

4.3.2 Corollary :

If $T_1,\ldots,T_n$ are independent NBU random variables then (a) $\underline{T} = (T_1,\ldots,T_n)$ is MNBU.

(b) $g(T_1,\ldots,T_n)$ is NBU whenever g is a nonnegative subhomogenous increasing function.

Proof :

(a) follows immidiately from property (P6).

(b) follows from (a) and (iv) of theorem 4.2.2.

4.4 Examples of MNBU [1] distributions :

(i) A replacement model : Suppose that devices $d_1,\ldots,d_5$ are available to perform tasks t_1,t_2,t_3. Upon failure of d_1 (which performs all three tasks simultaneously), it is replaced by d_2 (which performs tasks t_1 and t_2) and by d_3 (which performs only task t_3). When device d_2 fails, it is replaced by d_4 (which performs only task t_1)

and by d_5 (which performs task t_2). Let X_i be the life length of the i^{th} device i = 1,...,5 and let T_j be the time that t_j is performed using these devices j = 1,2,3.

Then $T_1 = X_1+X_2+X_4$, $T_2 = X_1+X_2+X_5$, $T_3 = X_1+X_3$.

It follows from property (P1) that if $X_1,\ldots,X_5$ are independent NBU, then (T_1,T_2,T_3) is MNBU [1]. Also $\tau(T_1,T_2,T_3)$ will be NBU where τ is the life function of a coherent system.

(ii) Freund's distribution : Suppose that devices d_1 and d_2 are placed in service together and are subjected to respective constant hazard rates λ_1 and λ_2 untill one or the other fails. From the earliest failure time on, the remaining device d_i is subjected to a new constant hazard rate $\mu_i > \lambda_i$, such that $\mu_2 \neq \lambda_1+\lambda_2$. If T_j is the life length of d_j j=1,2, the joint distribution of (T_1,T_2) as given in Brindley, Thompson (1972) is

$$\bar{F}(x,y) = e^{-(\lambda_1+\lambda_2)x} \left[\frac{\lambda_2-\mu_2}{\lambda_1+\lambda_2-\mu_2} e^{-(\lambda_1+\lambda_2)(y-x)} + \frac{\lambda_1}{\lambda_1+\lambda_2-\mu_2} e^{-\mu_2(y-x)} \right] \quad x \leq y$$

$$= e^{-(\lambda_1+\lambda_2)y} \left[\frac{\lambda_1-\mu_1}{\lambda_1+\lambda_2-\mu_1} e^{-(\lambda_1+\lambda_2)(x-y)} + \frac{\lambda_2}{\lambda_1+\lambda_2-\mu_1} e^{-\mu_1(x-y)} \right] \quad y \leq x$$

...(4.4.1)

Now consider the random variables X_i i=1,2,3,4 where $X_1 \sim \exp(\lambda_1)$, $X_2 \sim \exp(\lambda_2)$, $X_3 \sim \exp(\mu_1-\lambda_1)$ and $X_4 \sim \exp(\mu_2-\lambda_2)$. The joint survival function of $(\min(X_1,X_2+X_3), \min(X_2,X_1+X_4))$ is given by

$$P[\min(X_1,X_2+X_3) > x, \min(X_2,X_1+X_4) > y]$$

$$= P[X_1>x, X_2+X_3>x, X_2 > y, X_1+X_4 > y]$$

$$= P[X_1>\max(x,y-X_4)]\, P[X_2> \max(y,x-X_3)]$$

$$= P[X_1> \max(o,y-X_4-x)+x]\, P[X_2> \max(y-x, -X_3)+x]$$

$$= e^{-\lambda_2 y} \quad P[X_1 > \max (o, y-X_4-x)+x]$$

$$= e^{-\lambda_2 y} \left[\int_o^{y-x} e^{-\lambda_1(y-u)} (\mu_2-\lambda_2)\, e^{-(\mu_2-\lambda_2)u}\, du + \int_{y-x}^{\infty} e^{-\lambda_1 x} (\mu_2-\lambda_2)\, e^{-(\mu_2-\lambda_2)u}\, du\right]$$

$$= e^{-\lambda_2 y-\lambda_1 y}\left[\int_o^{y-x} (\mu_2-\lambda_2) e^{-(\mu_2-\lambda_2-\lambda_1)u}\, du + e^{\lambda_1(y-x)} e^{-(\mu_2-\lambda_2)(y-x)}\right]$$

$$= e^{-(\lambda_1+\lambda_2)y}\left[\frac{\mu_2-\lambda_2}{\mu_2-\lambda_2-\lambda_1} - \frac{\mu_2-\lambda_2}{\mu_2-\lambda_2-\lambda_1}\, e^{-(\mu_2-\lambda_2-\lambda_1)(y-x)} + e^{-(\mu_2-\lambda_2-\lambda_1)(y-x)}\right]$$

$$= e^{-(\lambda_1+\lambda_2)y}\left[\frac{\lambda_2-\mu_2}{\lambda_1+\lambda_2-\mu_2} + \frac{\lambda_1}{\lambda_1+\lambda_2-\mu_2}\, e^{-(\mu_2-\lambda_2-\lambda_1)(y-x)}\right]$$

$$= e^{-(\lambda_1+\lambda_2)x}\, e^{-(\lambda_1+\lambda_2)(y-x)}\left[\frac{\lambda_2-\mu_2}{\lambda_1+\lambda_2-\mu_2} + \frac{\lambda_1}{\lambda_1+\lambda_2-\mu_2}\, e^{-(\mu_2-\lambda_2-\lambda_1)(y-x)}\right]$$

$$= e^{-(\lambda_1+\lambda_2)x}\left[\frac{\lambda_2-\mu_2}{\lambda_1+\lambda_2-\mu_2}\, e^{-(\lambda_1+\lambda_2)(y-x)} + \frac{\lambda_1}{\lambda_1+\lambda_2-\mu_2}\, e^{-\mu_2(y-x)}\right]$$

if $x \le y$.

Similarly it can be shown that this probability equals

$$e^{-(\lambda_1+\lambda_2)y}\left[\frac{\lambda_1-\mu_1}{\lambda_1+\lambda_2-\mu_1}e^{-(\lambda_1+\lambda_2)(x-y)}+\frac{\lambda_2}{\lambda_1+\lambda_2-\mu_1}e^{-\mu_1(x-y)}\right]$$

if $y \leq x$.

We see that this consides with (4.4.1). Hence (T_1,T_2) has some joint distribution as $(\min(X_1,X_2+X_3), \min(X_2,X_1+X_4))$. Since X_i, $i=1,\ldots,4$ are independent, it follows that (T_1,T_2) have MNBU [1] distribution using property (P1) of theorem 4.3.1. It can also be seen that (T_1,T_2) has MIFRA distribution according to definition 3.6.1 of chapter III.

4.5 The MNBU [2] class :

In this section we introduce a multivariate version of the NBU distribution based on a physical model. Suppose shocks occur in time which cause the simultaneous failure of subsets of n componants. The interval of time until the occurence of a shock destroying a given subset of componants is governed by an NBU distribution. The occurance times are mutually independent.

Based on this shock model, F. Proschan and J. Sethuraman (1983) have proposed the following class of multivariate NBU distributions:

4.5.1 Definition :

A random vector $\underline{T} = (T_1,\ldots,T_n)$ is said to be a MNBU [2] random vector if it has a representation $T_i = \min_{i \in A} T_A$ where $\{T_A, A \in \mathcal{J}\}$ are independent NBU random variables and $\mathcal{J}$ is the class of nonempty subsets of $\{1,\ldots,n\}$.

Below we present an equivalent version of definition 4.5.1.

4.5.2 Definition :

A random vector $\underline{T} = (T_1,\ldots,T_n)$ is said to be a MNBU [2] random vector if it has a representation $T_i = \min_{j \in S_i} X_j$ where $X_1,\ldots,X_M$ are independent NBU random variables and $\emptyset \neq S_i \subset \{1,\ldots,M\}$ $i=1,\ldots,n$ and $\bigcup_{i=1}^{n} S_i = \{1,\ldots,M\}$. The equivalance of the definition can be easily demonostrated.

4.5.3 Some implications of definitions 4.5.1 and 4.5.2 :

Let $\bar{F}(t_1,\ldots,t_n) = P[T_1>t_1,\ldots,T_n>t_n]$ be the joint survival function of $T_1,\ldots,T_n$ where $\underline{T}$ is MNBU [2].Then

(i) $\bar{F}(t_1,\ldots,t_n) = \prod_{A \in \mathcal{J}} \bar{F}_A(\max_{i \in A} t_i),\quad t_i \geq 0,$ ... (4.5.1)

$i = 1,\ldots,n$ where $\bar{F}_A$ is the survival function of T_A, $A \in$. This follows easily from definition 4.5.1.

(ii) $\bar{F}(t_1+s,\ldots,t_n+s) \leq \bar{F}(t_1,\ldots,t_n)\ \bar{F}(s,\ldots,s)$

for all $s \geq 0$, $t_i \geq 0$ $i=1,\ldots,n$.

This follows since by (i)

$$\bar{F}(t_1+s,\ldots,t_n+s) = \prod_{A \in \mathcal{J}} \bar{F}_A\left(\max_{i \in A}(t_i+s)\right)$$

$$= \prod_{A \in \mathcal{J}} \bar{F}_A\left(\left(\max_{i \in A} t_i\right) + s\right)$$

$$\leq \prod_{A \in \mathcal{J}} \bar{F}_A\left(\max_{i \in A} t_i\right) \bar{F}_A(s)$$

since T_A is NBU for all A.

$$= \bar{F}(t_1,\ldots,t_n).\ \bar{F}(s,\ldots,s)$$

We note that (ii) can be expressed as

$P[T_1>t_1+s,\ldots,T_n>t_n+s/\ T_1>s,\ldots,T_n>s] \leq P[T_1>t_1,\ldots,T_n>t_n]$.

This implies that the joint survival probability of n componants each of age s is less than or equal to the joint survival probability of n new componants. Another alternative interpretation of (ii) may be obtained by rewriting it as

$P[T_1>t_1+s,\ldots,T_n>t_n+s/\ T_1>t_1,\ldots,T_n>t_n] \leq P[T_1>s,\ldots,T_n>s]$.

This implies that a series system of n componants of ages $t_1,\ldots,t_n$ is stochastically shorter lived than is a series system of n new componants.

4.5.4 Remark :

A multivariate new worse than used (MNWU) random vector $\underline{T}$ can be defined as in definition 4.5.1 (4.5.2) where now $T_A, A \in \mathcal{F}$, $(X_i, i=1,\dots,M)$ are assumed to be independent NWU random variables. It can be easily shown that in this case

$\bar{F}(t_1+s_1, \dots, t_n+s_n) \geq \bar{F}(t_1,\dots,t_n)\bar{F}(s_1,\dots,s_n)$. This follows since $\max_{i \in A}(t_i+s_i) \leq \max_{i \in A} t_i + \max_{i \in A} s_i$, therefore,

$\bar{F}_A(\max_{i \in A}(t_i+s_i)) \geq \bar{F}_A[(\max_{i \in A} t_i)+(\max_{i \in A} s_i)]$.

Now if each T_A is NWU, we have

$$\bar{F}_A[(\max_{i \in A} t_i) + (\max_{i \in A} s_i)] \geq \bar{F}_A(\max_{i \in A} t_i) . \bar{F}_A(\max_{i \in A} s_i).$$

the result now follows by using implication (i) of 4.5.3 which is also true for NWU case.

We note here that in the MNWU case, the s values may differ, while in the MNBU case, the s values must be the same

(iii) $\quad \bar{F}(t_1,\dots,t_n) \geq \prod_{i=1}^{n} \left(\prod_{i \in A} \bar{F}_A(t_i)\right)$

(iv) $\quad \bar{F}(t_1,\dots,t_n) \geq \prod_{i=1}^{n} [\, 1 - \prod_{i \in A} \bar{F}_A(t_i)]$

Since $T_1,\dots,T_n$ are increasing functions of independent random variables, they are associated. From wellknown inequalities for associated random variables(Cf. Borlow and Proscha n (1975) page 33) it follows that,

$$\bar{F}(t_1,\ldots,t_n) \geq \pi_{i=1}^{n} \bar{F}_i(t_i) = \pi_{i=1}^{n} [\pi_{i \in A} \bar{F}_A(t_i)] \text{ since}$$

$\bar{F}_i(t_i) = \pi_{i \in A} \bar{F}_A(t_i)$ from definition 4.5.1. Similarly

$$F(t_1,\ldots,t_n) \geq \pi_{i=1}^{n} F_i(t_i) = \pi_{i=1}^{n} [1- \bar{F}_i(t_i)] = \pi_{i=1}^{n} [1- \pi_{i \in A} \bar{F}_A(t_i)].$$

In the next section we discuss the properties of MNBU [2] class.

4.6 Closure properties of the class MNBU [2] :

4.611 Theorem :

The class MNBU [2] of multivariate NBU distributions possesses the following properties :

(P1) Let $\underset{\sim}{T}$ be an NBU random variable. Then T is 1- dimensional MNBU.

(P2) Let $T_1,\ldots,T_n$ be independent NBU random variables. Then $\underset{\sim}{T}$ is MNBU [2].

(P3) Let $\underset{\sim}{T}$ be MNBU [2]. Then $(T_{i_1},\ldots,T_{i_k})$ is k- dimensional MNBU [2], $1 \leq i_1<\ldots<i_k \leq n$ $k = 1,2,\ldots,n$.

(P4) Let $\underset{\sim}{T}$ be MNBU [2] and $T_j^* = \min_{i \in B_j} T_i$, $\emptyset \neq B_j \subset \{1,2,\ldots,n\}$, $j = 1,\ldots,m$. Then $\underset{\sim}{T}^*$ is MNBU [2].

(P5) Let $\underline{T}$ be MNBU [2] and $a_i > 0$, $i=1,\ldots,n$. Then $\min_{1\leq i\leq n} a_i T_i$ is NBU.

(P6) Let $\underline{T}$ be n-dimensional MNBU [2], $\underline{T}'$ be m-dimensional MNBU [2], and $\underline{T}$, $\underline{T}'$ be independent. Then $(\underline{T},\underline{T}')$ is (m+n) dimensional MNBU [2].

(P7) Let $\underline{T}$ be MNBU [2] and let τ be the life function of a coherent system. Then $\tau(\underline{T})$ is NBU.

(P8) Let $g:[0,\infty)\longrightarrow[0,\infty)$ be a nondecreasing contineous function such that $g(x+y) \leq g(x)+g(y)$ for all x,y. Let $\underline{T}$ be MNBU [2] such that each X_i of definition 4.5.2 is contineous , then $\underline{T}'= (g(T_1),\ldots,g(T_n))$ is MNBU [2] .

Proof :

(P1) and (P2) are obvious.

(P3) and (P4) : Since (P3) is a special case of (P4) by taking $B_j = \{i_j\}$ $j=1,\ldots,k$, we need only prove (P4). Let $T_i = \min_{l\in S_i} X_l$, $i=1,\ldots,n$. [By using definition 4.5.2]. Then $T_j^* = \min_{i\in B_j} \min_{l\in S_i} X_l = \min_{l\in S_j'} X_l$ where

$S_j' = \bigcup_{i\in B_j} S_i$ $j=1,\ldots,m$. Thus by definition 4.5.2, $\underline{T}^*$ is MNBU [2] .

(P5) : Let $T_i = \min_{i \in A} T_A$ $i=1,2,\ldots,n$. Then $\min_{1 \leq i \leq n} a_i T_i =$

$$\min_{1 \leq i \leq n} a_i \min_{i \in A} T_A = \min_{1 \leq i \leq n} \min_{i \in A} a_i T_A =$$

$$= \min_{A \in \mathcal{F}} \min_{i \in A} a_i T_A = \min_{A \in \mathcal{F}} \left\{ (\min_{i \in A} a_i) T_A \right\} \qquad \ldots (4.6.1)$$

Since $(\min_{i \in A} a_i) T_A$, $A \in \mathcal{F}$ are independent NBU random variables, 4.6.1 is life function of a series system formed cut of independent NBU (univariate) random variables and hence has NBU distribution.

(P6) : The proof is obvious .

(P7) : Let $\tau(\underline{T})$ be the life function of a coherent sys tem formed out of $T_1,\ldots,T_n$. Let $P_1,\ldots,P_p$ be the minimal path sets for the corrosponding structure function. Then we have $\tau(\underline{T}) = \max_{1 \leq i \leq p} \min_{j \in P_i} T_j$. But since $T_j = \min_{\ell \in S_j} X_\ell$ for $\emptyset \neq S_j \subset \{1,\ldots,M\}$ $j=1,\ldots,n$ and X_ℓ, $\ell=1,\ldots,M$ are independent NBU random variables, we get

$$\tau(\underline{T}) = \max_{1 \leq i \leq p} \min_{\ell \in A_i} X_\ell \quad \text{where } A_i = \bigcup_{j \in P_i} S_j \; i=1,\ldots,p.$$

Thus $\tau(\underline{T}) = \tau'(\underline{X})$ is a life function of coherent system formed out of independent NBU componants and hence has NBU distribution.

(P8) : Let $T_i = \min_{j \in S_i} X_j, \emptyset \neq S_i \subset \{1,\ldots,M\}$.

We note that since $g(x)$ is nondecreasing function on $(0,\infty)$, $g^{-1}(x)$ is also a nondecreasing function on $(0,\infty)$. Also $g(x+y) \leq g(x) + g(y)$ for all $x,y \geq 0$, therefore, operating g^{-1} on both sides we get

$$(x+y) \leq g^{-1}(g(x) + g(y)) .$$

Putting $x = g^{-1}(g(x))$ (since g is contineous) we get $g^{-1}(g(x))+ g^{-1}(g(y)) \leq g^{-1}(g(x)+ g(y))$ for all $x,y \geq 0$. Letting $g(x) = s$, $g(y) = t$ we get $g^{-1}(s)+ g^{-1}(t) \leq g^{-1}(s+t)$ for all $s,t \geq 0$. Therefore,

$$\begin{aligned} P[g(x) > x+y] &= P[X > g^{-1}(x+y)] \\ &\leq P[X > g^{-1}(x) + g^{-1}(y)] \\ &\leq P[X > g^{-1}(x)]. P[X > g^{-1}(y)] \end{aligned}$$ and

hence $g(x)$ is also NBU random variable. Now since g is increasing, we have

$g(T_i) = g(\min_{j \in S_i} X_j) = \min_{j \in S_i} g(X_j)$. Since $g(X_i)$, $i=1,\ldots,n$ are independent NBU random variables, the result follows.

Our next theorem gives various necessary and sufficient conditions for an MNBU [2] random vector to be MVE.

4.6.2 Theorem :

Let $\underline{T}$ be MNBU [2]. Then following conditions are equivalent.

(i) $\underline{T}$ is MVE.

(ii) $\min_{1\leq i\leq n} a_i T_i$ is exponential for all $a_i>0$ $i=1,\ldots,n$.

(iii) $\underline{T}$ has exponential minimums.

(iv) T_i is exponential for $i=1,\ldots,n$.

(v) $\min_{1\leq i\leq n} T_i$ is exponential.

Proof :

The equivalence of these conditions is established by showing that (i)$\Longrightarrow$(iii)$\Longrightarrow$(iv)$\Longrightarrow$(ii)$\Longrightarrow$(v)$\Longrightarrow$(i). The proofs (iii)$\Longrightarrow$(iv) and (ii)$\Longrightarrow$(v) are trivival.

(i)$\Longrightarrow$(iii) : Let $\underline{T}$ be MVE. Then

$$\bar{F}(t_1,\ldots,t_n)= \exp[-\{\sum_{i=1}^{n} \lambda_i t_i + \sum\sum_{i<j} \lambda_{ij} \max(t_1,t_j)+\ldots+ \lambda_{12,\ldots,m}\max(t_1,\ldots,t_n)\}]. \text{ Hence}$$

$$\bar{F}(t,\ldots,t)= \exp[-\{\sum_{i=1}^{n} \lambda_i + \sum\sum_{i<j} \lambda_{ij}+\ldots+ \lambda_{12\ldots n}\} t]$$

$$= \exp[-\lambda' t] \text{ say}$$

where $\lambda' = \sum_{i=1}^{n} \lambda_i + \sum\sum_{i<j=1}^{n} \lambda_{ij} + \dots + \lambda_{12\dots n}$. Hence $\min_{1\leq i\leq n} T_i$ has exponential distribution, Since every marginal distribution of $\underline{T}$ also has MVE distribution, it follows that $\min_{i\in A} T_i$ is exponentially distributed for every subset $S \subset \{1,\dots,n\}$.

(iv) $\Longrightarrow$ (ii) : Let T_i, $i=1,\dots,n$ have marginal exponential distribution. Since $T_i = \min_{j\in S_i} X_j$, $\emptyset \neq S_i \subset \{1,\dots,n\}$ and $X_1,\dots,X_m$ are independent, it follows that each X_j is also exponentially distributed.[If not, let $u,v>0$ be such that $\bar{F}_{X_j}(u+v) < \bar{F}_{X_j}(u) . \bar{F}_{X_j}(v)$. Then

$$P[T_i > u+v] = \prod_{j\in S_i} P[X_j > u+v] < \prod_{j\in S_i} P[X_j > u]. P[X_j > v].$$

If T_i has exponential distribution with parameter λ_i, this implies that $e^{-\lambda_i(u+v)} < e^{-\lambda_i(u)} . e^{-\lambda_i(v)}$ which is a contradiction]. Now,

$$\min_{1\leq i\leq n} a_i T_i = \min_{1\leq i\leq n} \min_{j\in S_i} a_i X_j = \min_{j=1}^{M}\left[\left\{\min_{j\in S_i} a_i\right\} X_j \right].$$

Since $\left\{\min_{j\in S_i} a_i\right\} X_j$, $j=1,\dots,M$ are independent exponential variables, it follows that $\min_{1\leq i\leq n} a_i T_i$ has exponential distribution.

(v)$\Longrightarrow$(i) : By a similar argument as above it follows that each X_j, j=1,...,M has exponential distribution. Thus each T_i has a representation $T_i = \min_{j \in S_i} X_j$ where X_j are independent exponential variables. Hence $\underline{T}$ has MVE distribution.

4.7 Relation between MNBU [1] and MNBU [2] :

4.7.1 Lemma :

Let $\underline{T}$ be MNBU [2]. Then $\underline{T}$ is also MNBU [1]. [i.e. MNBU [2] is a subclass of MNBU [1] distributions].

Proof :

By definition 4.5.1 it immidiately follows that $\bar{F}_{\underline{T}}(0) = 1$. Now by definition 4.5.2 we have $T_i = \min_{j \in S_i} X_j$ where $X_1,\ldots,X_M$ are independent NBU random variables and $\emptyset \neq S_i \subset \{1,\ldots,M\}$ i=1,...,n. By corollary 4.3.2(a) it follows that $\underline{X} = (X_1,\ldots,X_M)$ has MNBU [1] distribution. Since each T_i is nondecreasing, nonnegative homegenous function of $\underline{X}$ it follows by property (P1) of theorem 4.3.1 that $\underline{T} = (T_1,\ldots,T_n)$ has MNBU [1] distribution.

The following example shows that MNBU [1] and MNBU [2] classes are distinct.

4.7.2 Example :

Let $\bar{\bar{F}}(x,y) = e^{-\sqrt{x^2+y^2}}$ $x, y \geq 0$.

Since $P[a_1 X > t, a_2 Y > t] = e^{-t\sqrt{1/a_1^2 + 1/a_2^2}}$, it follows that $\min\{a_1 X, a_2 Y\}$ has exponential distribution for every choice of nonnegative a_i, hence by theorem 3.8.7 of chapter III, it follows that (X,Y) has multivariate IHRA distribution of Block and Savits [i.e. $(X,Y) \in \pi$]. From remark 4.2.4 it now follows that (X,Y) has MNBU [1] distribution. But according to theorem 4.6.2 (i) and (ii) it follows that (X,Y) can not be MNBU [2].

Next we introduce some more multivariate NBU classes and compare them.

Consider nonnegative random variables $T_1,\ldots,T_n$ whose joint distribution satisfies one of the following conditions.

[A] $T_1,\ldots,T_n$ are independent and each T_i is NBU random variable.

[B] $(T_1,\ldots,T_n)$ is MNBU [2].

[C] for all $a_i > 0$ $i=1,\ldots,n$ $\min_{i\leq i\leq n} a_i T_i$ is NBU.

[D] For each $\emptyset \neq A \subset \{1,\ldots,n\}$, $\min_{i \in A} T_i$ is NBU.

[E] Each T_i is NBU.

Each of these classes of multivariate distributions may be dèsignated as a class of multivariate new better than used distributions. We now compare these classes.

4.7.3 Lemma :

The following relations hold among the classes [A] to [E] : [A] $\not\Longrightarrow$ [B] $\Longrightarrow$ [C] $\not\Longrightarrow$ [D] $\not\Longrightarrow$ [E] .

Proof :

The proofs [A] $\not\Longrightarrow$ [B] and [B] $\not\Longrightarrow$ [C] are discussed in the earlier sections. [C] $\Longrightarrow$ [D] and [D] $\not\Longrightarrow$ [E] are trivi al.

The following examples show that no other relations hold among these classes.

4.7.4 Example :

Let $T_1 = \min(U,W)$, $T_2 = \min(V,W)$ where U,V,W are independent exponential random variables with parameters $\lambda_1 = \lambda_2 = \lambda_{12} = 1$. Then it is clear from property (P2) and (P4) of theorem 4.6.1 that (T_1,T_2) is MNBU [2], but T_1,T_2 are not independent. Thus [B] $\not\Longrightarrow$ [A].

4.7.5 Example :

Let $T_1' = 2T_1$, $T_2' = T_2$ where T_1,T_2 are defined in Example 4.7.4 . Now $\min(a_1T_1', a_2T_2'] = \min(2a_1T_1, a_2T_2)$

has NBU distribution [since $(T_1,T_2) \in$ MNBU [2]] for all $a_1,a_2 > 0$, hence$(T_1',T_2') \in [C]$. However $(T_1',T_2')$is not MVE and hence by Theorem 4.6.2,$(T_1',T_2')$is not MNBU [2]. Thus [C]   [B].

<u>4.7.6 Example</u> :

Let T_1,T_2 be as in Example 4.7.4 and let $(T_1^*,T_2^*) = (\min(U,W),\ 1/2\ W)$. Then $\bar{F}_{T_1^*,T_2^*}(t_1,t_2) =$

$P[\ \min(U,W) > t_1, \frac{1}{2}W > t_2\ ] = P[U > t_1,\ W > \max(t_1,2t_2)]$

$= \exp[-(t_1 + \max(t_1,2t_2))]$. Let

$\bar{F}(t_1,t_2) = p\ \bar{F}_{T_1,T_2}(t_1,t_2)+(1-p)\ \bar{F}_{T_1^*,T_2^*}(t_1,t_2)$ where $0<p<1$.

$$= p\ \exp -[t_1+t_2+\max(t_1,t_2)]+(1-p)\exp[t_1+ \max(t_1,2t_2)].$$

Let (T_1',T_2') be the bivariate random vector whose joint survival function is $\bar{F}(t_1,\ t_2)$. Now

$\bar{F}_{T_1'}(t_1) = \bar{F}_{T_1',T_2'}(t_1,0) = p.\exp(-2t_1) + (1-p)\ \exp(-2t_1);$
$= \exp\ (-2t_1);$

$\bar{F}_{T_2'}(t_2) = \bar{F}_{T_1',T_2'}\ (0,t_2) = p\ \exp[-2t_2]+(1-p)\ \exp[-2t_2] =$ $\exp\ (-2t_2)$ and $\bar{F}_{T_1',T_2'}(t,t)= p\ \exp[-3t]+(1-p)\ \exp(-3t) =$ $\exp(-3t)$. Thus T_1',T_2' and $\min\ (T_1',\ T_2')$ are exponentially distributed. Hence (T_1',T_2') satisfies D. But

$F(t) = P[\ \frac{1}{2}T_1'>t,\ T_2'>t] = P[T_1'>2t,\ T_2'>t] = p.\exp(-5t)+$ $(1-p)\ \exp(-4t)$ and

$\bar{F}(2t)-[\bar{F}(t)]^2 = e^{-8t}\, P[1-p)\; [e^{-t}-1]^2 > 0$ for $t>0$. Hence $\min[\frac{1}{2}T_1', T_2']$ is not NBU. Hence (T_1', T_2') does not satisfy [C]. Hence $[D] \not\Rightarrow [C]$.

4.7.7 Example :

Let U,V and W be as in Example 4.7.4.

Let $\bar{F}(t_1,t_2) = P\,\bar{F}_{U,V}(t_1,t_2)+(1-p)\,\bar{F}_{W,W}(t_1,t_2)$ where $0 < p < 1$.

$$= P[\exp-(t_1+t_2)]+(1-p)[\exp-(\max(t_1,t_2))].$$

Let (T_1,T_2) be the bivariate random vector whose joint survival function is $\bar{F}(t_1,t_2)$. Now $\bar{F}_{T_1}(t_1) = \bar{F}(t_1,0) = p.\exp(-t_1) + (1-p)\exp[-t_1] = \exp(-t_1)$; $\bar{F}_{T_2}(t_2) = \bar{F}(0,t_2) = P.\exp(-t_2) + (1-p)\exp(-t_2). = \exp(-t_2)$

Thus T_1, T_2 have marginal exponential and hence NBU distribution. Hence (T_1,T_2) satisfies [E]. Let $T^* = \min(T_1,T_2)$. Then

$\bar{F}_{T^*}(t) = P[T_1>t,\, T_2>t] = p.\exp(-2t)+(1-p)\exp(-t)$ and

$\bar{F}_{T^*}(2t)-[\bar{F}_{T^*}(t)]^2 = e^{-2t}\,p[1-p][e^{-t}-1]^2 > 0$ for $t > 0$.

Hence $\min(T_1,T_2)$ is not NBU. Hence (T_1,T_2) does not satisfy D. Hence $[E] \not\Rightarrow [D]$.

Finally we present an additional class of multivariate new better than used distributions due to Block and Savits (1981) and compare this with the MNBU [2] class.

<u>4.7.1 Definition</u> :

A random vector $\underline{T}$ is said to be multivariate new better than used.(according to Block and Savits) or MNBU [F] if $\underline{T}$ has a representation.

$T_i = \sum_{j \in S_i} X_j$ where $X_1, \ldots, X_M$ are independent NBU and $\emptyset \neq S_i \subset \{1, \ldots, M\}$, $i=1, \ldots n$.

The following two examples show that none of the classes MNBU [F] and MNBU [2] is a subclass of the other.

<u>4.7.8 Example</u> :

Let U,V and W be independent exponential random variables with parameters $\lambda_1 \neq \lambda_2$ and $\lambda_{12} > 0$. respectively. Let $T_1 = \min(U,W)$ and $T_2 = \min(V,W)$. Clearly (T_1,T_2) have MVE and hence MNBU [2] distribution. If (T_1,T_2) also satisfy definition 7.1, then it has the form $T_1 = X+Z$, $T_2 = Y+Z$ where X,Y and Z are independent NBU random variables. Since T_1 is exponential, it follows that either X is exponential and Z is degenerate at

0 or vice versa. Same is true about Y and Z since T_2 has exponential distribution. Thus either $X \equiv Y \equiv 0$ and Z has exponential distribution or $Z \equiv 0$. Consequently either T_1 and T_2 are independently distributed or identically distributed which is not possible. Hence (T_1,T_2) does not belong to the class MNBU [F]. Thus MNBU [2] $\not\Rightarrow$ MNBU [F].

4.7.9 Example :

Let X,Y and Z be independent with absolutely contineous distributions. Let $T_1 = X+Z$ and $T_2 = Y+Z$. It follows that (T_1,T_2) is MNBU [F]. But if T_1, T_2 has the form $T_1 = \min\{T_{A_1}, T_{A_{12}}\}$ and $T_2 = \min\{T_{A_2}, T_{A_{12}}\}$ where T_{A_1}, T_{A_2}, $T_{A_{12}}$ are independent NBU r.v., since (T_1,T_2) have joint absolutely contineous distribution, an agrument similar to that of 3.5.3 (a) of chapter III, it follows that T_1, and T_2 are independent, which is a contradiction. Hence (T_1,T_2) do not have MNBU [2] distribution.

4.7.1 Remark :

In Example 4.7.9 we observe that $(T_1,T_2) = (X,Y)+(Z,Z)$. This shows that MNBU [2] class is not closed under convolution.