CHAPTER 1

Introduction

It is very important to see that the farm machinery can be repaired and its life is as long as possible. However, from the point of view of interests of individual farms and society as a whole, of a particular significance is the reliability of these machines in the sense of high probability of failure free work in the course of a period of field work. The loss to a farm will be great if tractors and combines of great longevity often fail to function during sowing or harvesting and if time is lost in their repair during the most important time of year. Similarly, in the case of aeroplane, it is necessary to have the maximum probability of failure free functioning during the period of the entire flight.

In this dissertation we will consider the problem originated in the context of reliability of a component of strength X subjected to a stress Y. The component fails if and only if at any time the applied stress is greater than its strength. Since the stress is a function of enviornment to which the component is subjected, it can be treated as a random variable, also the strength of component is mass-produced, depends on material properties, manufacturing procedures etc and can be treated as a random varaible. The reliability of the component in successfully completing its mission time is defined as the probability that its strength exceeds the maximum stress encoured during its operation. If X is the strength and Y is maximum stress then $R = Pr(Y < X)$ will be the probability that systems will be in the functioning state under stress-strength model.

Under this stress-strength model, there will be various cases regarding the form of distribution, nature of the parameters, that is either some parameters are known or all are unknown and dependence or independence of X and Y etc.

In the case when X and Y are identically distributed, it is very easy to obtain that

$$R = P(Y < X) = 1/2$$

and there does not arise any problem of estimation.

When the form of the distributions of X and Y are unknown then based on $(X_1, Y_1), (X_2, Y_2), --- (X_n, Y_n)$, n pairs of observations from X and Y respectively, one can obtain the estimator as,

$$Z_i = \begin{cases} 1 & \text{if } Y_i < X_i \\ 0 & \text{otherwise,} \end{cases}$$

then $\bar{Z} = n^{-1} \sum_{i=1}^{n} Z_i$ is an unbiased estimator of R and it has variance $n^{-1} P(X_i \leq Y_i). P(X_i > Y_i)$.

But, when there is further information regarding the form of the distributions of X and Y that is, form of the distributions is known but the parameters are unknown, then it is possible to find better estimators.

In this dissertation we discuss the method of obtaining MVUE of R, when the form of distribution of X and Y is same and it is known but the parameters for X and Y are different and we consider the cases when some parameters are known and when all the parameters are unknown. And we assume that X and Y are independent of each other.

In manycases we observe that after the period of random failures the unit or the system may gradually deteriorate with time, that is,unit or system is subject to wear-out. For example the break system of a vehicle. The wear-out. Failure distribution is close to normal distribution and that the use of this distribution for predicting the reliability is valid. So in Chapter 2 we consider the case when X and Y are independent normal variates. In section 2.2 we obtain the MVUE of R when the distribution of Y is known with one case dealing with the mean μ of X is known and variance σ^2 unknown and in section 2.3 we suppose that distribution of Y is known and X has mean μ and variance σ^2 with two cases; one is when σ^2 is known and remaining when σ^2 is unknown. In section 2.4 we obtain the MVUE of R when the parameters of X and Y are unknown.

In some cases we observe that the failures are of random nature for example the failure of electrical instruments. These kinds of failures are well described by an exponential law. In chapeter 3 we obtain the MVUE of R when X and Y follow exponential law with parameter λ and μ. In section 3.2 we obtain MVUE of R when both λ and μ are unknown and in 3.3 we assume that μ is known.

In chapter 4, we obtain the MVUE of R when the r.v.s X and Y belong to the truncated exponential family. Here the results are derived for only the lower truncated exponential families, because our main interest was to obtain the MVUE of R for the normal and exponential distribution. However, the corresponding results for the upper truncated and double truncated distribution can similarly be derived.

In section 4.2 we obtain the conditional density of X_1 given the sufficient and complete statistic by using Laurent's approach. In section 4.3 we obtain the MVUE of R. In section 4.4 we obtain the MVUE of R for two particular cases of the exponential family, namely, the two parameter exponential and two parameter pareto distributions.

In this dissertation we often use the following results :

<u>Theorem 1.1</u> (Rao-Blackwell-Lehmann-Scheffe) :

Let $X_1,\ldots,X_n$ be i.i.d r.v.s. with p.d.f. $h(\cdot, \theta)$, $\theta \in \Theta$, and let $T = (T_1,T_2,\ldots,T_m)'$, $T_j = T_j(X_1,\ldots,X_n)$, $j = 1,\ldots,m$ be a sufficient statistic for θ. Let $f(\cdot\, ; \theta)$ be its p.d.f. set $F = \{f(\cdot\, ; \theta), \theta \in \Theta\}$ and assume that F is complete. Let $\breve{g} = g(X_1,\ldots,X_n)$ be an unbiased statistic of a real-valued function g of θ, which is not a function of T alone, (with probability 1). Set $\phi(t) = E_\theta(\breve{g} | T=t)$ then we have,

i) the r.v. $\phi(T)$ is a function of sufficient statistic T alone.

ii) $\phi(T)$ is an unbiased statistic for $g(\theta)$.

iii) $\sigma_\theta^2 [\phi(T)] < \sigma_\theta^2(\breve{g})$, $\theta \in \Theta$, provided $E_\theta \breve{g}^2 < \infty$.

iv) $\phi(T)$ is the unique unbiased statistic for $g(\theta)$ with the smallest variance in the class of all unbiased statistic for $g(\theta)$, in the sense that if $V = V(t)$ is another unbiased statistic for $g(\theta)$, then $\phi(T) = V(T)$, a.e. $\theta \in \Theta$.

Proof:

(i) That $\phi(T)$ is a function of the sufficient statistic T alone and does not depend on Θ is a consequence of the sufficiency of T.

Now (ii) is also true because,

$$g(\Theta) = E(\breve{g}) = E_{\Theta}\{E_{\Theta}(\breve{g} \mid T)\}$$

$$= E_{\Theta}[\phi(T)]$$

To prove (iii) part, we use the Jensen's inequality,

let C be a continuous convex function of a real variable, if u is a r.v., then $C\{E(u)\} \leq E\{C(u)\}$, this inequality holds for conditional expectations also,

Now

$$E_{\Theta}\{C(g)\} = E_{\Theta}\{E_{\Theta}[C(\breve{g}) \mid T]\}$$

so, by Jensen's inequality,

$$E_{\Theta}[E_{\Theta}\{C(\breve{g}) \mid T\}] \geq E_{\Theta}[C\{E_{\Theta}(\breve{g} \mid T)\}]$$

$$= E_{\Theta}[C(\phi(T))]$$

If we now set, $C(u) = \{u - g(\Theta)\}^2$, we get the result that,

$$\sigma_{\Theta}^2(\breve{g}) \geq \sigma_{\Theta}^2(\phi(T)), \quad \Theta \in \Theta, \text{ provided } E_{\Theta}(\breve{g}^2) < \infty.$$

Finally to prove (iv) by unbiasedness of $\phi(T)$ and $V(T)$ we have,

$$E_{\Theta}(\phi(T)) = E_{\Theta}(V(T)) = g(\Theta), \quad \Theta \in \Theta$$

or equivalentely,

$$E_{\Theta}[\phi(T) - V(T)] = 0, \quad \Theta \in \Theta$$

Then by completeness of F, we have

$$[\ \phi(T) - V(T)\ ] \equiv 0$$

i.e. $\phi(t) = V(t)$, for all $t \in R^m$.,

except possibly on a set N of t's such that

$$P_\theta(T \in N) = 0 \qquad \text{for all } \theta \in \Theta .$$

This proves the theorem.

Definition 1.2.

A statistic V(X) is said to be ancillary if its distribution does not depend on θ,

Lemma 1.3. (D. Basu)

If T is a sufficient and complete statistic for the family $P = \{P_\theta, \theta \in \Theta\}$, then any ancillary statistic V is independent of T.

Proof :

If V is ancillary, the probability,

$P_A = P(V \in A)$ is independent of θ for all A.

Let

$n_A(t) = P(V \in A | T=t)$, then Notation

$$\begin{aligned} E_\theta[n_A(t)] &= E_\theta \{P[V \in A | T=t]\} \\ &= \int P(V \in A | T=t)\, dF_T(t) \\ &= P(V \in A) \\ &= P_A \end{aligned}$$

that is $E_\theta[n_A(t) - P_A] = 0.$

but T is complete statistic, which implies that

$$n_A(t) = P_A$$

or

$$P[V \in A | T=t] = P(V \in A)$$

This establishes the independence of V and T.

Hence the lemma.