C H A P T E R - III

INEQUALITIES ON THE INTEGRAL OF A SYMMETRIC UNIMODEL FUNCTION OVER A SYMMETRIC CONVEX SET

This chapter is based on one of the published works by T.W.Anderson [11]. Even though the concept of majorization does not appear any where in this paper, still we include this topic as the concepts found in his work forms the basis for latter studies in inequalities via majorization.

3.A CONCEPTIONAL BACKGROUND

Let $f(x)$ be a function on the real line. Also $f(x)$ is symmetric about origin and $f(k\,x) \geq f(x)$; $0 \leq k \leq 1$, (known as unimodel). Consider the integral of $f(x)$ over a fixed length. It is obvious that if the region of integration is centered at the origin, then the integral would have a maximum value, as is illustrated bellow.

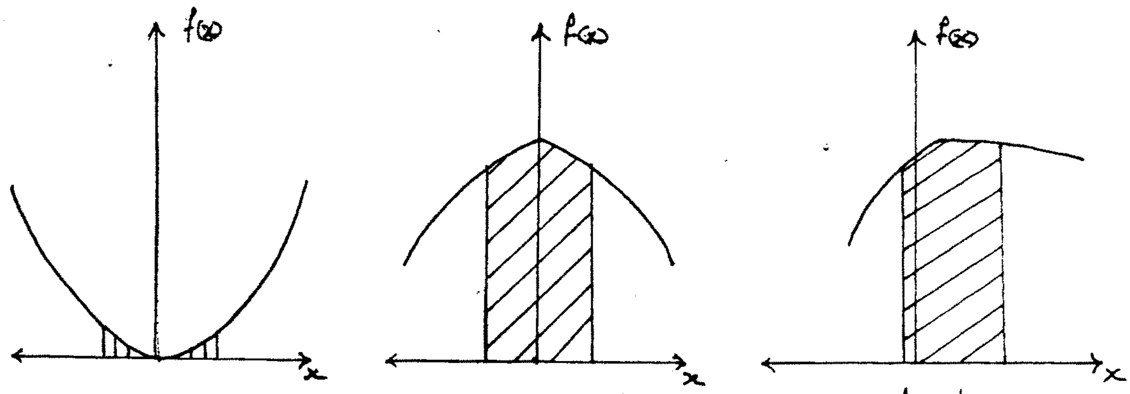

fig 1. a   fig 1. b   fig 1. c

Fig.1.a. : f(x) is symmetric but not unimodel. Hence the optimal area-region is not centered at the origin.

Fig.1.b. : f(x) is symmetric and unimodel. Hence the optimal area-region is centered at the origin.

Fig.1.c. : f(x) is unimodel but not symmetric. Thus the optimal area-region is not centered at the origin.

This result could be generalised into n-space by considering a symmetric convex set in place of the interval. Also the condition of unimodelity could be modified as the set of points for which the function is at least equal to a given value is convex.

3.B ANDERSON'S THROREM AND RELATED RESULTS

3.B.1 Theorem : Let E be a convex st in n-space, symmetric about origin. Let $f(x) \geq 0$ be a function such that

i) $f(x) = f(-x)$

ii) $k_u = \{x \mid f(x) \geq u\}$ is convex for every u, $(0 < u < \infty)$

iii) $\int_E f(x)\,dx < \infty$ (in the Lebesgue sense)

Then

$$\int_E f(x + ky)\,dx \geq \int_E f(x + y)\,dx \qquad (1)$$

for $0 \leq k \leq 1$

Outline of the proof :

Let $E + ky$ and $E + y$ be two sets translated from E by the vector ky and y. Thus $\int_E f(x + ky)\, dx$ is equivalent to $\int_{E+ky} f(x)\, dx$.

Also $\int_E f(x+y)\, dx = \int_{E+y} f(x)\, dx$

Thus the inequality

$$\int_E f(x+ky)\, dx \geq \int_E f(x+y)\, dx$$

could be written as

$$\int_{E+ky} f(x)\, dx \geq \int_{E+y} f(x)\, dx$$

The theorem is proved by showing that

i) For every u

$$V\{(E + ky) \cap ku\} \geq V\{(E + y) \cap ku\}$$

where $V\{\ \}$ indicates the volume of the set.

ii) Taking the integrals of the function over $(E + ky) \cap ku$ and $(E + y) \cap ku$ for all possible values of u would result in the inequality

$$\int_E f(x+ky)\, dx \geq \int_E f(x+y)\, dx$$

This could be explained with the help of fig.2 given for single variable case.

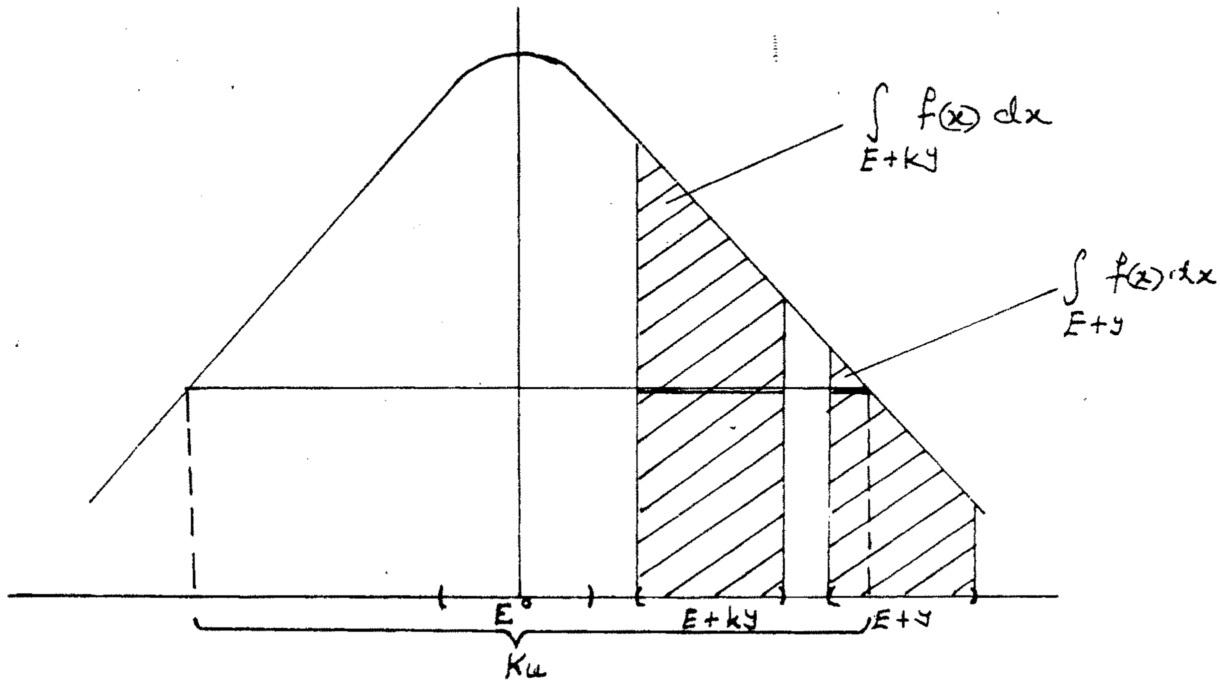

fig.2

As the example is of one dimensional case the volume reduces to the length the strip shown in the diagram.

It can be seen that the length of the strip $(E+ky) \cap ku$ is greater than or equal to the length of the strip $(E+y) \cap ku$ for all possible u's. Then take the sums of areas over both sets for different du's (where du is a small strip of u). Which is nothing but the integral of the function over the two sets (shown by the shaded region).

As $V\{(E + ky) \cap ku\} \geq V\{(E + y) \cap ku\}$ for every du

$$\int_{E+ky} f(x)\, du \geq \int_{E+y} f(x)\, dx$$

as was to be shown.

<u>Brunn - Minkowski Theorem</u> :

Let E_0 and E_1 be two non-empty sets. Then

$$V^{1/n}\left\{(1-\theta)\, E_0 + \theta\, E_1\right\} \geq (1-\theta)\ V^{1/n}(E_0) + \theta\, V^{1/n}(E_1)$$

whenever $0 \leq \theta \leq 1$.

<u>Proof of Anderson's Theorem</u> :

The inequality

$$\int_{E} f(x+ky)\, dx \geq \int_{E} f(x+y)\, dx$$

is equivalent to

$$\int_{E+ky} f(x)\, dx \geq \int_{E+y} f(x)\, dx$$

where $E + y$ is the set E transformed by y.

(Let $x \in E$. Use the transformation

$z = x + ky$

Hence $dz = dx$ (As y is fixed)

If $x \in E$ then $z \in E + ky$

Thus $$\int_{E} f(x+ky)\, dx = \int_{E+ky} f(z)\, dz$$

$$= \int_{E+ky} f(x)\, dx$$

The same argument holds good for the integral over $E + y$).

Initially we would show that

$$V\{(E + ky) \cap ku\} \geq V\{(E + y) \cap ku\}$$

for every u.

Let $\alpha\ [(E + y) \cap ku\} + (1 - \alpha)\ [(E - y) \cap ku]$

denotes the set obtained by taking all linear combinations

$\alpha z + (1 - \alpha) w$.

Where $z \in (E + y) \cap ku$ and $W \in (E - y) \cap ku$

and $0 \leq \alpha \leq 1$.

Let $\alpha = (1 + k) / 2$ so that

$$\alpha y + (1 - \alpha)(- y) = ky \tag{1}$$

Then $(E + ky) \cap ku > \alpha[(E + y) \cap ku] + (1 - \alpha)[(E - y) \cap ku]$

Because ku is convex and

$$(E + ky) \supset \alpha (E + y) + (1 - \alpha)(E - y) \tag{2}$$

We have

$$V\{(E+ky) \cap ku\} \geq V\{\alpha[(E+y) \cap ku] + (1- \alpha)[(E-y) \cap ku]\} \tag{3}$$

[Statement (2) could be proved as follows. Consider a point from the set

$D = \alpha (E + y) + (1 - \alpha)(E - y)$. We ought to show that is also belongs to $E + ky$.

If $z \in D = \{\alpha u + (1 - \alpha) v : u \in E + y, v \in E - y\}$

$$= \{\alpha (e_1 + y) + (1 - \alpha)(e_2 - y);\ e_1, e_2 \in E\}$$

$$= \{\alpha e_1 + (1-\alpha) e_2 + (2\alpha - 1) y;\ e_1, e_2 \in E\}$$

$$= \{e + ky;\ e \in E\} \equiv E + ky$$

(since $2\alpha - 1 = k$ from (1)

Hence $E + ky \supset \alpha (E + y) + (1 - \alpha) (E + y).]$

$(E + y) \cap ku$ is a mirror immage through the origin of $(E - y) \cap ku$ and therefore these two sets have the same volume.

Using Brann Minkowski theorem we get

$$V \{ \alpha[(E + y) \cap ku] + (1 - \alpha) [(E + y) \cap ku]\}$$

$$\geq \alpha V \{(E + y) \cap ku\} + (1 - \alpha) V \{(E - y) \cap ku\}$$

$$= \alpha V \{(E + y) \cap ku\} + (1 - \alpha) V \{(E + y) \cap ku\}$$

$$= V \{(E + y) \cap ku\} \qquad (4)$$

From (3) and (4) it follows that

$$H(u) = V \{(E + ky) \cap ku\} \geq V \{(E + y) \cap ku\} = H^*(u) \qquad (5)$$

as was to be shown.

The theorem would be proved if it could be shown that

$$\int_0^\infty H(u)\, du \geq \int_0^\infty H^*(u)\, du$$

Definition of Lebesge and Lebesgue-Stieltjes, integrals show

$$\int_{E+ky} f(x)\, dx - \int_{E+y} f(x)\, dx = - \int_0^\infty u\, d\, H(u) + \int_0^\infty u\, d\, H^*(u)$$

(As $H(u)$ is a decreasing function of u, $d\,H(u)$ would be negative in sign).

$$= \int_0^{\infty} u\, d\left[H^*(u) - H(u)\right]$$

Consider the integral

$$\int_a^b u\, d\left[H^*(u) - H(u)\right] = u\,(H^*(u) - H(u))\,\Big|_a^b - \int_a^b (H^*(u) - H(u))\, du$$

$$= b\left[H^*(b) - H(b)\right] - a\left[H^*(a) - H(a)\right] + \int_a^b \left[H(u) - H^*(u)\right] du \qquad (6)$$

Since $f(x)$ has finite integral over E (as is assumed in the theorem)

$b\,H(b) \longrightarrow 0$ as $b \longrightarrow \infty$; so also

$b\,H^*(b) \longrightarrow 0$ as $b \longrightarrow \infty$.

Therefore the first term on R.H.S of (6) can be made arbitarilly small in absolute value.

If $a \geq 0$ the second term above is non-negative as well as the third.

Thus $\int_0^{\infty} u\, d\left[H^*(u) - H(u)\right] \geq 0$

Hence $\int_{E+ky} f(x)\,dx - \int_{E+y} f(x)\,dx \geq 0.$

i.e., $\int_{E+ky} f(x)\,dx \geq \int_{E+y} f(x)\,dx$

Hence the proof.

3.B.2 Remark : Consider the integral $\int_E f(x + y)\,dx$ as a function of y.

Write $\varphi(y) = \int_E f(x + y)\,dx$

It is to be observed that $\varphi(y)$ possess certain properties of $f(x)$,

(a) $\varphi(y) = \int_E f(x + y)\,dx = \int_{E+y} f(x)\,dx = \int_{E-y} f(x)\,dx = \varphi(-y)$

(This follows from the proof if Anderson's theorem).

Thus $\varphi(y)$ is symmetric.

(b) $\varphi(y)$ is unimodal in the sense that along any given ray through the origin the integral is a non-decreasing function of the distance from the origin. However $\varphi(y)$ does not necessarily satisfy the condition of unimodality imposed on $f(x)$.

i.e., $\{y \,/\, \varphi(y) \geq u\}$ is not necessarily Convex.

3.C PROBABILITY INEQUALITIES

If a probability density function satisfies all the assumptions of Anderson's theorem it would give rise to the inequalities anticipated in the theorem. Hence the following corollary.

3.C.1 Corollary :

Let X be a random vector with density $f(x)$ such that

i) $f(x) = f(-x)$ and

ii) $\{u \mid f(x) \geq u\}$ is convex for every u $(0 \leq u \leq \infty)$.

If E is a convex set, symmetric about the origin,

$P\{X + ky \in E\} \geq P\{X + Y \in E\}$

for $0 \leq k \leq 1$.

If $h(x)$ is a symmetric function such that $\{x \mid h(x) \leq v\}$ is convex, then $P\{h(X + ky) \leq v\} \geq P\{h(X + Y) \leq v\}$

3.C.2 Remark :

$h(x)$ could be the cumulative distribution. Hence the corollary implies that the cumulative distribution of $h(X + Y)$ is bounded by that of $h(X + ky)$ and by choosing $k = 0$ one would get the upper limit as $h(X)$.

3.C.3 Example :

A particular distribution which satisfies the conditions of Anderson's theorem and hence the corollary is the Normal distribution. This is illustrated below.

Let $f(x)$ be the joint probability density of n iid normal random variables, with mean zero and variance one.

Hence $f(x) = (2\pi)^{-n/2} \exp\{-\frac{1}{2} \sum_{i=1}^{n} x_i^2\}$

i) It is obvious that $f(x) = f(-x)$

ii) To varify unimodality consider the set

$$ku = \{x \mid f(x) > u\}$$

$$= \{x \mid (2\pi)^{-n/2} \exp\{-\frac{1}{2} \sum_{i=1}^{n} x_i^2 > u\}$$

$$= \{x \mid \exp\{-\frac{1}{2} \sum_{i=1}^{n} x_i^2 > u'\}$$

where $u' = u.(\pi)^{n/2}$

$$= \{x \mid -\frac{1}{2} \sum_{i=1}^{n} x_i^2 > u''\}$$

where $u'' = \log u'$

$$= \{x \mid \sum_{i=1}^{n} x_i^2 < u'''\}$$

where $u''' = -2u''$.

It is easy to see that the set ku is a sphere in n-space which will always be a convex set.

Hence $f(x)$ satisfies both the conditions stipulated by Anderson's theorem.

<u>3.C.4 Theorem</u> :

Let X be a random vector with density $f(x)$ such that

i) $f(x) = f(-x)$

ii) $\{x \mid f(x) \geq u\}$ is convex for every u $(0 \leq u < \infty)$.

Let Y be independently distributed. If E is a convex set, symmetric about origin, then

$$P\{(X + kY) \in E\} \geq P\{X + Y \in E\}$$

for $0 \leq k \leq 1$. if $h(x)$ is a symmetric function such that

$\{x \mid h(x) \leq v\}$ **is** convex then

$$P\{h(X + kY) \leq v\} \geq P\{h(X + Y) \leq v\}$$

for $0 \leq k \leq 1$.

<u>Proof</u> : Let the cumulative distribution of Y be $G(y)$.

Then the density of $X + kY$ is $\int_R f(z - ky)\, d\, G(y)$, where R - denotes the entire n-space.

Thus

$$P\{X + kY \in E\} = \int_E \int_R f(z - ky)\, d\, G(y)\, dz$$

$$= \int_R \int_E f(z - ky)\, dz\, d\, G(y).$$

$$= \int_R \int_{E-ky} f(w)\, dw\, d\, G(y).$$

Then Anderson's theorem yields the implied results.

<u>3.C.5 Remark</u> : 3.C.4 shows that in certain sense the distribution of $X + Y$ is more spread out than that of $X + kY$ $(0 \leq k \leq 1)$.