CHAPTER - 2

APPROXIMATIONS

APPROXIMATIONS

2.1) Introduction :-

An ideal filter has a zero loss in its pass band and infinite loss in its stop band. Unfortunately , ideal filter response is not practical because linear networks can not produce the discontinuties. However ,it is possible to obtain a practical response that approximates the ideal response by using special design techniques and high speed op.amps (LM312 or ICL8017).Such special design technique is called approximation.

The following are some of the most commonly used (practical filters that approximates the ideal response) approximations.

(i) Butterworth (maximally flat magnitude)

(ii) Chebyshev (T chebycheff or equal ripple)

(iii) Bessel (Thomson or maximaly flat dealy)

(iv) Elliptic-Function (cauer or biquadratic)

(v) Optimal (Legendre or monotonic L)

(vi) Parabolic (P)

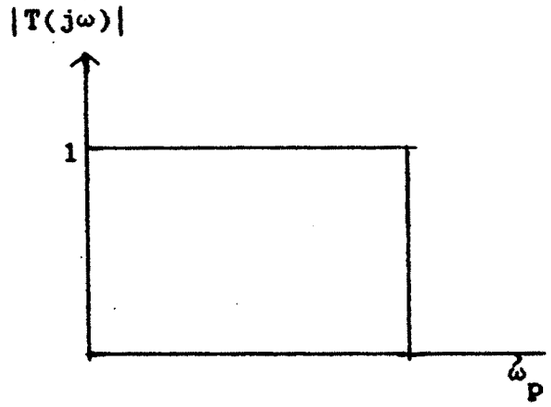

fig 2.1 (a)

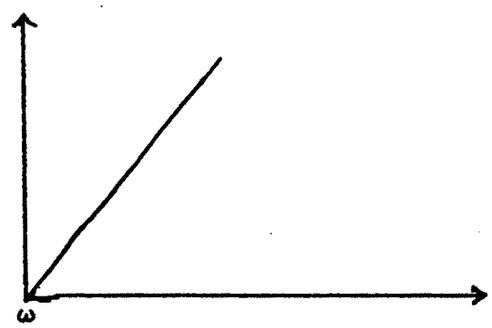

fig.2.1(b)

Fig(a) shows the characteristics of an ideal LPF & fig(b).shows the phase of the filter network is to be linear fuction of ω .But these specifications cannot be achieved by the physical network.It can be shown that any band limited frequency response results in a non causal time response.

In order to circumvent this problem the filter specifications are modified.Fig 2.2(a) shows the specifications for a realizable low-pass filter.

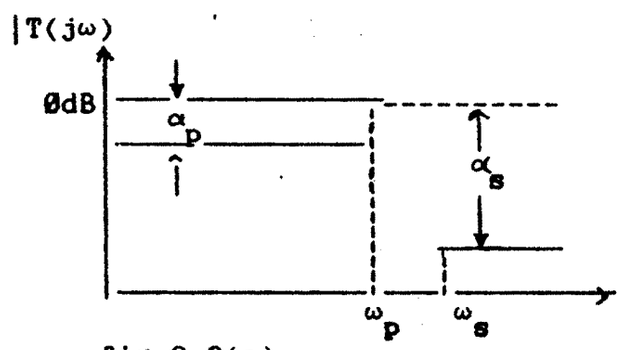

fig 2.2(a)

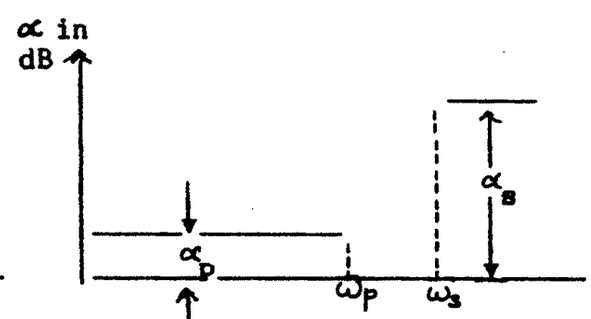

fig 2.2(b)

$\omega_p \longrightarrow$ pass band edge

$\omega_s \longrightarrow$ stop band edge

$\omega_p < \omega < \omega_s \longrightarrow$ transition band

Maximum pass-band attenuation α_p dB

Minimum stop-band attenuation α_s dB

i.e. $\alpha \le \alpha_p \qquad \omega \le \omega_p$

$\alpha \ge \alpha_s \qquad \omega \ge \omega_s$

Where α is the attenuation of the filter in dB.

Fig 2.2(b) shows the plot of loss (attenuation) in dB versus ω[1].

The approximation methods solve the problem of selecting a realizable rational function whose frequency response approximates that in fig 2.2.

Let f(ω) be the required frequency response function and t(ω) the function which approximates f(ω).The method of approximating is to have t (ω) exactly equal to f(ω) at a prescribed point $\omega = \omega_o$. Let f(ω) and t(ω) be expanded in Taylor series about ω_o,i.e.

$$f(\omega)=f(\omega_o)+f'(\omega_o)(\omega - \omega_o)+\frac{f''(\omega_o)}{2!}(\omega - \omega_o)^2+\dots\dots\dots$$

$$\text{and } t(\omega)=t(\omega_o)+t'(\omega_o)(\omega - \omega_o)+\frac{t''(\omega_o)}{2!}(\omega - \omega_o)^2+\dots\dots$$

(Where ′ indicates differentiation w.r. t. ω).If f(ω_o) = t(ω_o) and as many as possible derivatives at $\omega = \omega_o$ are set to zero.This represents a "Taylor series approximation".

In all approximations the transfer function is selected as

$$|T(j\omega)|^2 = \frac{1}{1+\epsilon^2 k_n^2(\omega)} \qquad (2.1a)$$

with

$$0 < k_n^2(\omega) << 1, \quad 0 < \omega < \omega_p$$

and $k_n^2(\omega) >> 1, \ \omega > \omega_s$

With this selection, $|T(j\omega)|^2 \approx 1$ in the pass-band and approximately zero in the stop-band. From equation 2.1a the attenuation function is

$$\alpha(\omega) = 10 \log\left(1 + \epsilon^2 k_n^2(\omega)\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (2.1b)$$

and the constant $\epsilon(\omega)[= \int_{\omega a}^{\omega b} |f(\omega)-t(\omega)|^2 d\omega]$ determines the pass-band and/or stop-band attenuation.

The following are the well known approximations.

1) Butterworth approximation

2) Chebyshev approximation

3) Bessel approximation

4) Eliptic-function [cauer] approximation.

2.2) BUTTERWORTH APPROXIMATION :-

Butterworth approximation is a special form of Taylor series approximation in which the approximating function $t(\omega)$ and the specified function $f(\omega)$ are identical at $\omega = 0$. For this approximation $k_n(\omega)$ is selected as

$$k_n(\omega) = \beta_0 + \beta_1\omega + \beta_2\omega^2 + \cdots\cdots + \beta_n\omega^n.$$

For a Tayior series approximation the function $k_n(\omega)$ must be"maximally flat" at the origin (i.e.$\omega = 0$).Hence as many derivatives

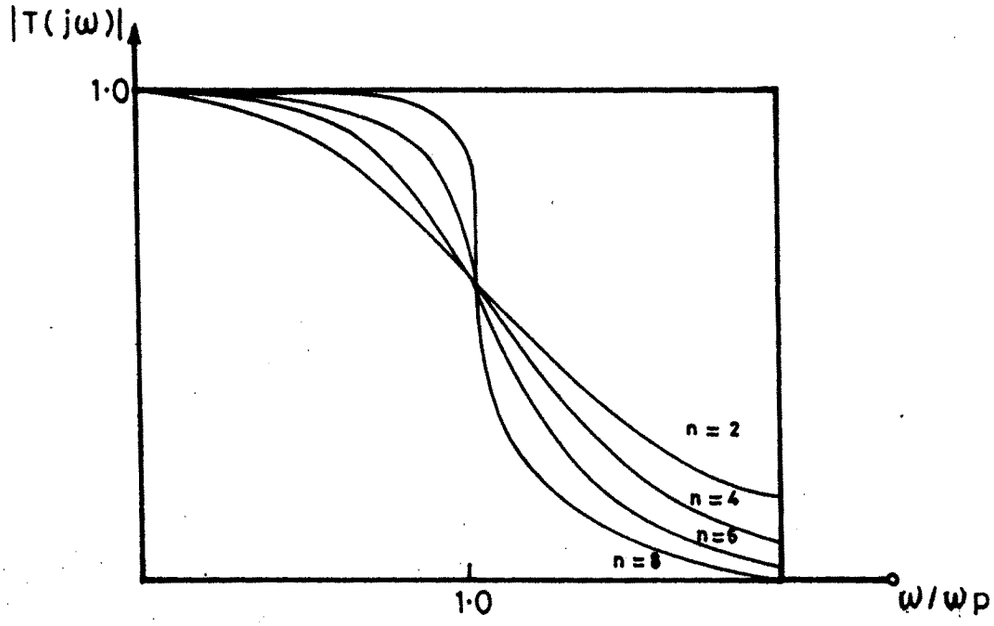

Fig. 2·3 - Shows the frequency response of a Butterworth filter for various values of n. All the curves pass through the same point at $\omega = \omega p$ and this point is determined by αp.

of $k_n(\omega)$ as possible must vanish at $\omega = 0$.Hence for Butterworth approximation

$$k_n(\omega) = \omega^n$$

The magnitude function $|T(j\omega)|^2$ and the attenuation function $\alpha(\omega)$ are then given by

$$|T(\omega)|^2 = \frac{1}{1+c^2\omega^{2n}}$$

$$\alpha(\omega) = 10 \log[1+c^2\omega^{2n}] \qquad \text{.....} \qquad (2.2)$$

(Here we have used c in place ϵ in equation 2.1).In equation (2.1) & (2.2) ω should be interpreted as the frequency normalized with respect to the pass-band edge ω_p ,

i.e.

$$|T(j\omega)|^2 = \frac{1}{1+c^2(\frac{\omega}{\omega_p})^{2n}} \qquad \text{.....} \qquad (2.3a)$$

and $$\alpha(\omega) = 10 \log [1+c^2(\frac{\omega}{\omega_p})^{2n}] \qquad \text{.....} \qquad (2.3b)$$

Fig 2.3 shows the frequency response of a Butterworth filter for various values of n . All the curves pass through the same point at $\omega = \omega_p$ and this point is determined by α_p .

If the specification are barely satisfied , then from equation 2.3b we have .

$$\alpha_p = 10 \log \left(1+c^2(\frac{\omega_p}{\omega_p})^{2n} \right)$$

$$\alpha_s = 10 \log \left(1+c^2(\frac{\omega_p}{\omega_p}) \right) \qquad \text{.....} \qquad (2.4)$$

The factor ω_p / ω_s, is called "selectivity parameter" and is represented by k

i.e. $$k = \frac{\omega_p}{\omega_s} \qquad (2.5)$$

equation 2.4 can be written as

$$c^2 = 10^{0.1\alpha_p} - 1$$

and $$\frac{c^2}{k^{2n}} = 10^{0.1\alpha_s} - 1 \qquad (2.6)$$

Dividing one part by the other we have

$$k^{2n} = \frac{10^{0.1\alpha_p} - 1}{10^{0.1\alpha_s} - 1}$$

If we define

$$\left[\frac{10^{0.1\alpha_p} - 1}{10^{0.1\alpha_s} - 1} \right] = k_1 \qquad (2.7)$$

as the "discrimination parameter", we have

$$n = \frac{\log k_1}{\log k}$$

The order of the filter "n" should be selected such that

$$n \geq \frac{\log k_1}{\log k}, \quad n \text{ is an integer.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.8)$$

If 'n' happens to be equal to $\log k_1 / \log k$ then the value of c obtained from both the equetion in (2.6) are the same. If $n = \log k_1 / \log k$, then c can be selected to satisfy either the pass band edge or stop-band edge requirments exactly.

In the first case ; $$c = \left(10^{0.1\alpha_p} - 1\right)^{1/2} \qquad \ldots\ldots\ldots (2.9a)$$

In the second case ; $$c = \left[k^{2n}(10^{0.1\alpha_s} - 1)^{1/2} \right] \qquad \ldots\ldots (2.9b)$$

In order to realize a filter we have to determine its transfer function . From realizability conditons, the poles of the transfer function must be in the left half of s-plane .The poles of the transfer function are determined by factorizing the denominator of $|T(j\omega)|^2$and properly allocating its roots. To this end let

$$|T(j\omega)|^2 = \frac{1}{|H(j\omega)|^2}$$

Here, $|H(j\omega)|^2 = 1+\omega^{2n}$,and by analytical continuation we can write

$$|H(j\omega)|^2 = H(s)\ H(-s)|_{s=j\omega} = 1+(-s^2)^n|_{s^2=-\omega^2}$$

The 2n zeros of H(s) H(-s)are obtained by solving

$$1+(-1)^n s^{2n} = 0 \qquad \ldots\ldots\ldots\ldots (2.10)$$

As can be easily seen , the roots of H(s) H(-s) appear in quadrantal symmetry . Hence , the roots in the LH plane can be separated and assigned as the poles of T(s).Two cases arise depending upon whether n is even or odd.

(i) If n is even equetion 2.10 reduces to

$$s^{2n} = -1 = e^{j(2k-1)\pi}$$

The 2n-roots are

$$p_k = \exp\left[j\frac{2k-1}{2n}\pi\right],\ k = 1,2,\ \ldots\ldots 2n \qquad (2.11)$$

i.e

$$p_k = \cos\left(\frac{2k-1}{2n}\pi\right) + j\sin\left(\frac{2k-1}{2n}\pi\right);\ k = 1,2,..2n$$

(ii) If n is odd : equation 2.10 reduces to

$$s^{2n} = 1 = e^{j2k\pi}$$

The 2n roots are

$$p_k = \exp\left[j\frac{k}{n}\pi\right], \quad k = 0, 1, 2, \ldots (2n-1) \qquad (2.12)$$

i.e

$$p_k = \cos\left(\frac{k}{n}\pi\right) + j\sin\left(\frac{k}{n}\pi\right); \quad k = 0, 1, 2, \ldots (2n-1)$$

the transfer function is constructed from the LH plane roots and is given by

n even :

$$T(s) = \frac{1}{\prod_{k=1}^{n/2}(s^2 + 2\cos\theta_k s + 1)} \qquad (2.13\ a)$$

$$\text{with } \theta_k = \frac{2k-1}{2n}\pi$$

n odd :

$$T(s) = \frac{1}{(s+1)\prod_{k=1}^{(n-1)/2}(s^2 + 2\cos\theta_k s + 1)} \qquad (2.13\ b)$$

$$\text{with } \theta_k = \pi/n$$

For example , For n = 2.

$$T(s) = \frac{1}{s^2 + 2\cos(\pi/4)s + 1}$$

$$= \frac{1}{s^2 + \sqrt{2}\,s + 1}$$

and for n = 3

$$T(s) = \frac{1}{(s+1)(s^2 + 2\cos(\pi/3)s + 1)}$$

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

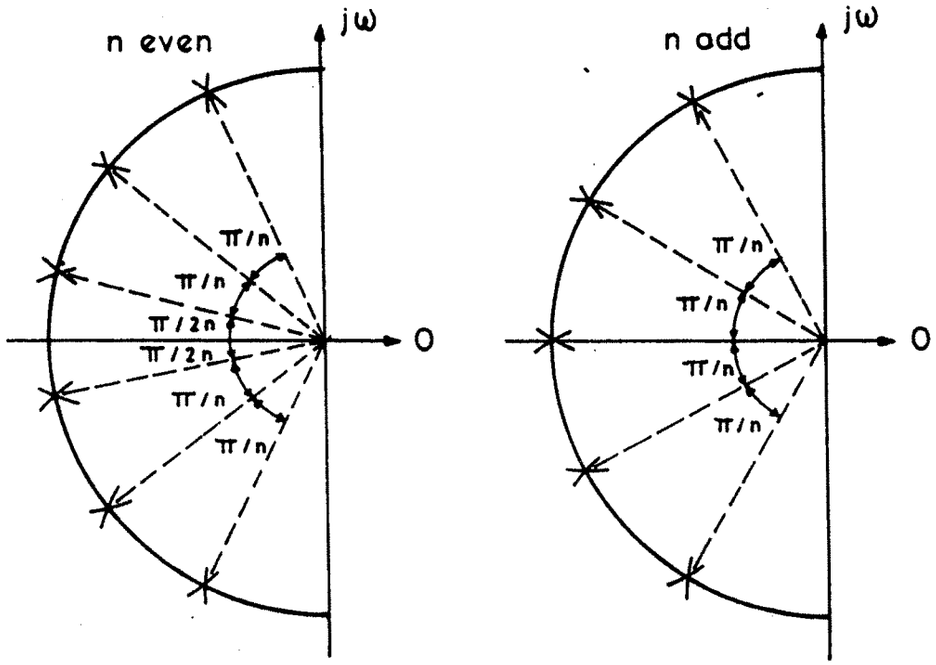

Fig. 2·4 – Shows the poles of T(s) for even and odd n It can be seen that for odd n, these is always a pole on the negative real axis and for both the cases these can not be a pole on the jω axis.

Fig.(2.4) shows the poles of T(s) for even and odd "n".It can be seen that for odd n ,there is always a pole on the negative real axis and for both the cases there can not be a pole on the $j\omega$ axis.

2.3) CHEBYSHEV APPROXIMATION :-

A steeper rate of descent to the stop-band than Butterworth filters is obtainable from the chebyshev approximation .This is obtained at the expense of pass-band ripples as shown . in fig 2.5 .

For a particular filter of order n , increasing the ripple magnitude increases the rate of descent from pass band to stop band and make the response in the cutoff region more rectangular . However , the transient properties are poorer than those of Butterworth approximation and rapidly deteriorate with increasing ripple .

The poles are located on an elipse . In contrast to the Butterworth filters , the stop band rate of descent exceeds nx6 decidels per octave for the first few octaves and eventually attains nx6 decibels per octave well in to the stop band .

2.4) BESSEL APPROXIMATION :-

These filters provide an excellent approximation to constant delay , especially for high-order n's . The step response exhibits no overshoot .

The frequency-response characteristics are extremely poor in comparision with Butterworth and Chebyshev types . The pass band is rounded & the rate of roll off in the first few

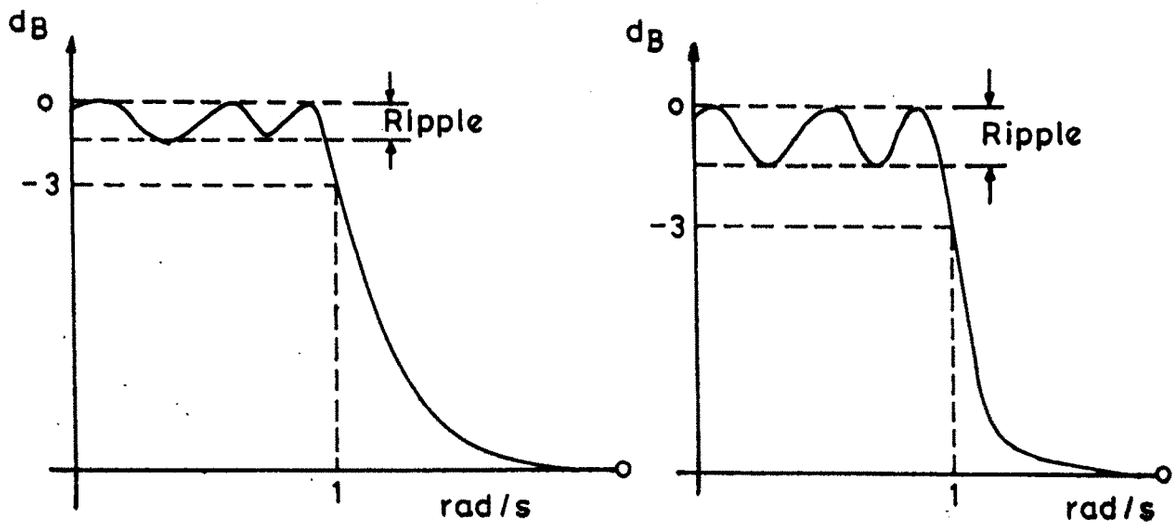

Fig. 2·5 - Normalized Chebyshev low pass filter response.

octaves is inferior . Nevertheless , the superior transient properties make this apprximation extremely useful . A Bessel filter , which has constant delay , passess all frequencies below the cutoff frequency with equal time delay . A constant time delay device is said to be "non dispersive'. An example of a non duspersive circuit is a wide band co-axial cable which has been carefully terminated and passes all signal frequencies "without time delay distortion ".

2.5) ELLIPTIC-FUNCTION [CAUER] APPROXIMATION :-

All of the previous network approximations discussed are all pole types. That is the numerator of the transfer function is unity or a constant multiplier. Elliptic-function filter type cotains zero as well as poles . The introduction of zeros results in finite frequencies of infinite attenuation . These poles and zeros are located so that the pass band exhibits equal ripples similar to the Chebyshev approximation and the stop band has equal return lobes .

The introduction of transmission zeros results in the steepest possible transition from the pass band to the stop band for a given number of poles . This is achived by choosing $k_n^2(\omega)$in equetion 2.1 as

$$k_n(\omega)= R_n(\omega) \qquad \ldots\ldots\ldots\ldots \qquad (2.14)$$

Where $R_n(\omega)$ is the chebyshev rational function .

This function introduced in network theory by Cauer is given by

$$R_n(\omega)= \frac{\omega(\omega_1^2-\omega^2)\ (\omega_2^2-\omega^2)\ldots\ldots\ldots(\omega_k^2-\omega^2)}{(1-\omega_1^2\omega^2)\ (1-\omega_2^2\omega^2)\ldots\ldots(1-\omega_k^2\omega^2)} \quad n \text{ odd} \quad \ldots\ldots (2.15)$$

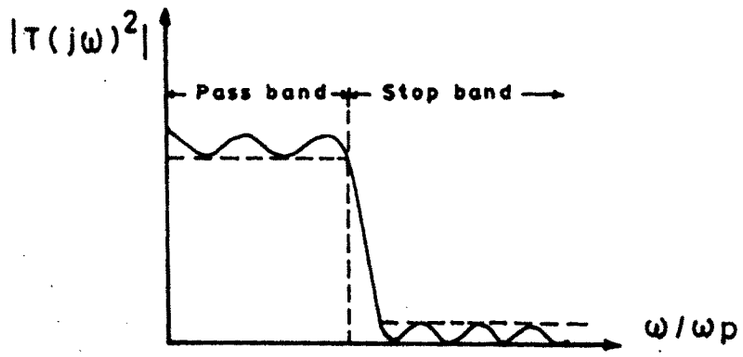

Fig. 2·6 – Elliptic filter response.

$$= \frac{(\omega_1^2-\omega^2)\ (\omega_2^2-\omega^2)\ldots\ldots(\omega_k^2-\omega^2)}{(1-\omega_1^2\omega^2)\ (1-\omega_2^2\omega^2)\ldots\ldots(1-\omega_k^2\omega^2)} \quad n \text{ even} \qquad \ldots (2.16)$$

It can be observed from equation (2.15) or (2.16) that

$$R_n\left(\frac{1}{\omega}\right) = \frac{1}{R_n(\omega)}$$

and that its poles and zeros are reciprocal of each other.

This ensures that the value of $R_n(\omega)$at a frequency ω_i in $0<\omega<1$ is the reciprocal of its value at $1/\omega_i$,$1<\omega<\alpha$.The ω_k^2 are related to Jacobi Elliptic functions and hence the filter is said "Elliptic filter".[As the function $R_n(\omega)$ is introduced by Cauer this kind of filter is also known as Cauer filter .]

Fig 2.5 shows the response of such a filter

2.6) <u>COMPARISION</u> :-

Comparision of three most commonly used practical filters that approximates the ideal response .

Butterworth	Chebyshev	Cauer
1) Flat pass band & flat stop band (so called flat-flat filter.)	Ripple pass band & flat stop band.	Ripple pass band & Ripple stop band.
2) Lead to all pole transefer function. i.e. $T(s)=\frac{1}{s+a_{n-1}s^{n-1}+..+a_1s+a_o}$	All pole type	Contains poles as well as zeros.

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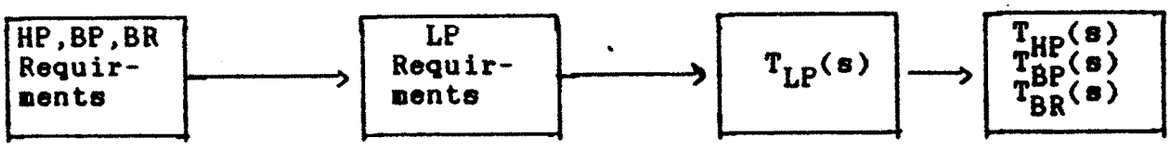
3) Order (n) to satisfy a set of specification is more.	It is less compared with Butterworth.	It requires lower order than Butterworth & Chebyshev. Hence this leads to least expensive filter realization.
4) The transition band is broder than Chebyshev for a given order of filter transfer function.	Transition band is narrower than Butt.for a given order of the filter transfer function.	Transition band is narrowest of all.
5) Phase response is fairly linear than others	Phase response is less linear than Butterworth as order of filter increases it becomes more non-linear.	
6) The roots (poles) of the transfer function denominator all fall on a circle having a radius of unity when plotted in the $j\omega$ plane.	The (roots) poles are located on an ellipse.	

2.7) FREQUENCY-TRANSFORMATION :-

The approximation described so far are directly applicable to low-pass filters. These approximations can be adopted to HPFs,symmetrical BPFs, and symmetrical band rejection filter by using "frequency transformation".



[fig(2.6)Block diagram of the freq. transformation procedure]

This block diagram shows the steps in the approximation of these filters.

The first step is to translate the given HP,BP or BR requirement to related low-pass requirement by using a frequency transformation function . The resulting LP requirement is then approximated using the methods described so far. Finally ,the LP approximation function is transformed to the desired HP, BP, or BR approximation function.

Once the low-pass prototype filter has been designed,the other filters can be obtained by a simple frequency transformation.The frequency variable of the low pass filter is transformed by a" reactance transformation"to obtain high-pass band-pass and band elimination characteristics.In the reactance transformation the low-pass frequency variable Ω is replaced by a realizable, rational function $X(\omega)$ of another frequency variable ω.By proper choice of this function we can transform the low-pass characteristic to other characteristics.

The transformation $X(\omega)$ is selected as a reactance function.

Let $\Omega = X(\omega)$

be the required transformation where.

$$X(\omega) = A \frac{\omega\ (\omega_2^2-\omega^2)\ (\omega_4^2-\omega^2)}{(\omega_1^2-\omega^2)\ (\omega_3^2-\omega)\cdots\cdots}$$

and $\emptyset < \omega_1 < \omega_2 < \omega_3 < \cdots\cdots\cdots$.

The elements of the transformed filter can be obtained by applying the transformation to the reactive elements of the prototype. An inductor of the prototype "L_p" is transformed by the rule

$$\Omega\ L_p \longrightarrow L_p X(\omega)$$

and capacitance C_p by the rule

$$\frac{1}{\Omega\ C_p} \longrightarrow \frac{1}{C_p X(\omega)}$$

By selecting the poles and zeros of $X(\omega)$ properly we can transform a LPF to other types of filters.

For example, the transformation.

$$\Omega = A\ \omega$$

Transforms a low-pass filter to another low-pass filter,whose frequency scale is compressed or elongated depending on A.(this is equivalent to frequency scaling).

2.7.1) <u>LOW-PASS TO HIGH-PASS</u> :-

To transform a low-pass to a high-pass filter, select $X(\omega)$ as $-A/\omega$, i.e. the transformation is

$$\Omega = -\ A/\omega \qquad (2.17)$$

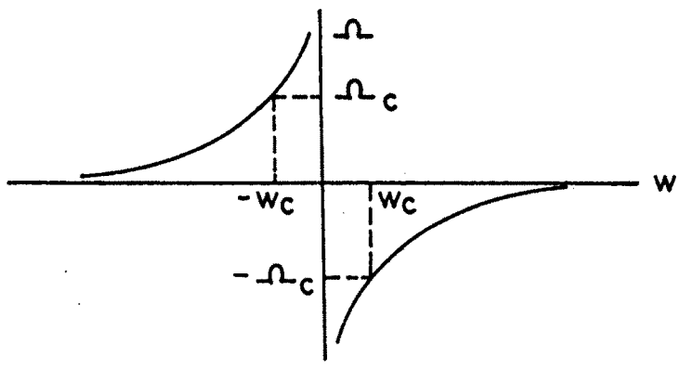


Fig. 2·[illegible] — Low-pass to high pass.

This transformation is represented in fig.(2.7).From the figure it can be observed that the pass band $(-\Omega_c, \Omega_c)$ of the low pass is transformed to two pass bands (ω_c, α) and $(-\omega_c, -\alpha)$. Thus a lowpass is transformed to a high pass.

The transformation of equation (2.17) is a simple frequency inversion. The inductance (capacitance) of the low pass is transformed to a capacitance (inductance) in the HP.

$$j\,\Omega\,L_p \longrightarrow jL_p\left(-\frac{A}{\omega}\right) = \frac{1}{j\,C_\omega}$$

$$\frac{1}{j\,\Omega\,C_p} \longrightarrow \frac{1}{jC_p\,(-A/\omega)} = j\,L_\omega$$

Where

$$C = \frac{1}{A\,L_p}$$

and

$$L = \frac{1}{A\,C_p}$$

are the capacitance and inductance of the HP.

Figure (2.8) shows a LP prototype and the transformed HP network.

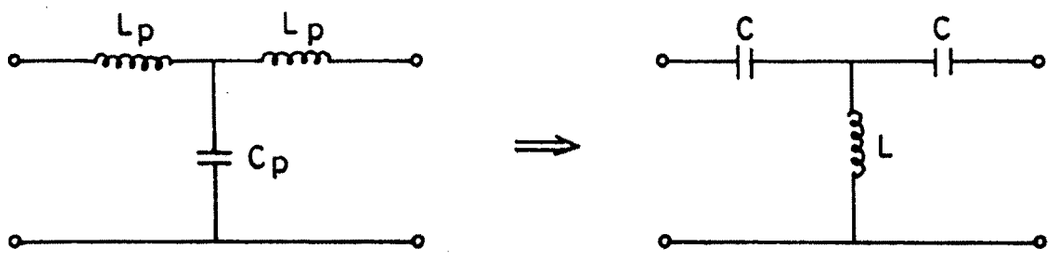


Fig. 2·8 – Shows Lp prototype and the transformed HP network.