# CHAPTER - I I

## THEORY OF FILTER

# CHAPTER II

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## 2.1 BASIC THEORY OF FILTERS

In practice, it is observed that a single section whether a passive type or active type will not provide the filter action. The actual response is far from the ideal one. The drawback is mainly the roll-off in the stopband. To achieve the performance of the ideal filter, large number of sections are to be connected in series i:e. order of the filter must be increased. In simple terms, the order of the filter is generally the number of RC combinations used in the circuit. It also represents the number of energy storing elements (capacitor or inductor) used in the circuit.

In the Laplace transform terminology, the order is related to the "number of poles". The actual mathematical significance of "order" is related to the transfer function of the filter. For a given filter type, the performance of the filter becomes closer to the ideal characteristic as the number of poles (i.e. the order) increases.

In the stop band the gain roll-off is determined by the order (n=1,2,3 ... etc.) of the filter. Each unit increase in order, increases the roll-off by 20 dB/decade. This is shown in Fig.(2.1) for a LP configuration with order

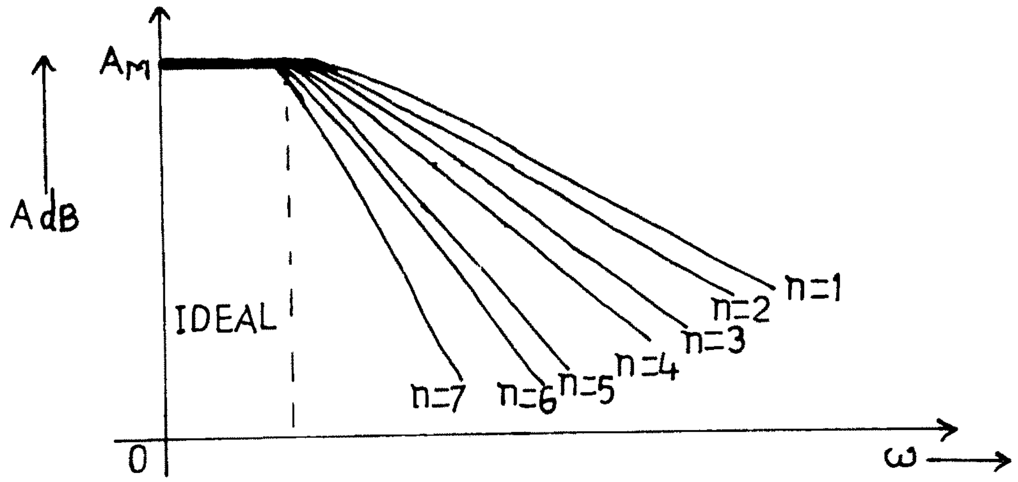



FIG. (2·1): GAIN ROLL-OFF FOR LOW PASS FILTER.

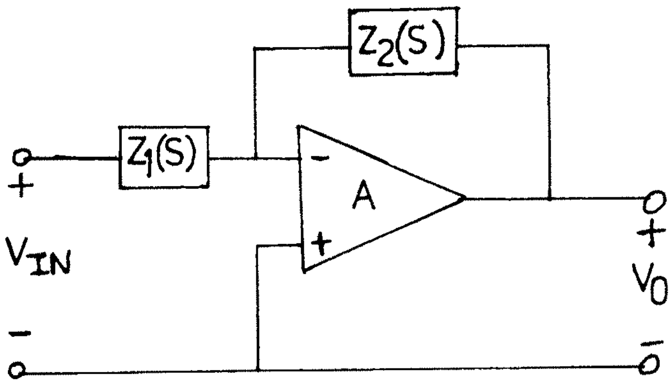



FIG. (2·2): FIRST ORDER LOW PASS FILTER.

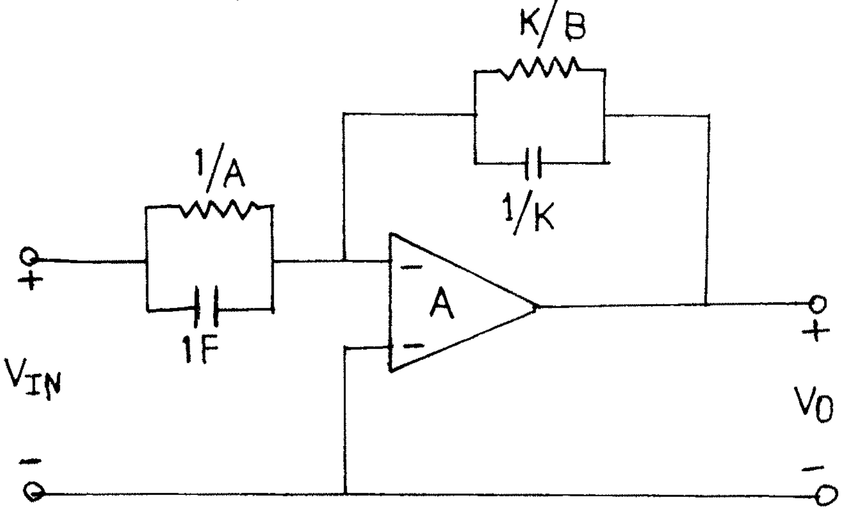



FIG. (2·3): REALIZATION OF FIRST ORDER LOW PASS FILTER.

of six or more. The larger the order, the nearer is response approaches more closely towards ideal response. However, each 20 dB increment in the roll-off is achieved by a phase angle change of -45° at $w_o$.

Generally higher order filters are realized by the "direct method" where a single circuit is connected in series to realize the entire transfer function. The technique is known as "cascade method". The transfer function to be realized, is first factored into a product of first-order and/or second order terms. Each term is then individually realized by an active RC circuit. The cascade connection of individual circuit realizes the overall transfer function.

### 2.1.1 THE FIRST-ORDER FILTER

The first order low pass filters are often used to perform a running average of a signal having high frequency fluctuations superimposed upon a relatively slow mean variation. For this purpose, it is necessary to make the filter time constant RC much greater than the period of the high frequency fluctuations.

The first-order low pass filter is, in inverting amplifier structure drawn in Fig.(2.2). The transfer function is,

$$T(S) = -K \frac{s + d}{s + b} \quad ....(2.1)$$

where a and b are constant.

The voltage transfer function for the structure shown in Fig.(2.2) can be written as,

$$T(S) = \frac{V_O(s)}{V_{IN}(s)} = -\frac{Z_2(s)}{Z_1(s)} \quad ....(2.2)$$

One has to find the impedances $Z_1$ and $Z_2$ from the right hand side of equation (2.1). The impedances require that all poles and zeros lie on the negative real axis. The impedances $Z_1$ and $Z_2$ are identified as,

$$Z_1(s) = \frac{1}{s + a} \text{ and } Z_2(s) = \frac{K}{s + b} \quad ....(2.3)$$

The resulting circuit realization is shown in Fig.(2.3).

### 2.1.2 THE SECOND ORDER FILTER

The second order filter has a response whose magnitude falls at 40 dB/decade in stop band region. The sharpness of the transition between the pass and stop bands depends upon the choice of filter constants which are fixed by circuit parameters.

A band pass characteristics can be obtained by combination of a high and low pass filters. The Fig.(2.4) shows the second order LP filter realization and its voltage transfer function is,

$$T(s) = \frac{K.\ 1/R_1R_2C_1C_2}{s^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} - K\frac{1}{R_2C_2}\right)s + \frac{1}{R_1R_2C_1C_2}} \quad ....(2.3a)$$

2.1.3 **BIQUAD SECOND ORDER FILTER**

For special types of filtering applications which needs critical phase requirements. A biquadratic network function in which the location of the complex conjugate poles and zeros can be independently specified may be required; such biquadratic transfer function has the form,

$$\frac{V_o(s)}{V_1(s)} = \frac{m_2s^2 + m_1s + m_o}{s^2 + (W_{o/Q})s + w_o^2} \quad ....(2.4)$$

or

$$T(s) = K\frac{s^2 + cs + d}{s^2 + as + b}$$

$$= K\frac{s^2 + (w_Z/Q_Z)s + w_Z^2}{s^2 + (w_p/Q_p)s + w_p^2}$$

The frequency $w_Z$ is known as the zero frequency and $Q_Z$ is the zero 'Q'.

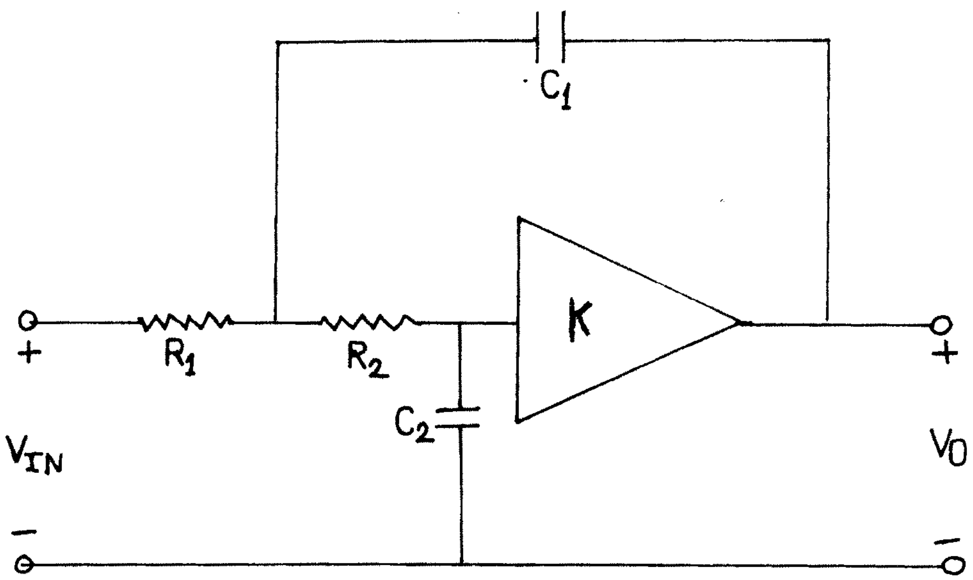



FIG.(2.4): SECOND ORDER LOW PASS FILTER.

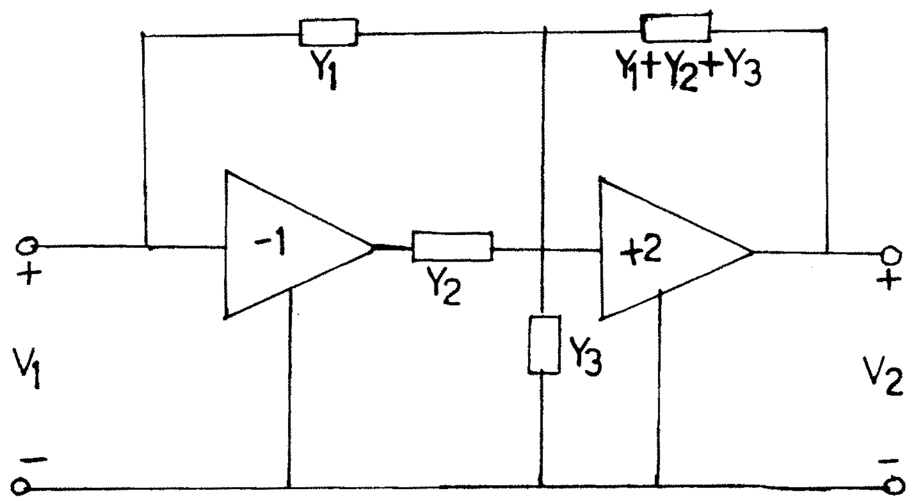



FIG.(2.5): GENERAL BIAQUADRATIC FILTER.

Similarly, the frequency $w_p$ is pole frequency and '$Q_p$' the pole 'Q'. The numerator coefficients determine the filter bandwidth and the denominator coefficients determine the filter response.

The general biquadratic filter is sketched in Fig.(2.5) which requires two VCVSs. The first is an inverting amplifier with gain -1 and the second is non-inverting one with gain of 2. The voltage transfer function for such filter is,

$$\frac{V_2(s)}{V_1(s)} = \frac{2(Y_1 - Y_4)}{Y_3 - Y_4} \qquad \ldots .(2.5)$$

The element values are determined from equations (2.4). e.g. consider the realization of normalized all-pass (constant magnitude) transfer function.

$$\frac{V_2(s)}{V_1(s)} = \frac{s^2 - 3s + 2}{s^2 + 3s + 2} \qquad \ldots .(2.6)$$

It is shown that the biquadratic function can be characterized by its zero and pole frequencies and zero and pole Q's. Depending on these parameters, the biquadratic function can be classified into low-pass, high pass, band reject and all pass functions.

### 2.1.4 HIGHER ORDER FILTERS

The higher order filters have very excellent gain roll-off characteristics which may be obtained by cascading, coupling or direct method. In direct method, a single circuit is used to realize the all transfer function.

#### (A) CASCADE APPROACH

Most of the filtering applications however, require a filter of order higher than second order either to provide greater stop band attenuation and sharper cut-off at the pass band edge in the low pass or high pass case or to provide a broad passband in the band pass filter.

A standard method of realizing higher order transfer function is to express it as product of second order transfer function and realize each second-order transfer function as a single op-Amp or a state variable filter. The final filter is obtained by cascading the individual second order filter blocks. For example, a third order LP filter may be obtained by cascading $1^{st}$ order and $2^{nd}$ order LP filters. A fourth order filter may be obtained by cascading two second order filters. A fourth order band pass filter is obtained by cascading (combination) LP filter and HP filter with cut off frequencies properly adjusted.

The cascade approach as the name implies, consists of realizing each of biquadratics by an appropriate circuit and connecting these circuits in cascade. This is shown in Fig.(2.6). The important advantage is that in cascade approach the realization of a higher-order transfer function is reduced to much simpler realization of second order function. Additional feature is that the change in one biquad doesnot affect the change in any other biquad. Such property is very useful in adjusting the network performance at the time of manufacture.

The general transfer function for cascade approach is[134].

$$T(s) = \sum_{i=1}^{N} T_i(s) \qquad \ldots.(2.6)$$

where $T_i(s)$ is of the form,

$$T_i(s) = K_i \frac{m_i s^2 + C_i s + d_i}{n_i s^2 + a_i s + b_i} \qquad \ldots.(2.7)$$

The cascade topology is shown in Fig.(2.6). The output voltage of the block $T_1$ is,

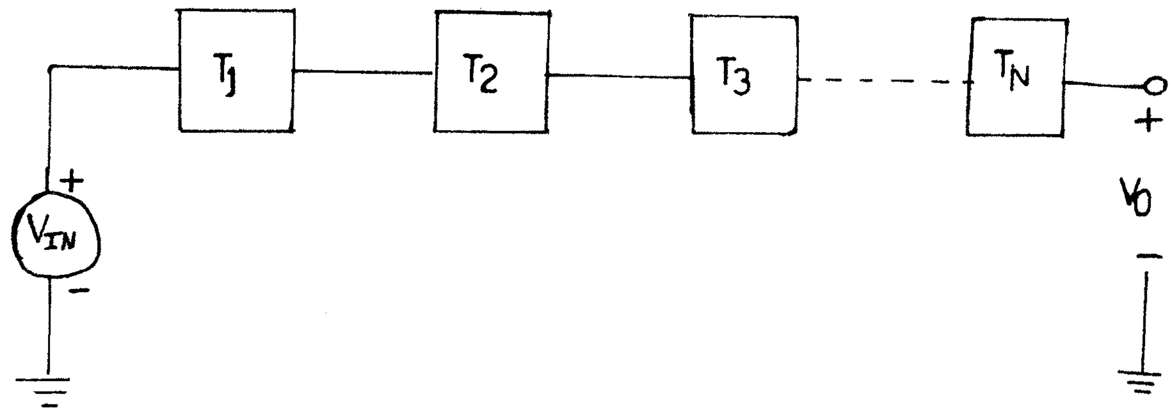
$$V_{0_1} = T_1 V_{IN}$$

The output voltage of the block $T_2$ is,

$$V_{0_2} = T_2 V_{0_1} = T_1 \cdot T_2 \cdot V_{IN}$$

Extending this argument to cascade of 'N' sections.

$$V_o = T_1 \cdot T_2 \cdot T_3 \; \ldots\ldots \; T_N \cdot V_{1N}$$




FIG(2.6): THE CASCADE TOPOLOGY.

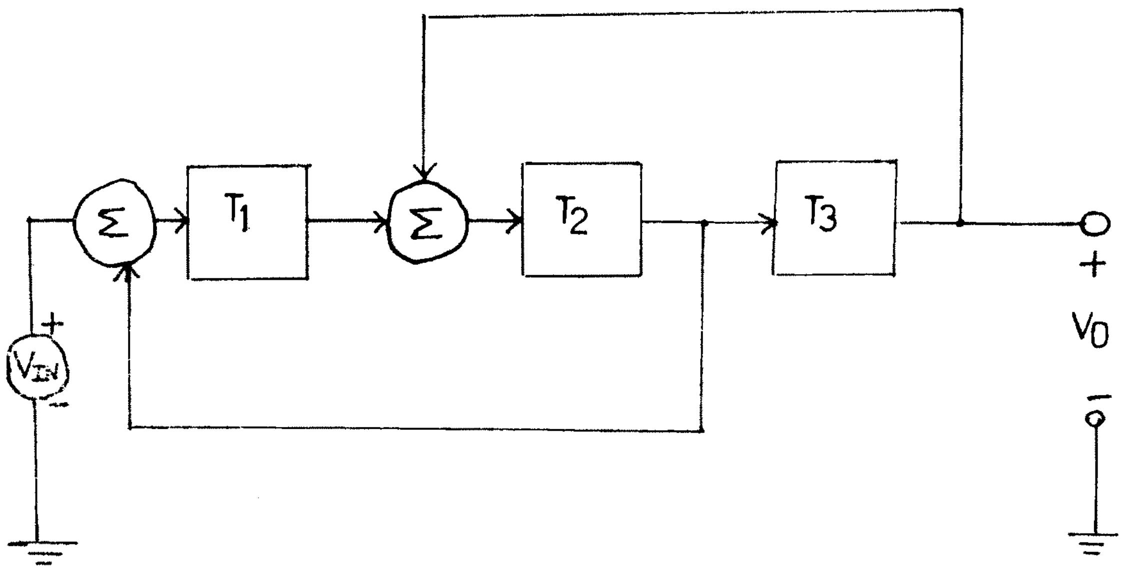



FIG.(2.7): MULTILOOP FEEDBACK COUPLED STRUCTURE.

$$\frac{V_O}{V_{IN}} = T_1 \cdot T_2 \cdot T_3 \; \ldots\ldots \; T_N$$

$$\frac{V_O}{V_{IN}} = \sum_{i=1}^{N} T_i(s) \qquad \ldots.(2.8)$$

Thus the transfer function of a cascade of networks is the product of the individual transfer functions provided that the input impedance of each network is very large as compared with the output impedance of the preceding network.

**(B) COUPLED STRUCTURE**

In coupled structures the individual biquadratic blocks are coupled to each other via feedback path. The coupled structure is shown in Fig.(2.7).

This structure is more complex than the cascade structure. The change in one biquad affects the currents and voltages in all the biquads. Moreover, this lack of isolation between the blocks makes their tunning more difficult. On the other hand, one distinct advantage of using coupled structures is that the sensitivity is usually lower than that off the equivalent cascaded realization. The biquad is the basic block used in both cascaded and coupled realizations. Therefore, biquad circuits are of the fundamental importance in the design of active filters.

## 2.2 SENSITIVITY

Active filters are designed to perform certain specific functions such as wave shaping or signal processing. In the filter design process the choice of the circuit and determination of element values are important. Given perfect components, there would be little difference among the many possible designs. In practice, all components will deviate from their nominal values due to manufacturing tolerance, changes in the environmental conditions such as temperature, humidity and chemical changes due to the aging of the components. Consequently, the performance of the filters differs from the nominal design. This causes the network transfer function to drift away from its nominal value. The relationship between the network element variation and the resulting changes in the network transfer function is known as the "sensitivity". In order to minimize this change or to reduce sensitivity, we have to select the components with low tolerances, low temperature, and humidity coefficients. However, this approach increases the cost of the network. A more practical solution is to design a network. that has a low sensitivity to element changes. The lower the sensitivity of the circuit, the less will be the deviation in performance.

Sensitivity is one of the more important criteria used for comparison of various circuit realizations. This is important in active filter design where the active element such as op-Amp is much more sensitive to environmental changes. A good understanding of sensitivity is essential in the design of the practical active filter.

Defination : The defination of sensitivity was given by Bode H.W.[134].

Sensitivity function is defined as "the ratio of the fractional change in network function to the fractional change in an element for the situation when all changes concerned are differentially small".

The sensitivity is denoted by symbol 'S'. Notice that a superscript character is used to indicate the performance characteristics that changes and subscript character is used to indicate the specific network element that causes the change.

Classically, the sensitivity of a network function F(s) with respect to a parameter 'X' is defined as,

$$S_X^{F(s)} = \frac{\partial \ln F(s)}{\partial \ln X} = \frac{X}{F} \frac{\partial F}{\partial X} \qquad ....(2.9)$$

If the transfer function (open-circuit voltage transfer ratio) of a network N is T(s) = p(s)/q(s) then the sensitivity of T(s) with respect to the parameter X is given as,

$$S_X^{F(s)} = S_X^{P(s)} - S_X^{q(s)}$$

$$S_X^{F(s)} = X \left[ \frac{1}{P} \frac{\partial P}{\partial X} - \frac{1}{q} \frac{\partial q}{\partial X} \right] \qquad \text{....(2.10)}$$

2.2.1 **W AND Q SENSITIVITY**

In qualitative sense, the sensitivity of a network is a measure of the degree of variation of its performance from nominal due to changes in the elements constituting the network. A biquadratic filter function can be expressed in terms of the parameters $w_p$, $w_z$, $Q_p$, $Q_z$ and K as,

$$T(s) = K \frac{s^2 + \dfrac{w_z}{Q_z} s + w_z^2}{s^2 + \dfrac{w_p}{Q_p} s + w_p^2} \qquad \text{....(2.11)}$$

Let us first consider the sensitivity of the pole frequency $w_p$ to a change in a resistor R. Pole sensitivity is defined as the per unit change in the pole frequency $\Delta w_p/w_p$ caused by a per unit change in the resistor, $\Delta R/R$. Mathematically,

$$S^{w_p}_{R} = \lim_{R \to 0} \frac{\frac{\Delta w_p}{w_p}}{\frac{\Delta R}{R}}$$

$$S^{w_p}_{R} = \frac{R}{w_p} \cdot \frac{\partial w_p}{\partial R}$$

This is equivalent to,

$$S^{w_p}_{R} = \frac{\partial(\ln w_p)}{\partial(\ln R)} \qquad \ldots.(2.12)$$

Note that the cost of a manufacturing a component is a function of the percentage change rather than absolute change of the component. Due to this reason it is desirable to measure sensitivity in terms of the relative changes in components. The sensitivities of the parameters $w_z$, $Q_p$, $Q_z$ and K to any element of the network are defined in a similar way.

$$S^{w_p}_{C} = \frac{c}{w_p} \frac{\partial w_p}{\partial c}, \quad S^{Q_p}_{R} = \frac{R}{Q_p} \frac{\partial Q_p}{\partial R}$$
$$S^{k}_{R} = \frac{R}{K} \cdot \frac{\partial K}{\partial R} \qquad \ldots.(2.13)$$

### 2.2.2 MAGNITUDE AND PHASE SENSITIVITY

The computation of the sensitivity functions for the magnitude and phase functions are given below. First convert a transfer function in polar form and substitute 's' by jw to get.

$$H(jw) = |H(jw)|\, e^{J\Phi(w)} \qquad \ldots.(2.14)$$

Then sensitivity function becomes as,

$$S_X^{H(jw)} = \frac{X}{H(jw)} \cdot \frac{\partial}{\partial X}[|H(jw)|\, e^{J\Phi(w)}] \qquad \ldots.(2.15)$$

This can be solved by the use of product rule of differentation of a product. Thus,

$$S_X^{H(jw)} = S_X^{|H(jw)|} + J\Phi(w)\, S_X^{\Phi(w)} \qquad \ldots.(2.16)$$

It shows that the magnitude and phase sensitivity of a transfer function with respect to an element are simply related to the real and imaginary parts of the transfer function sensitivity with respect to the same element.

### 2.2.3 GAIN SENSITIVITY

In the filter design, filter requirements are frequently stated in terms of the maximum allowable deviation in gain over a specified bands of frequencies. In such a situation, it is convenient to consider the logarithm of the transfer function of a network operating under the sinusoidal steady state input. Thus the magnitude of the transfer function in dB can be written as,

$$G(w) = 20 \log |T(jw)| \qquad \ldots.(2.17)$$

where,

$$T(jw) = K \frac{S^2 + \frac{w_{z_i}}{Q_{z_i}} S + w_{z_i}^2}{S^2 + \frac{w_{p_i}}{Q_{p_i}} S + w_{p_i}^2} \qquad \ldots.(2.18)$$

$$G(w) = \sum_{i=1}^{N} 20 \log \left| S^2 + \frac{W_{z_i}}{Qz_i} S + w_{z_i}^2 \right|$$

$$- \sum_{i=1}^{N} 20 \log \left| S^2 + \frac{W_{p_i}}{Qp_i} S + w_{p_i}^2 \right|_{s=jw} + 20 \log|K| \;.(2.19)$$

Gain sensitivity is defined as the gain in dB due to per unit change in an element (or parameter) X

$$S_X^{G(w)} = \frac{\partial G(w)}{\partial X/X}$$

$$= X \frac{\partial G(w)}{\partial X} \text{ dB} \qquad \ldots.(2.20)$$

From this equation,

$$\Delta G(w) = \lim_{\Delta x \to 0} S_X^{G(w)} \frac{\Delta X}{X}$$

for small changes in X,

$$\Delta G(w) \approx \lim_{\Delta x \to 0} S_X^{G(w)} \cdot \frac{\Delta X}{X} \qquad \ldots.(2.21)$$

Gain variation is affected by,

1. The approximation function.
2. The choice of the circuit topology.
3. The types of the component used in realization.

### 2.2.4 ROOT SENSITIVITY

The functions and the filter transfer functions are often specified by their poles and zeros. The location of the poles of an active filter determines the stability of the network. The poles and zeros themselves, being a functions of network parameter get perturbed due to variation in these parameters. The change in pole-zero location changes the frequency response characteristics of a filter and may indicate potential instability of the filter. In order to study this aspect the sensitivity of a root (a pole or a zero) is found to be useful.

Let $S_i$ be a root of either the numerator or denominator. Then the root sensitivity $S_X^{S_i}$ is defined as,

$$S_X^{s_i} = X \left. \frac{ds_i}{dX} \right|_{s=s_i} \qquad \ldots.(2.22)$$

The root sensitivity, in contrast to the transfer function sensitivity, is a complex constant. When $s_i$ is the numerator polynomial of the network function, equation (2.22) defines the "zero sensitivity". Similarly, if $s_i$ is the denominator polynomial, equation (2.22) is referred to as the "pole sensitivity".

### 2.2.5 THE MULTI-PARAMETER SENSITIVITY

We have considered the situation that for the changes in biquadratic parameter or (network parameter) due to change in a particular network element. The change in a resistance causes the pole frequency to change by

$$\Delta w_p = \lim_{\Delta R \to 0} S_R^{w_p} \frac{\Delta R}{R} . w_p \qquad \ldots.(2.23)$$

for small deviations in R.

$$\Delta w_p \approx S_R^{w_p} \frac{\Delta R}{R} . w_p \qquad \ldots.(2.24)$$

This is the change due to one element. In general, we extend this change in network function due to the simultaneous variation of all elements in the network. If change in $w_p$ due to simultaneous variation of all the network elements $x_j$ (where the elements can be resistors, capacitors, inductors or parameters describing the active devices) the change $\Delta w_p$ can be obtained by expanding it in a Taylor series as,

$$\Delta w_p = \frac{\partial w_p}{\partial X_1} \Delta x_1 + \frac{\partial w_p}{\partial X_2} \Delta x_2 + \dots + +$$

$$\dots \frac{\partial w_p}{\partial X_m} . \Delta x_m + \text{second and higher order terms} \quad .(2.25)$$

where 'm' is total number of elements in the circuit. If the changes in the components $\Delta x_j$ are assumed to be small then the second and higher-order terms can be ignored. Thus,

$$\Delta w_p \approx \sum_{j=1}^{m} \frac{\partial w_p}{\partial X_j} . \Delta X_j \quad \dots .(2.26)$$

To bring the sensitivity term into evidence (2.26) equation can be written as,

$$\Delta w_p \approx \sum_{j=1}^{m} \left(\frac{\partial w_p}{\partial X_j} . \frac{X_j}{w_p}\right) \left(\frac{\Delta X_j}{X_j}\right) w_p$$

$$\Delta w_p \approx \sum_{j=1}^{m} S_{x_j}^{w_p} . V_{x_j} w_p \quad \dots .(2.27)$$

where $V_{X_j} = \Delta X_j / X_j$ is the per unit change in the element of X. From equation (2.27) the per unit change in $w_p$ is,

$$\frac{\Delta w_p}{w_p} = \sum_{j=1}^{m} S_{x_j}^{w_p} . V_{x_j} \quad \dots .(2.28)$$

In similar manner we will obtain the per unit changes in pole Q, $w_z$ $Q_z$ and K due to simultaneous variations of all network elements. The multi-parameter sensitivity is complex and doesn't take into account the random element variations. A more precise and accurate measurement is known as "statistical multi-parameter sensitivity".

Until now, we have defined several sensitivity functions and they are related to each other. In final analysis, we have to minimize the deviation of the filter response due to incremental variation in network parameters. In high 'Q' networks (highly selective networks) the pole sensitivity is important factor for the stability of the network. In the design of the active filters, the sensitivity of the filter transfer function due to variation in active parameters is also an important considerations.

## 2.3 VARIOUS APPROXIMATIONS IN FILTER CIRCUITS

The approximation problem consists of finding a function whose loss characteristic lies within the permitted region. The function which is to be chosen must be realizable using passive and/or active components. In order to minimize the components required for design of a filter, the order of the function must be as small as possible. The modern filter design is based on the selection of the transfer function to satisfy the given filter specification and then realization of this function by synthesis techniques. The steps involved in modern filter design can be described as below [9].

1. Selection of filter specification.
2. Selection of a realizable rational function which satisfies this specification.
3. Realization of transfer function and calculation of component values of the chosen filter structure.
4. Construction and testing of the filter.

Filter performance is prescribed graphically in the frequency or time domain. Since electrical network reactances have continuous frequency characteristics (except at resonance), the abrupt cut-off inherent in the ideal response of Fig.(2.8) can't be actually attained by any finite connection of elements. This gives rise to the so called "approximation problem" i:e. the determination of the system functions which approximate the gain curve within specified tolerances and which are at the same time realizable as a physical networks[131].

For LP filter, the filter network is required to pass all frequencies below '$w_p$' with least attenuation and frequencies above $w_p$ with infinite attenuation. The phase of the filter network is to be a linear function of w. These

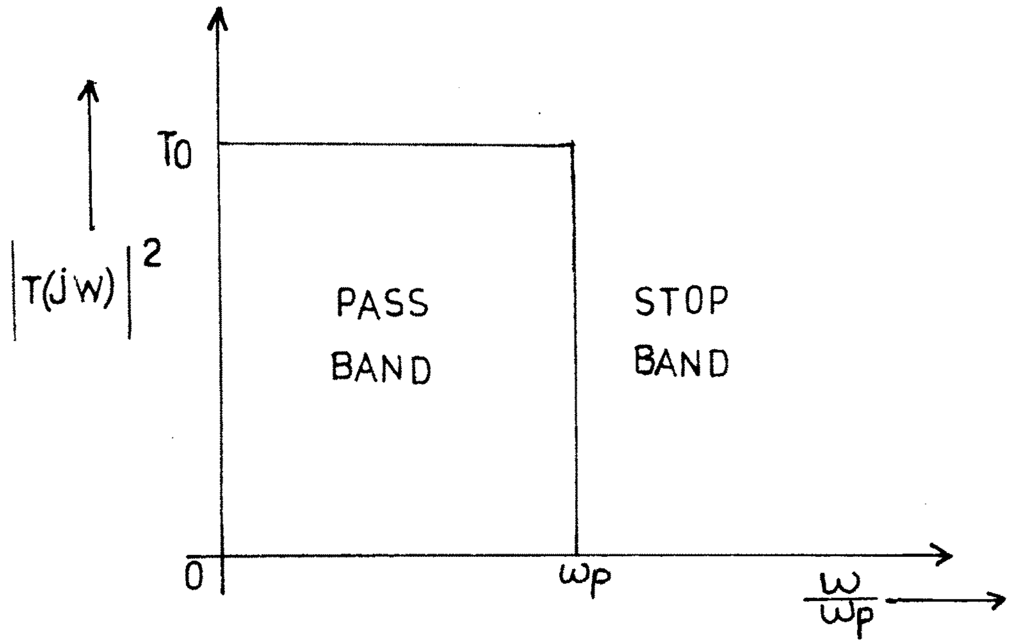



FIG.(2·8): AN IDEAL LOW PASS FILTER.

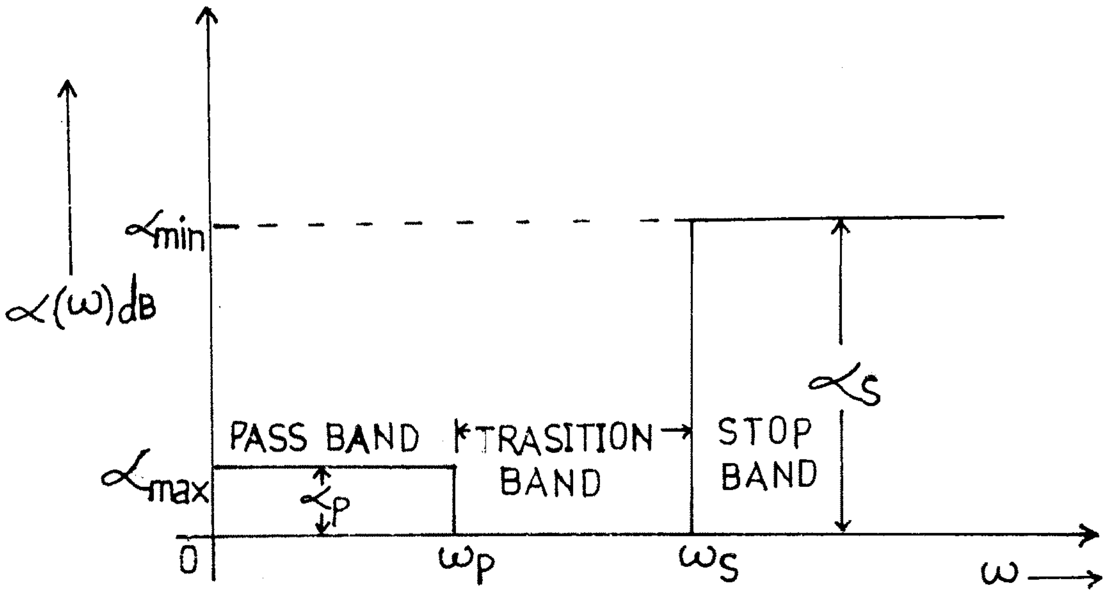



FIG.(2·9): THE SPECIFICATION FOR A REALIZABLE LOW PASS FILTER.

specifications can not be achieved by a physical network. If we consider the magnitude as a fourier transform the corresponding time function is of the form $\sin\alpha t/\alpha$ and exists for all 't'. This makes the impulse response non-causal and hence non realizable. In order to overcome this problem, the filter specifications are modified as shown in Fig.(2.9) for LP filter. When we specify $\alpha_p$ and $\alpha_s$ we should interpret them as, the maximum permissible loss or attenuation over a given frequency band of interest called 'passband' and minimum allowable loss over a another frequency band called 'stopband'. The selectivity or tolerable interval between these two bands is called "the transitional band". The approximate method solves the problem of selecting a realizable rational function whose frequency response approximates the response shown in Fig.(2.9).

A method of approximation based on Bode plots[134] is suitable for low order simple filter designs. More complex filter characteristics are approximated by using some well described rational functions whose roots have been tabulated. The most popular approximations are the Butterworth, Chebyshev, Bessel and elliptic types. These approximations are directly applicable to low pass filters. However, they can be used to design high pass filters and symmetrical band pass and band reject filters by using the frequency transformation technique.

### 2.3.1 BUTTERWORTH APPROXIMATION

Butterworth approximation[9] is a special form of Taylor series approximation in which the approximating function t(w) and the specified function f(w) are identical at w = 0. For this approximation $K_n(w)$ is selected as,

$$K_n(w) = \beta_o + \beta_1 w + \beta_2 w^2 + \ldots + \beta_n w^n \qquad \ldots .(2.29)$$

For Taylor series approximation the function $K_n(w)$ must be maximally flat at the origin (i.e. w=0). Hence, as many derivatives of $K_n(w)$ as possible must vanish at w=0. Hence for Butterworth approximation.

$$K_n(w) = w^n \qquad \ldots .(2.30)$$

The magnitude function $|T(jw)|^2$ and attenuation function $\alpha(w)$ are given as,

$$|T(jw)|^2 = \frac{1}{1 + c^2 w^{2n}} \qquad \ldots .(2.31)$$

where c is attenuation constant. The w should be interpreted as the frequency normalized with respect to the pass band edge $w_p$.

$$\therefore \; |T(jw)|^2 = \frac{1}{1 + c^2 \left(\frac{w}{w_p}\right)^{2n}} \qquad \ldots .(2.32)$$

And $$\alpha(w) = 10 \log \left[ 1 + c^2 \left(\frac{w}{w_p}\right)^{2n} \right] \qquad \ldots.(2.33)$$

The frequency response of a Butterworth filter for various values of 'n' is shown in Fig.(2.10). From Fig., it is seen that all curves pass through the same point at $w=w_p$ and this point is determined by $\alpha_p$. Then from equation (2.33) we get,

$$\left.\begin{aligned} \alpha_p &= 10 \log \left[ 1 + c^2 \left(\frac{w_p}{w_p}\right)^{2n} \right] \\ \alpha_s &= 10 \log \left[ 1 + c^2 \left(\frac{w_s}{w_p}\right)^{2n} \right] \end{aligned}\right\} \qquad \ldots.(2.34)$$

The factor $(w_p/w_s)$ is called selectivity parameter represented by,

$$K = w_p/w_s \qquad \ldots..(2.35)$$

Solving above equations, we get,

$$K_1 = \left[ \frac{10^{0.1\alpha_p} - 1}{10^{0.1\alpha_s} - 1} \right] \qquad \ldots.(2.36)$$

The discriminator parameter is given as,

$$n = \frac{\log K_1}{\log K}$$

The order of the filter 'n' should be selected such that,

$$n \geq \frac{\log K_1}{\log K}, \text{ n-an integer} \qquad \ldots.(2.37)$$

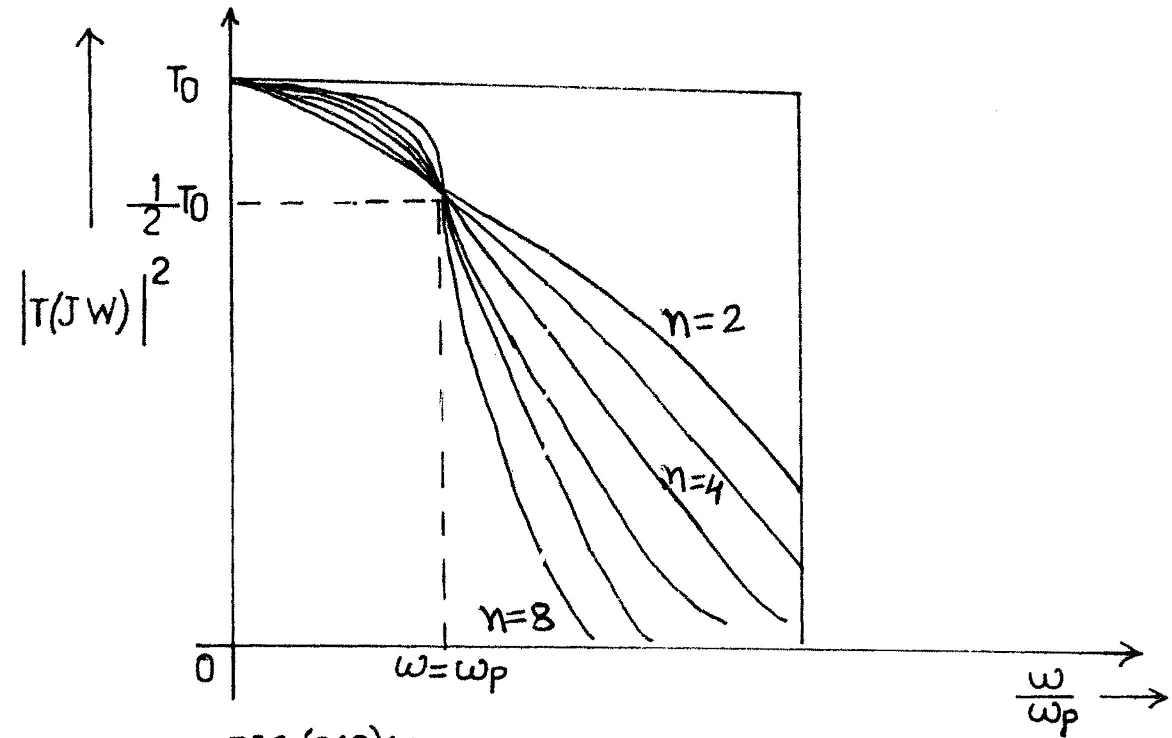



FIG.(2.10): BUTTERWORTH FILTER FOR VARIOUS VALUES OF 'n'

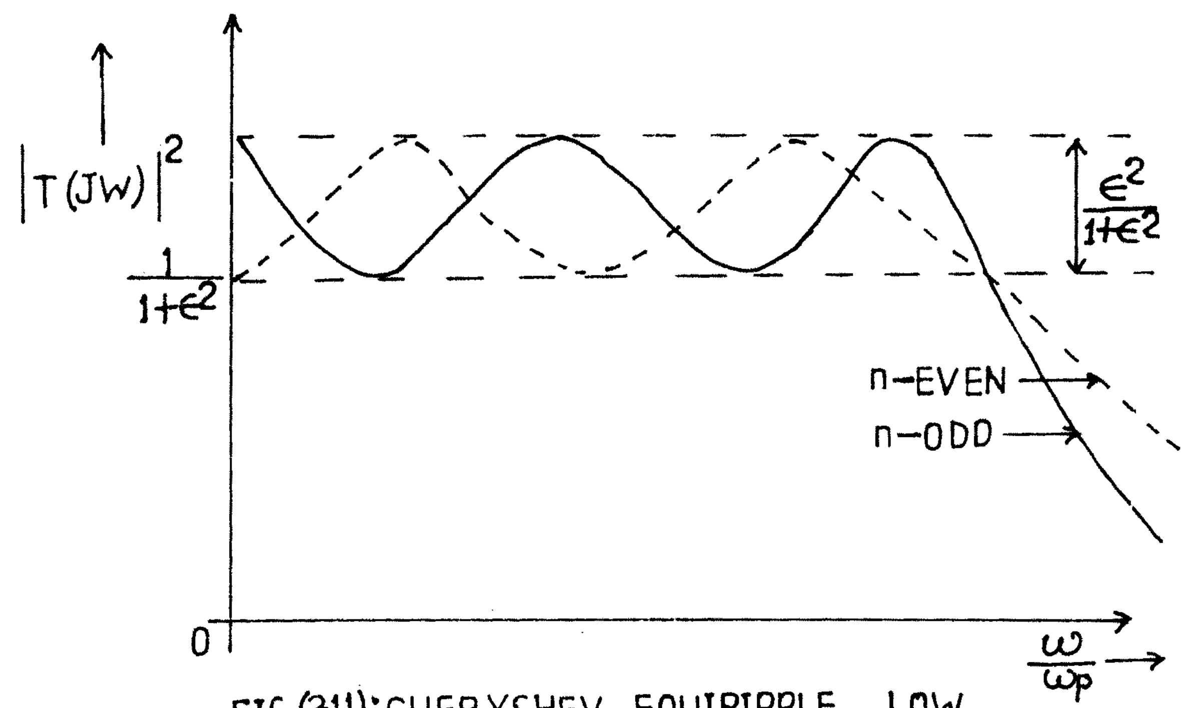



FIG.(2.11): CHEBYSHEV EQUIRIPPLE LOW PASS FILTER.

If $n = \dfrac{\log K_1}{\log K}$, then the value of 'c' obtained from equation (2.34) is same If,

$n \neq \dfrac{\log K_1}{\log K}$, then the value of 'c' can be selected to satisfy either pass band edge or stop band edge exactly.

So the $n^{th}$ order Butterworth polynomial and approximation K(s) satisfies the following conditions[134].

1. $K_n(s)$ is an $n^{th}$ order polynomial.
2. $K_n(o) = 0$.
3. $k_n(s) = 1$ is maximally flat at the origin.
4. $k_n(1) = 1$.
5. As n (order of filter) is increased, the pass band is flat over a wider interval.
6. As n is increased the stop band loss is increased.

**2.3.2 CHEBYSHEV APPROXIMATION**

The Butterworth approximation concentrates on the polynomial at w = 0 instead of distributing it over a range $0 < w < 1$. This yields maximally flat low pass filters. This network has a disadvantage that it requires a very high degree polynomial for a sharp transition region.

A good result in this regard may be obtained, if we look for a rational function that approximates the constant value unity throughout this range in an oscillatory manner, rather than in a monotonic manner. Chebyshev approximation does exactly this.

Chebyshev approximation can be defined as follows: "A function c(w) is a Chebyshev approximation of F(w), if the available parameters are adjusted such that the magnitude of the largest error is minimized". Since the Chebyshev approximation minimizes the maximum error, it is often called as "min-max" approximation. The stop band attenuation is increased by changing the approximation conditions in the pass band. The criterion used is to minimize the maximum deviation from the ideal flat characteristic. We should get the equiripple characteristic shown in Fig.(2.11).

$$|H(jw)|^2 = \frac{H_o}{1 + \varepsilon^2 (C_n^2 (w/w_c)} \qquad \ldots.(2.38)$$

where, $C_n(w)$ is the $n^{th}$ order Chebyshev polynomial of the first kind and $\varepsilon^2 < 1$ and $H_o$ is constant. The response of equation (2.38) is called "$n^{th}$ order Chebyshev or equiripple response".

In other way, the Chebyshev polynomials are defined as linearity independent solutions of the differential equation.

$$(1-w^2)\ddot{y} - w\dot{y} - n^2 y = 0 \qquad \ldots.(2.39)$$

one of the solution is a $n^{th}$ order Chebyshev polynomial is,

$$y = T_n(w) = \cos(n\cos^{-1}w) \quad |w| \leqslant 1 \qquad \ldots.(2.40)$$

$$y = \cosh(n\cosh^{-1}w) \quad |w| \geqslant 1 \qquad \ldots.(2.41)$$

In fact, these two expressions are completely equivalent each being valid for all 'w'.

The properties of Chebyshev polynomial are given below(134).

1. The zeros of the polynomials are all located in the interval $|w| \leqslant 1$.

2. The Chebyshev polynomials posses special values at w=0, 1 or -1.

$$T_n(0) = (-1)^{n/2}, \text{ for n even} \qquad \ldots(2.42a)$$
$$= 0 \quad , \text{ for n odd}$$

$$Tn(\pm 1) = 1 \quad , \text{ for n even} \qquad \ldots(2.42b)$$
$$= \pm 1 \quad , \text{ for n odd}$$

3. $T_n(w)$ is either even or odd functions depending on whether n is even or odd. Thus, we can write,

$$T_n(-w) = T_n(w), \text{ n even} \qquad \ldots(2.42c)$$

$$T_n(-w) = -T_n(w), \text{ n odd} \qquad \ldots.(2.43)$$

4. Every coefficient of $T_n(w)$ is an integer and the one associated with $w_n$ is $2^{n-1}$. Thus in the limit w approaches infinity,

$$T_n(w) \longrightarrow 2^{n-1} . w^n \qquad \ldots.(2.44)$$

5. In the range $-1 < w < 1$, all of the Chebyshev

polynomials have the equal ripple property vary between a maximum 1 and minimum -1 and outside of this interval. Their magnitude increases monotonically as w is increased and approaches infinity in accordance with equation (2.44).

There is practical problem that arizes when one attempts to realize an even-order Chebyshev low-pass filter with a passive network. The even order Chebyshev low pass filters have a zero-frequency loss which is equal to the pass band ripple, maximum gain. However, this implies that the source resistance can't be equal to the load impedance. One restriction around this is to use a frequency transformation which changes the loss at d.c. It is also found that the active and digital filters can easily realize low pass characteristics that have non-zero loss at d.c.

### 2.3.3 ELLIPTIC APPROXIMATION

The Chebyshev approximation has an equiripple pass band. It yields a greater stop band loss than the maximally flat Butterworth approximation. In both approximations the stop band loss keeps increasing at the maximum possible rate of 6ndB/octave for an $n^{th}$ order function. Therefore, these approximations provide increasingly more loss than the flat $A_{min}$ needed above the edge of the stop band.

If we have to improve the performance of a filter which is derived from the Chebyshev filter, we have to allow equiripple response in both pass and stop band. This leads to a narrower transition band. These filters are designed by using Elliptic function and referred to as "elliptical filters". The approximation is called "Elliptical Approximation". They are also known as "Cauer or Zolotarev" approximation. These filters are also called "Darlington filters" as S. Darlington did much original work. The typical elliptic approximation function is drawn in Fig.(2.12).

The distinguishing feature of elliptic approximation is that it has poles of attenuation in the stop-band. Thus elliptic approximation is rational function with finite poles and zeros. The Butterworth and Chebyshev are polynomials and such have all their loss poles at infinity. In particular, in elliptic approximation the location of the poles must be chosen to provide the equiripple stop band characteristics shown in Fig.(2.12). The pole closest to the stop band edge ($w_{p_1}$) significantly increases the slope in the transition band. The further poles ($w_{p_2}$ and infinity) are needed to maintain the required level of stop band attenuation. By using the finite poles, the elliptic approximation is able to provide a considerably higher flat level of stop band loss than the Butterworth and Chebyshev

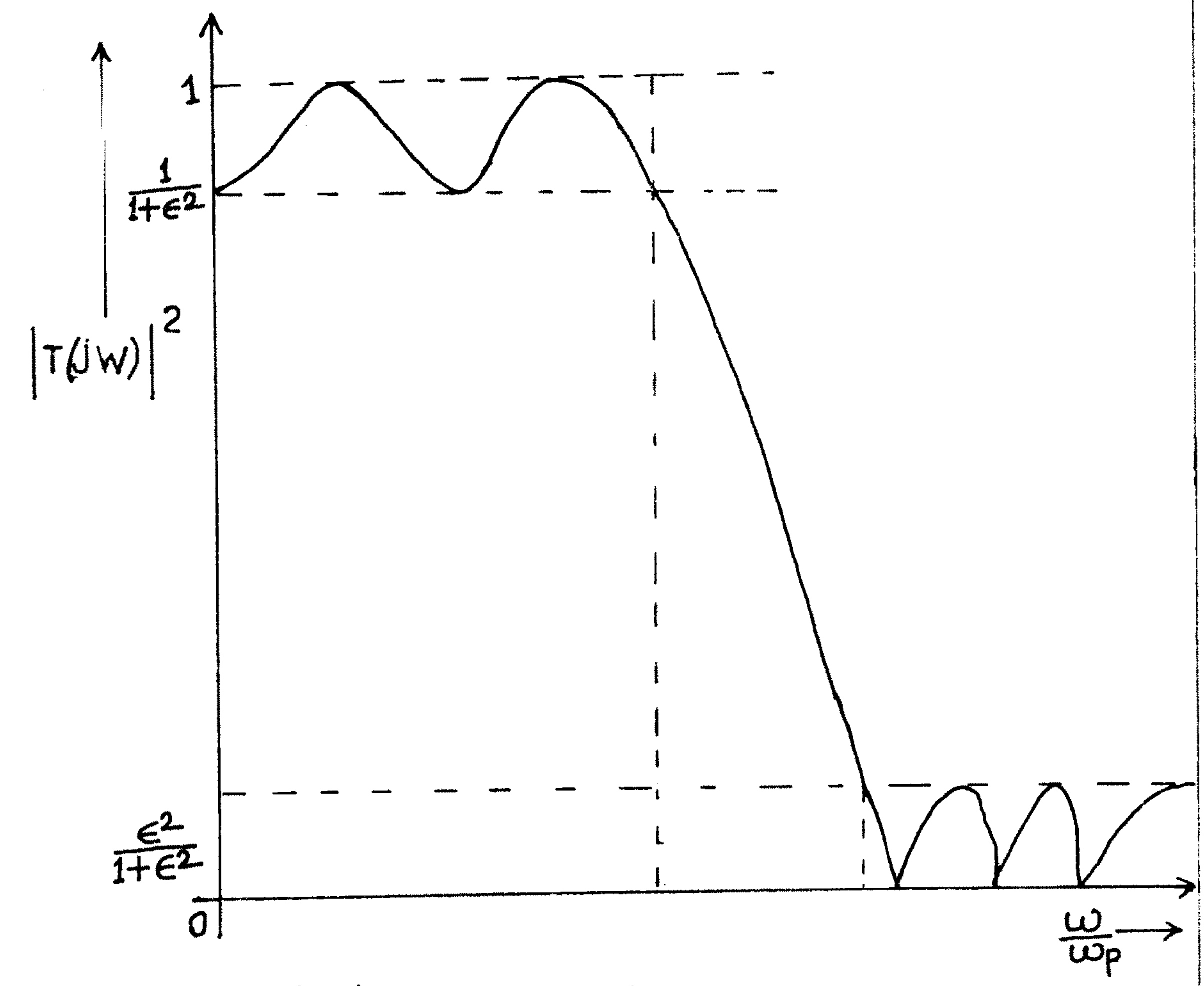



FIG.(2·12): ELLIPTICAL LOW PASS FILTER.

approximations. Thus for a given requirement the elliptic approximation will, in general, require a lower order than the Butterworth or the Chebyshev[134]. Since lower order corresponds to less number of components in the filter circuit, the elliptic approximation will lead to the least expensive filter realization.

### 2.3.4 BESSEL'S APPROXIMATION

Untill now, we have discussed the all approximation techniques. But we have concentrated on approximating the magnitude of the transfer function. In all signal processing requirements linearity of the phase or constant phase delay is an important factor. The phase distortion is more in the Chebyshev filter than in the Butterworth filter. It can be seen that a sudden change in the amplitude is accompanied by a similar change in the phase. An equiripple filter has greater amount of phase distortion than the maximally flat filter. Besssel's approximation deals with phase and delay characteristics.

A constant delay response can be approximated by a maximally flat delay at w=0 by using Bessel polynomials. The coefficients of the polynomials used in the transfer function H(s) are closely related to Bessel polynomials and at the beginning Thomson has used these polynomials in the approximation for this response.

The loss function for the ideal delay characteristics is given by[134].

$$H(s) = e^{ST_o} \qquad \ldots.(2.45)$$

even lower than that for the Butterworth.

For Bessel approximation, the stopband has poor response so that Bessel approximation is not useful for practical implementation for most filtering applications. Therefore the alternative solution to this problem is attaining a flat delay characteristics by the use of delay equalizers. The Bessel approximation is a polynomial that approximates this ideal characteristic. In this approximation the delay at the origin is maximally flat i:e. as many derivatives as possible are zero at the origin. It is convenient to consider the approximation of the normalized function, with the dc delay $T_o = 1$, that is,

$$H(s) = e^{S} \qquad \ldots.(2.46)$$

The Bessel approximation to this normalized function is,

$$H(s) = \frac{B_n(s)}{B_n(o)} \qquad \ldots.(2.47)$$

where $B_n(s)$ is the $n^{th}$ order Bessels polynomial which is defined by following equation.

$$B_o(s) = 1$$

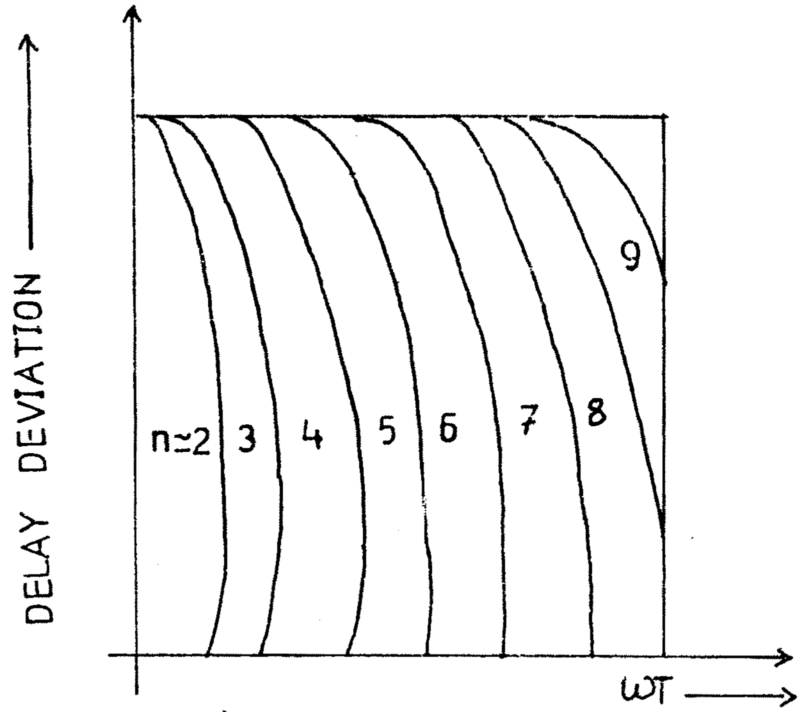



FIG.(2.13): DEALY ERROR OF BESSEL APPROXIMATION.

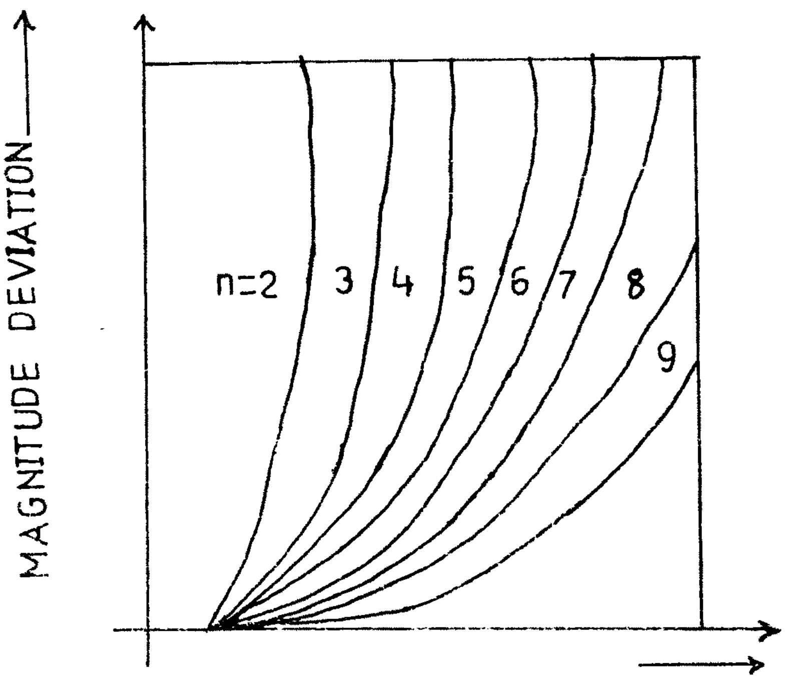



FIG.(2.14): MAGNITUDE ERROR OF BESSEL APPROXIMATION.

$$\beta_n(s) = s+1$$

and $\beta_1(s) = (2n-1)\beta_{n-2}(s) + s^2\beta_{n-2}(s)$ ....(2.48)

The polynomial β(s) is called Bessel's polynomial of order 'n'. The delay and magnitude of Bessel approximation are shown in Fig.(2.13) and Fig.(2.14) respectively.

The delay characteristic of the Bessel approximation is superior to the Butterworth and Chebyshev. It concentrates on the requirements of flatness of the time delay. Secondly, the step response is also superior having no peaking. However, the flat delay is achieved at the stopband attenuation.

## 2.4 STATE VARIABLE ANALYSIS

A network may be classified into three basic categories[9].

1) Linear system : A system containing linear components e.g. resistor, capacitor and inductor etc.
2) Non-linear system : A system containing non linear elements e.g. diode, transistors, FET, Tunnel diode etc.
3) Time varying system : The value of element changes with time e.g. capacitor microphone, mass of rocket.

For the analysis of the system, there are two usual circuit analysis methods.

(a) Mesh analysis KVL

(b) Nodal analysis KCL

These analysis methods are not convenient for the higher order systems, non-linear systems and time varying systems. So all these difficulties are overcome by "state variable analysis technique".

A "state variable" is the term used to define the effect of an energy storing element in a physical system. There is one state variable for each energy storage element represented by the capacitor voltages and inductor currents[9].

$$E_C = \frac{1}{2} cV^2 \quad \text{and} \quad E_L = \frac{1}{2} LI^2$$

These currents and voltages inform us about energy stored in system. Response to a given input depends on the zero-input response. The zero input response in an RLC circuit (or network) is completely determined once the initial inductor currents and capacitor voltages are known. Hence, we call the initial capacitor voltages and inductor currents (initial conditions) as the initial states of the system. The knowledge of capacitor voltages and inductor currents at a given time is sufficient to calculate any of the network variables (current and voltages) at that particular time. Hence, we call the capacitor voltages and the inductor currents at a specified time, as the "state variables" of the network.

The state of a network as a set of real or complex quantities that satisfy the following conditions[9].

a) The state at any time $t_1$ and inputs from $t_1$ to t $(t>t_1)$ uniquely determine the state at time t.

b) The state at time t and inputs at time t determine uniquely the value at time 't' of any network variable.

The equations are written in the form of a set of first order differential equations called "state equations". The concept of state and state equations is very useful for the study of control systems. The state equations of a linear time-invariant network can be written as,

$$\frac{dX(t)}{dt} = A\,X(t) + B\,U(t) \qquad \ldots.(2.49)$$

and the output equation

$$Y(t) = C\,X(t) + D\,U(t) \qquad \ldots.(2.50)$$

where X is the state vector, U(t) is input vector and Y(t) the output vector A, B, C and D are matrices of appropriate dimensions. If the network is time varying, then the elements of these matrices are functions of time. An advantage of the state equations is that similar equations can be written even for a non-linear network where the conventional network function technique is in general, not applicable. In writing the equations (2.49) and (2.50). We have made the tacit assumption in that the network does not have any circuit (cut-set) of capacitor (inductors) or capacitor (inductor)-voltage (current) sources. If it were not the case, the state and output equations would take the general form (here time 't' is not written to simplify notations).

$$\frac{dX}{dt} = AX + BU + E\frac{dU}{dt} \qquad \text{....(2.51)}$$

$$Y = CX + DU + E\frac{dU}{dt} \qquad \text{....(2.52)}$$

## 2.5 STATE VARIABLE FILTER

State variable is the one approach representing the effect of the energy storing element in any physical system. The electrical method is used to implement an analog computer simulation circuit where the integrator output voltages represent the state variables. We require one integrator to represent each energy-storing element. Thus, we count the number of integrators and know how many energy storing elements are in the system being represented. The analog computer having a biquadratic transfer function of Fig.(2.15) is called state-variable biquad".

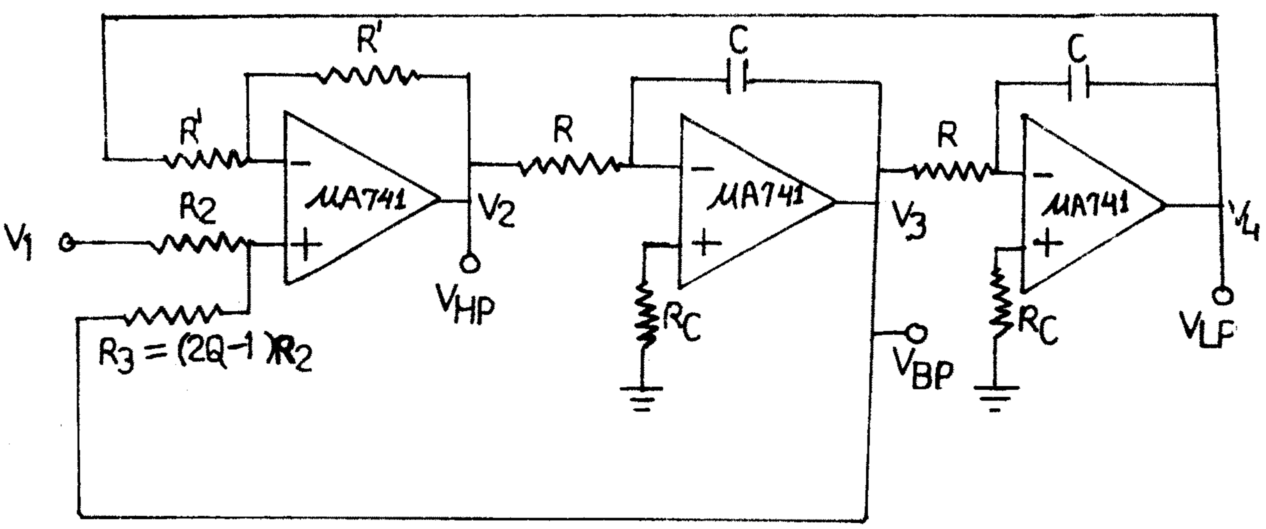



FIG.(2.15): KHN STATE VARIABLE BIAQUAD CIRCUIT.

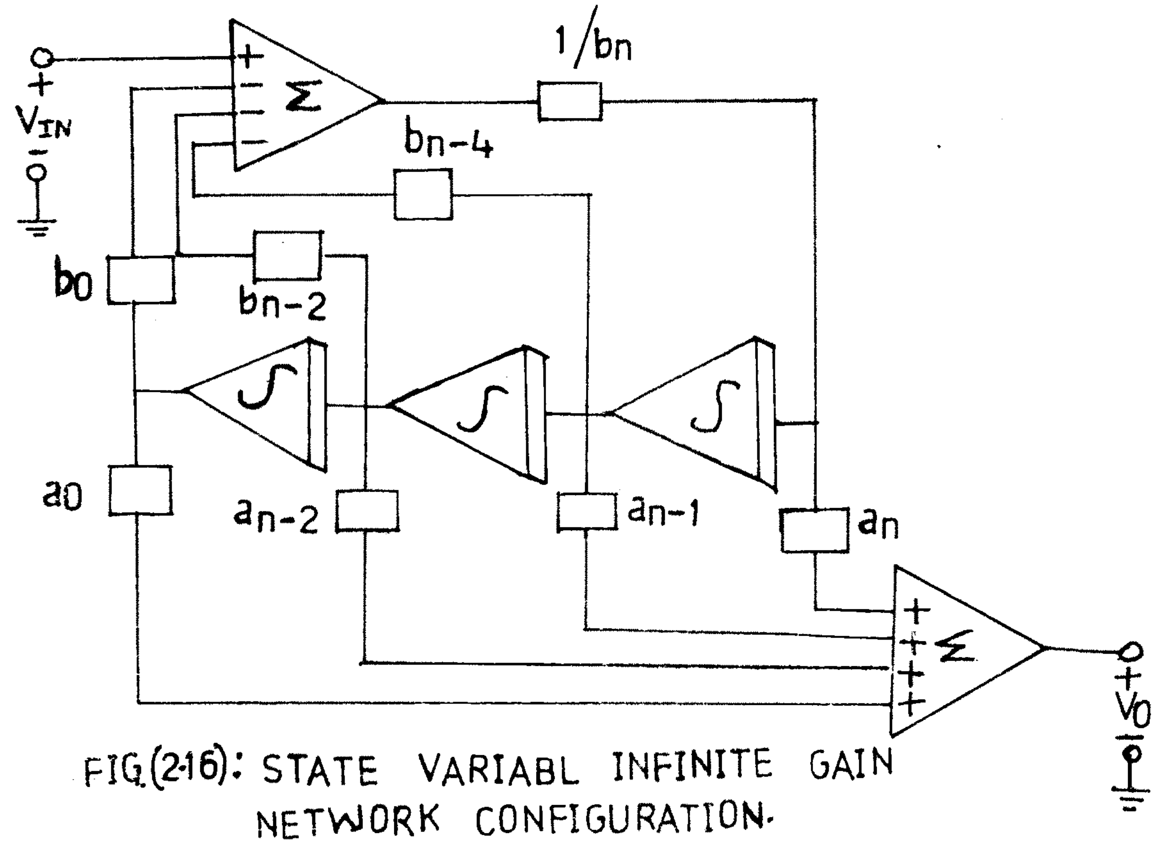



FIG.(2.16): STATE VARIABL INFINITE GAIN NETWORK CONFIGURATION.

A number of circuits are available for implementing the biquadratic equations. The first is the kHN state variable circuit named after Kerwin, Huelsman and New-comb which is sketched in Fig.(2.15). This simultaneously represents usual low pass, high pass and band pass filters at three different output points.

An infinite-gain state variable network configuration is shown in Fig.(2.16). This configuration makes use of operational amplifiers in the same way they would be used in an analog computer realization of transfer functions (using summer and integrator). Second order realization is shown in Fig.(2.16). The voltage transfer function has the form[7].

$$\frac{E_o}{E_1}(s) = \frac{a_o + a_1 s + \ldots + a_{n-1} s^{n-1} + a_n s^n}{b_o + b_1 s + \ldots + b_{n-1} s^{n-1} + b_n s^n} \qquad \ldots.(2.53)$$

The state variable realization in general provides lower 'Q' sensitivity to the element variation than a single amplifier realization. Due to this, it is sometimes used for high 'Q' band pass applications (Q > 50). It has a disadvantage that it requires three op-Amps. It is useful for low pass and high pass applications with low 'Q' value. It is a rather expensive circuit to use. The most widely used method in industry for higher order filter is cascade approach because the synthesis procedure required for determining the element values of a biquad is relatively simple and of low sensitivity.

## 2.6 FILTER TOPOLOGY

Topology formalizes the formulation of the network equilibrium equations (loop equations, node equations and the like) most of the computer aided analysis and design methods utilize topological formulation. The derivation of the state equations of a network inherently depends on the topological matrices of the network. The topology or geometry of the network is connected with the interconnections of the element in the network. The network is represented by a linear graph, so the study of topology is very useful in circuit or network analysis.

In this article, the commonly used single-amplifier biquad topologies are discussed. These structures require an RC network in addition to one op-Amp and used to realize the complex pole-zero pair. The transfer function of the RC network will have poles on the negative real axis while the zeros can be any where in the 's' plane. Since the general biquad circuit must realize complex poles as well as complex zeros, the op-Amp must be used to realize complex poles in spite of the fact that the RC network poles are real. There are many circuits that can accomplish this.

The majority of these circuits can be classified into two basic categories such as negative feedback topology and positive feedback topology. The classification depends on to which input terminal of the op-Amp the RC network is connected.

### 2.6.1 NEGATIVE FEEDBACK TOPOLOGY

The negative feedback topology is so called because the RC network associated with it provides a feedback path to the negative input terminal of the op-Amp. The topology is sketched in Fig.(2.17). The transfer function of this general structure can be expressed in terms of the feedforward and feedback transfer functions of the passive RC network. The transfer function of the negative feedback topology can be written as,

$$T_V = - \frac{N_{FF}}{N_{FB}} \qquad \ldots.(2.54)$$

where $N_{FF}$ and $N_{FB}$ represent the zeros of the RC network observed from feedforward and feedback ports. The main characteristics of negative feedback topology is summarized as,

1. The zeros of the feedback network determine the poles of the transfer function.
2. The zeros of the feedforward network determine the zeros of the transfer function.

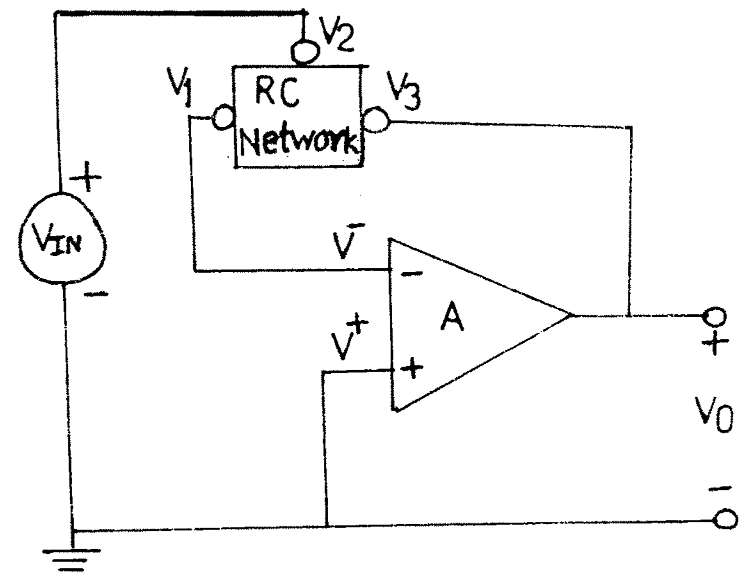



FIG.(2·17): THE NAGATIVE FEEDBACK TOPOLOGY.

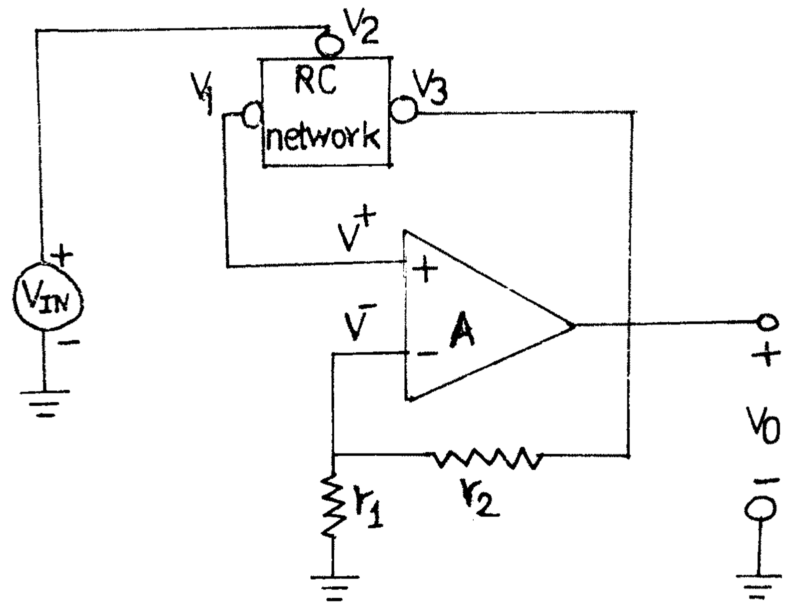



FIG.(2·18): POSITIVE FEEDBACK TOPOLOGY.

3. The poles and zeros can be complex. For stable network the poles can't lie in the right-half of 's' plane.
4. The poles of the RC network do not contribute to the transfer function (assuming op-Amp to be ideal).

### 2.6.2 POSITIVE FEEDBACK TOPOLOGY

The positive feedback topology is shown in Fig.(2.18). The topology is so called because the RC network provides a feedback path to the positive terminal of the op-Amp. However, some part of the output voltage is feedback to the inverting terminal via resistors $r_1$ and $r_2$ which form potential divider. In this sense, this is really a mixed topology. The negative feedback is used to define a positive gain for the VCVS.

In case of positive feedback topology, the transfer function of the network of Fig.(2.18) can be characterized in terms of the feedforward and feedback transfer function of the RC network. The transfer function can be written as,

$$T_V = \frac{K\ N_{FF}}{D - kN_{FB}} \quad \ldots.(2.55)$$

$$\text{But} \quad K = 1 + \frac{r_2}{r_1} \quad \ldots.(2.56)$$

where $N_{FF}$ and $N_{FB}$ represents the zeros of the RC network which can be complex. 'D' represents the poles of 'RC' network which must be real. The zeros of $T_V$ are determined by $N_{FF}$ while the poles of the $T_V$ are the roots of $D-kN_{FB}$.

Summarizing, for positive feedback topology.

1. The zeros of the transfer function are the zeros of the feedforward RC network which can be complex.
2. The poles of the transfer function can be located anywhere in the left half of 's' plane, being determined by the poles of the RC network and the factor $K=1 + r_2/r_1$.

## 2.7 IMEDANCE AND FREQUENCY SCALING

In most of the examples we have considered, the values of the elements R, L and C are of the order of unity. It is very difficult to build a capacitor of 1 farad. Since the practical value of the capacitor available is of the order of microfarads, in the circuits considered so far, we have to normalized the element values. There are two reasons for resorting to normalized designs. The first reason is simplicity of numerical calculations (computation). It is easier to manipulate numbers of the order of unity. The round-off errors that occur in normalized designs are small. The second reason is that if we have a normalized design of say, a band pass filter, then it is easy to generate band-pass filters of similar characteristics of varying center frequencies and impedance levels without re-designing the complete circuit. To obtain the element values of the required band pass filter, one has to impedance and frequency scale the normalized design.

After obtaining the nominal design, the 'impedance scaling' is used to change the element values of the circuit in order to make the circuit practically realizable. The impedance normalizing factor ($Y_n$) is given by[1].

$$Y_n = \frac{\text{desired impedance level}}{\text{normalized impedance level}} \qquad \ldots.(2.57)$$

Frequency scaling is used to shift the frequency response of a filter to a different part of the frequency axis. This is useful in designing the filters using normalized frequency requirements such as those given in the standard tables.

One example of frequency scaling is in denormalization of an LP transfer function which has cut-off frequency at $w_p$ rad/sec to realize a LP function with cut-off frequency at $w_p$ rad/sec.

In general, the frequency response of a given active filter can be scaled up by a factor $\alpha$ by decreasing all capacitors or resistors value by the factor $\alpha$.

The frequency normalizing factor ($\Omega n$) is given by[1].

$$\Omega n = \frac{\text{desired frequency}}{\text{normalized frequency}} \qquad \ldots.(2.58)$$