Definitions and Results

SECTION-I

Definitions and Results

In this section we give some basic definitions and results which will be used in further sections.

1.1: Definitions

Def. 1.1.1 Partially ordered set or poset [5]: Let p be a nonvoid set. Define a relation $\leqslant$ on P satisfying the following for all $a, b, c \in P$.

i) $a \leqslant a$ (reflexivity)

ii) $a \leqslant b$ and $b \leqslant a \implies a = b$ (antisymmetry)

iii) $a \leqslant b$ and $b \leqslant c \implies a \leqslant c$ (transitivity)

The ordered pair $(P, \leqslant)$ is called a poset or a partially ordered set.

A poset $(P, \leqslant)$ is called as chain (or totally ordered set or linearly ordered set) if it satisfies following condition :

iv) $a \leqslant b$ or $b \leqslant a$ for all $a, b, c \in P$ (linearity)

Let $H \subseteq P$, $a \in P$ is an upper bound of H if $h \leqslant a$ for all $h \in H$. An upper bound of H is the least upper bound of H or supremum of H (join) if for any upper bound b of H we have $a \leqslant b$. We shall write $a = \text{Sup } H$ or $a = \vee H$. The concept of lower bound or infimum is similarly defined. The latter is denoted by inf H or $\wedge H$.

Def.1.1.2 <u>Zero element or unit element of poset</u> [5] :

A zero of poset P is an element $0 \in P$ with $0 \leq x$ for all $x \in P$. A unit element of a poset P is an element $1 \in P$ with $x \leq 1$ for all $x \in P$.

<u>Def.1.1.3</u> <u>Lattice as poset</u> [5] : A poset $(L, \leq)$ is a lattice if $\sup\{a,b\}$ or $a \vee b$ and $\inf\{a,b\}$ or $a \wedge b$ exists in L for all $a, b \in L$.

<u>Def. 1.1.4</u> <u>Lattice as an algebra</u> [5] : An algebra $\langle L; \wedge, \vee \rangle$ is called as lattice if L is nonvoid set with two binary operations $\wedge$ and $\vee$ satisfying following properties for all $a,b,c \in L$

i)	$a \wedge a = a, \quad a \vee a = a$	(idompotency)
ii)	$a \wedge b = b \wedge a, \quad a \vee b = b \vee a$	(commutativity)
iii)	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	
	$(a \vee b) \vee c = a \vee (b \vee c)$	(associativity)
iv)	$a \wedge (a \vee b) = a$	
	$a \vee (a \wedge b) = a$	(absorption identities)

<u>Def. 1.1.5</u> <u>Complete lattice</u> [5] : A lattice L is said to be complete if $\wedge H$ and $\vee H$ exists for any subset H of L.

<u>Def. 1.1.6</u> <u>Semilattice</u> [5] : A poset P is a meet semilattice (dually join semilattice) if infimum $\{a,b\}$ or $a \wedge b$

(dually supremum $\{a, b\}$ or $a \vee b$) exists for any two elements $a, b \in P$.

<u>Def. 1.1.7</u> <u>O-distributive lattice</u> [5] : Let L be a lattice with O. L is said to be O-distributive if $a \wedge b = 0$, $a \wedge c = 0$ $(a, b, c \in L) \Longrightarrow a \wedge (b \vee c) = 0$

<u>Def.1.1.8</u> <u>O-distributive semilattice</u> [9] : A meet semi-lattice $< S, \wedge >$ with O is said to be O-distributive if $a \wedge x_1 = 0$, $a \wedge x_2 = 0$, ..., $a \wedge x_n = 0$; $x_1, x_2, \ldots x_n \in S$ (n finite) and $x_1 \vee x_2 \vee \ldots \vee x_n$ exists in S then $a \wedge (x_1 \vee x_2 \vee \ldots \vee x_n) = 0$.

<u>Def. 1.1.9</u> [6] For any subset $A \subseteq S$ define,

$$A^* = \{x \in S \,/\, x \wedge a = 0 \quad \text{for all } a \in A\}$$

If $A = \{x\}$ then $A^* = \{x\}^* = (x)^* = \{y \in S \,/\, x \wedge y = 0\}$ and $A^{**} = (A^*)^*$.

<u>Def. 1.1.10</u> <u>Ideal in a lattice</u> [5] : Let L be a lattice and let J be a non-empty subset of L, J is called as an ideal in L if a, b $\in$ J implies that $a \vee b \in J$ and $a \in J, x \in L, x \leq a$ imply $x \in J$

<u>Def. 1.1.11</u> <u>Prime ideal in a lattice</u> [5]: A proper ideal I of L is a prime if $a, b \in L$ and $a \wedge b \in I$ imply that $a \in I$ or $b \in I$

<u>Def. 1.1.12</u> <u>Maximal ideal in a lattice</u> [5] : Let L be

a lattice. A proper ideal of L is called as maximal if it is not contained in any other proper ideal of L.

Filter, prime filter, maximal filter in a lattice are defined dually.

Def. 1.1.13 Ideal in a semilattice [9] : A nonempty subset I of a semilattice S is called as an ideal if

i) $x \leq y$ in S and $y \in I \implies x \in I$

ii) if join of any finite number of elements of I exists in S then it must be in I.

Def. 1.1.14 Prime ideal in a semilattice [9] : A proper ideal I in a semilattice S is called as prime if $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Def. 1.1.15 Maximal ideal in a semilattice [9] : A proper ideal I in a semilattice S which is not contained in any other proper ideal in S is called as maximal ideal.

Def. 1.1.16 Minimal prime ideal in a semilattice [9] : A minimal element in the set of all prime ideals of S is called as a minimal prime ideal.

Def.1.1.17 Filter in a semilattice [9] : A filter F in S is a nonempty subset of S satisfying; ~~a,b in F implies that a∧b is in F.~~ $a \wedge b \in F \Leftrightarrow a, b \in F$

Def. 1.1.18 Maximal filter in a semilattice [9] : A proper filter which is not contained in any other proper filter is called as a maximal filter.

Def. 1.1.19 Prime filter in a semilattice [9] : A proper filter F is called as prime if for any finite subset A of S, $\vee A$ exists and is in F then $a \in F$ for some $a \in A$.

Def.1.1.20 Semifilter in a semilattice [12] : Let F be any non-empty subset of S. F is said to be semi filter in S if $x \leq y$, $x \in F$, $y \in S \Longrightarrow y \in F$.

Def.1.1.21 Dense ideal in a semilattice [6] : An ideal I in S is said to be dense idense if $I^* = \{0\}$.

An ideal which is not dense is called as nondense ideal in S.

Def.1.1.22 dense element in a semilattice [6] : An element a in a semilattice S with 0 is said to be dense if $(a)^* = \{0\}$.

Def.1.1.23 Annihilator ideal in ~~semi~~lattice [6] : An ideal J in ~~semi~~lattice L with 0 is said to be annihilator if $J = J^{**}$. Same definition we carry in semilattice.

Def.1.1.24 [5] : Let $< L_1; \wedge, \vee >$ and $< L_2; \wedge, \vee >$ be any two lattices. Then $f : L_1 \longrightarrow L_2$ is called as homomorphism if

i) $f(a \vee b) = f(a) \vee f(b)$

ii) $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in L_1$

Def.1.1.25 Isotone map [5] The map $\emptyset : P_0 \longrightarrow P_1$ is an isotone map (or order preserving map) of the poset P_0 into the poset P_1 if $a \leq b$ in P_0 implies that $\emptyset(a) \leq \emptyset(b)$ in P_1

Def.1.1.26 Closure operation :

A closure operation on a set I is an operator $X \longrightarrow \bar{X}$ on the subsets of I such that :

C1. $X \subset \bar{X}$ (Extensive)

C2. $\bar{X} = \bar{\bar{X}}$ (Idempotent)

C3. If $X \subset Y$, then $\bar{X} \subset \bar{Y}$ (Isotone)

§ 1.2 : Results

Result 1.2.1 [9] : Let S be semilattice with 0. Then S is 0-distributive if and only if A^* is an ideal for every $A \subseteq S$, $A \neq \emptyset$.

<u>Result 1.2.2</u> [9] : Let S be a O-distributive semilattice. A subset M of S is minimal prime ideal if and only if its set complement S-M is a maximal filter.

<u>Result 1.2.3</u> [9] : Let S be semilattice with O. A proper filter M in S is maximal if and only if (*) for any element $a \notin M$ $(a \in S)$, there exists an element b in M such that $a \wedge b = 0$.

<u>Result 1.2.4</u> [9] : Let S be semilattice with O. S is O-distributive if and only if for any filter F disjoint with $\{x\}^*$ $(x \in S)$, there exists a prime filter containing F and disjoint with $\{x\}^*$.

<u>Result 1.2.5</u> [13] : Set of all dense elements is filter.

<u>Result 1.2.6</u> [9] : Let S be semilattice with O. If $x \neq 0$ then there exists maximal filter F containing x.

<u>Result 1.2.7</u> [1] : The set of all ideals I(S) in a semilattice S form a lattice under the operations π and $\underline{\vee}$

where i) $I \pi J = I \cap J$

ii) $I \underline{\vee} J = (I \cup J]$

Let L_1 and L_2 be any two lattices. A bijective map f from L_1 to L_2

<u>Result 1.2.8</u> [1] : $f : L_1 \rightarrow L_2$ is an isomorphism if and only if i) f is onto

ii) both f and f^{-1} are isotone maps.