Characterization of α-ideals in a 0-distributive semilattice

Section-III

Characterization of α-ideals in a O-distributive semilattice.

Throughout this section S denotes a O-distributive semilattice.

Let $\mathfrak{m}$ be the set of all minimal prime ideals in a O-distributive semilattice. For a subset $\mathcal{P}$ of $\mathfrak{m}$ define as usual kernel of $\mathcal{P}$ by

$$\text{Kernel of } \mathcal{P} = K(\mathcal{P}) = \cap \{ R \,/\, R \in \mathcal{P} \}$$

For a subset A of S define the hull of A as,

$$\text{Hull of } A = h(A) = \{ M \in \mathfrak{m} \,/\, A \subseteq M \} .$$

Let $\mathcal{T}$ denotes the hull-kernel topology on $\mathfrak{m}$. First we prove two lemmas that we need for characterizing α-ideals in S.

<u>Lemma 3.1</u> : For any $x \in S$,

$$(x)^* = \cap \{ M \in \mathfrak{m} \,/\, x \notin M \}$$

<u>Proof</u> - Let $y \in (x)^*$. Then $x \wedge y = 0$. For any minimal prime ideal M we have, $x \wedge y \in M$. If $x \notin M$ then $y \in M$. Therefore $y \in \cap \{ M \,/\, x \notin M \}$. Thus $y \in (x)^*$ implies that $y \in \cap \{ M \,/\, x \notin M \}$ proving that $(x)^* \subseteq \cap \{ M \,/\, x \notin M \}$. Now let if possible, $(x)^* \subset \cap \{ M \,/\, x \notin M. \}$ Then there exists

$z \in \cap\{M / x \notin M\}$ such that $z \notin (x)^*$. Hence $x \wedge z \neq 0$. But then $z \wedge x \in F$, for some maximal filter F (See Result 1.2.6). Here $z \in F$. But then S-F is a minimal prime ideal not containing z (See Result 1.2.2). This contradicts with the fact that, $z \in \cap\{M / x \notin M\}$. Hence $(x)^* \subset \cap\{M/x \notin M\}$ is not possible. Therefore $(x)^* = \cap\{M/ x \notin M\}$. //

Denote $h(\{x\}) = h(x)$.
As $(x)^* = \cap\{M \in \mathfrak{m} / x \notin M\}$, we get $(x)^* = K[\mathfrak{m} - h(x)]$.

In the following lemma we give a necessary and sufficient condition for $h(x) = h(y)$ where $x \neq y$ in S.

<u>Lemma 3.2</u> : For any x, y in S, $h(x) = h(y)$ if and only if $(x)^* = (y)^*$

<u>Proof</u> - Let $h(x) = h(y)$. Then $\mathfrak{m} - h(x) = \mathfrak{m} - h(y)$. Hence $\cap[\mathfrak{m} - h(x)] = \cap[\mathfrak{m} - h(y)]$. This implies that $\cap\{M \in \mathfrak{m} / M \notin h(x)\} = \cap\{M \in \mathfrak{m} / M \notin h(y)\}$
Hence $\cap\{M \in \mathfrak{m} / x \notin M\} = \cap\{M \in \mathfrak{m} / y \notin M\}$.
This in turn implies that $(x)^* = (y)^*$ (by Lemma 3.1).

Conversely let $(x)^* = (y)^*$. Then by Lemma 3.1we get $\cap\{M \in \mathfrak{m} / x \notin M\} = \cap\{M \in \mathfrak{m} / y \notin M\}$. Hence $h[\cap\{M \in \mathfrak{m} / x \notin M\}] = h[\cap\{M \in \mathfrak{m} / y \notin M\}]$ i.e.

$h[\,K\{M \in \mathfrak{M} / x \notin M\}\,] = h[\,K\{M \in \mathfrak{M} / y \notin M\}\,]$. But as $\mathcal{T}$ is hull-kernel topology we get, $h[K\{M \in \mathfrak{M} / x \notin M\}\,] = \{M \in \mathfrak{M} / x \notin M\}$ and $h[\,K\{M \in \mathfrak{M} / y \notin M\}\,] = \{M \in \mathfrak{M} / y \notin M\}$.

Thus we get, $(x)^* = (y)^* \Longrightarrow \{M \in \mathfrak{M} / x \notin M\} = \{M \in \mathfrak{M} | y \notin M\}$

But this in turn implies that,

$$\mathfrak{M} - \{M \in \mathfrak{M} / x \notin M\} = \mathfrak{M} - \{M \in \mathfrak{M}\ y \notin M\}.$$

$$\text{i.e. } \{M \in \mathfrak{M} / x \in M\} = \{M \in \mathfrak{M} / y \in M\}.$$

Therefore $h(x) = h(y)$. This completes the proof of if past.//

Now we state our main result

<u>Theorem 3.3</u> : For any ideal I in S, the following conditions are equivalent :

(1) I is an α-ideal.

(2) $I \bigcup\limits_{x \in I} (x)^{**}$

(3) For $x, y \in S$, $(x)^* = (y)^*$ and $x \in I \Longrightarrow y \in I$.

(4) For $x, y \in S$, $h(x) = h(y)$ and $x \in I \Longrightarrow y \in I$.

<u>Proof</u> (1) ===> (2)

Since I is an α-ideal, for each $x \in I$ we get

$(x)^{**} \subseteq I$. Hence $\bigcup_{x \in I} (x)^{**} \subseteq I$. As $x \in (x)^{**}$ always $I \subseteq \bigcup_{x \in I} (x)^{**}$. Therefore by combining both the inclusions we get, $I = \bigcup_{x \in I} (x)^{**}$

(2) ===> (1)

By (2) we have, $I = \bigcup_{x \in I} (x)^{**}$. This implies that, $(x)^{**} \subseteq I$ for each $x \in I$. Therefore I is an α-ideal.

(2) ===> (3)

Let $(x)^{*} = (y)^{*}$ $(x, y \in S)$ and $x \in I$. We want to prove that $y \in I$. Assume that $y \notin I$. Then by (2) we get, $y \notin \bigcup_{t \in I} (t)^{**}$. This implies that $y \notin (t)^{**}$ for all $t \in I$. As $x \in I$ we get, $y \notin (x)^{**}$. Hence $y \wedge z \neq 0$ for some $z \in (x)^{*}$. But $(x)^{*} = (y)^{*}$. Therefore $z \in (x)^{*}$ implies that $z \in (y)^{*}$ i.e. $y \wedge z = 0$ which is a contradiction. Hence $y \notin I$ is not possible. Therefore $y \in I$.

(3) ===> (2)

For any $x \in S$ we know that, $x \in (x)^{**}$ always. Hence $I \subseteq \bigcup_{x \in I} (x)^{**}$. Now let $y \in \bigcup_{x \in I} (x)^{**}$. Then $y \in (x)^{**}$ for some $x \in I$. This implies that, $y \wedge z = 0$ for all $z \in (x)^*$. Now $z \in (x)^*$ implies $y \wedge z = 0$. But then $z \in (y)^*$ and hence $(x)^* \subseteq (y)^*$. Therefore $(x)^* \veebar (y)^* = (y)^*$. But $(x)^* \veebar (y)^* = (x \wedge y)^*$ (See Result 1.2.7). Hence $(x \wedge Y)^* = (y)^*$. As $x \wedge y \leqslant x$ and $x \in I$ we get $x \wedge y \in I$. Using (1) we get $y \in I$. This proves that $\bigcup_{x \in I} (x)^{**} \subseteq I$. Combining both the inclusions we get, $I = \bigcup_{x \in I} (x)^{**}$.

(3) ===> (4)

Let $h(x) = h(y)$ and $x \in I$. ~~Then~~ We want to prove that $y \in I$. Since $h(x) = h(y)$ therefore by Lemma 3.2, $(x)^* = (y)^*$. Thus $h(x) = h(y) \implies (x)^* = (y)^*$. Hence by using (3) we get, $y \in I$.

(4) ===> (3)

Let $(x)^* = (y)^*$ and $x \in I$. Then we want to prove that $y \in I$. Since $(x)^* = (y)^*$ therefore again by Lemma 3.2 we get $h(x) = h(y)$. Thus $(x)^* = (y)^* \implies h(x) = h(y)$. Hence by using (4) we get, $y \in I$.

Thus (1) <==> (2) <==> (3) <==> (4), completing the proof of Theorem. //

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