The α-map

Section IV

The α-map

In this section we change our setting from O-distributive semilattice to O-distributive lattice for the study of α-ideals. Throughout this section L will stand for O-distributive lattice. As L is O-distributive lattice, the set of annihilators of the type $\{x\}^* = (x)^*$ $(x \in L)$ will form a lattice called as lattice of annulets of L in this direction we only state,

Theorem 4.1 : $A_o(L) = \{(x)^* / x \in L\}$ is a lattice under the binary operations $\dot{\wedge}$ and $\dot{\vee}$ which are defined as,

$$(x)^* \dot{\wedge} (y)^* = (x \vee y)^* \quad \text{and}$$

$$(x)^* \dot{\vee} (y)^* = (x \wedge y)^*$$

First we establish a relation between the set of all ideals I(L) of L and the set of all filters $\mathcal{F}(A_o(L))$ of the lattice $A_o(L)$.

<u>Theorem 4.2</u> : For any ideal I in L,

$$\{(x)^* / x \in I\} \text{ is a filter in } A_o(L)$$

<u>Proof</u> Let $F = \{(x)^* / x \in I\}$

(i) Let $(x)^* \subseteq (y)^*$ and $(x)^* \in F$ $(x, y \in L)$

As $(x)^* \in F$, there exists $t \in I$ such that $(x)^* = (t)^*$. Then by assumption $(t)^* \subseteq (y)^*$. Hence $(t)^* \dot{\vee} (y)^* = (y)^*$. Therefore $(t \wedge y)^* = (y)^*$. As $t \wedge y \leq y$ and $t \in I$ we get $t \wedge y \in I$. Hence $(t \wedge y)^* \in F$. Therefore $(y)^* \in F$. Thus $(x)^* \subseteq (y)^*$, $(x)^* \in F \implies (y)^* \in F$.

(ii) Let $(x)^*, (y)^* \in F$ $(x, y \in L)$. As $(x)^* \in F$ there exists $t \in I$ such that $(x)^* = (t)^*$ and as $(y)^* \in F$ there exists $s \in I$ such that $(y)^* = (s)^*$. I being an ideal $t \in I$, $s \in I$ imply $t \vee s \in I$. Hence $(t \vee s)^* \in F$. This implies that $(t)^* \dot{\wedge} (s)^* \in F$ i.e. $(x)^* \dot{\wedge} (y)^* \in F$. Thus $(x)^*, (y)^* \in F \implies (x)^* \dot{\wedge} (y)^* \in F$.

From (i) and (ii) we get, F is a filter in $A_0(L)$. //

In the reverse direction we have the following theorem :

<u>Theorem 4.3</u> : For any filter F in $A_0(L)$,

$$\left\{ x \in L \,/\, (x)^* \in F \right\} \text{ is an ideal in } L.$$

<u>Proof</u> Denote $J = \left\{ x \in L \,/\, (x)^* \in F \right\}$

(i) Let $x \leq y$ and $y \in J$ $(x, y \in L)$.

Since $x \leq y$, $x \vee y = y$ and hence $(x \vee y)^* = (y)^*$ is in F. Thus we get $(x)^* \dot{\wedge} (y)^* \in F$ and $(x)^* \dot{\wedge} (y)^* \subseteq (x)^*$.

F being a filter, $(x)^* \in F$. Hence $x \in J$. Thus $x \leq y$ and y

$y \in J \implies x \in J$.

(ii) Let $x, y \in J$. Hence $(x)^*, (y)^* \in F \implies (x)^* \wedge (y)^* \in F$

$\implies (x \vee y)^* \in F$

$\implies x \vee y \in J$

Thus $x, y \in J \implies x \vee y \in J$.

By (i) and (ii), we get J is an ideal in L. //

Define the map,

$\alpha : I(L) \longrightarrow \mathcal{F}(A_o(L))$, $I \in I(L)$ by,

$\alpha(I) = \{ (x)^* / x \in I \}$

By Theorem 4.2, α is a well defined map. Now we prove α is an isotone map.

<u>Theorem 4.4</u> : For any two ideals I_1 and I_2 in L,

$I_1 \subseteq I_2 \implies \alpha(I_1) \subseteq \alpha(I_2)$.

<u>Proof</u> $(Z)^* \in \alpha(I_1) \implies (Z)^* \in \{ (x)^* / x \in I_1 \}$

$\implies (Z)^* = (y)^*$ for some $y \in I_1$

$\implies (Z)^* = (y)^*$ for some $y \in I_2$

(since $I_1 \subseteq I_2$)

$\implies (Z)^* \in \alpha(I_2)$

Hence $\alpha(I_1) \subseteq \alpha(I_2)$ //

Define the map,

$$\beta : \mathfrak{F}(A_o(L)) \longrightarrow I(L),\ F \in \mathfrak{F}(A_o(L)) \text{ by,}$$

$$\beta(F) = \left\{ x \in L \,/\, (x)^* \in F \right\}$$

By theorem 4.3, β is well defined map.

Clearly β is an isotone map.

As $\alpha : I(L) \longrightarrow \mathfrak{F}(A_o(L))$ and

$\beta : \mathfrak{F}(A_o(L)) \longrightarrow I(L)$, the composition map $\beta o \alpha$ maps $I(L)$ into $I(L)$.

<u>Theorem 4.5</u> : The map, $\beta o \alpha : I(L) \longrightarrow I(L)$ is a closure operator.

<u>Proof</u> i) $I \subseteq \beta o \alpha(I)$

For if $z \in I$ then $(z)^* \in \alpha(I)$ and hence $z \in \beta o \alpha(I)$.

ii) $I \subseteq J \Longrightarrow \beta o \alpha(I) \subseteq \beta o \alpha(J)$

Let $z \in \beta o \alpha(I)$. Then $(z)^* \in \alpha(I)$. Since the map α is an isotone map by Theorem 4.4, therefore $(z)^* \in \alpha(J)$. This implies that $z \in \beta o \alpha(J)$. Hence $I \subseteq J \Longrightarrow \beta o \alpha(I) \subseteq \beta o \alpha(J)$.

iii) $\beta o \alpha[\beta o \alpha(I)] = \beta o \alpha(I)$ for some ideal I in L.

Obviously $\beta o \alpha(I) \subseteq \beta o \alpha[\beta o \alpha(I)]$ (by (i))

Next let $z \in \beta o \alpha[\beta o \alpha(I)]$. Then $(z)^* \in \alpha[\beta o \alpha(I)]$ implies $(z)^* = (y)^*$ for some $y \in \beta o \alpha(I)$ and hence

$(z)^* \notin \alpha(I)$ (since $y \notin \beta o\alpha(I) \Longrightarrow (y)^* \notin \alpha(I)$) proving that $z \notin \beta o\alpha(I)$. Hence $\beta o\alpha\,[\beta o\alpha(I)\,] \subseteq \beta o\alpha(I)$. Thus by combining both the inclusions we get

$$\beta o\alpha(I) = \beta o\alpha\,[\beta o\alpha(I)].$$

From (i), (ii) and (iii) we get, $\beta o\alpha : I(L) \longrightarrow I(L)$ is a closure operator. //

As by (iii) in the Theorem 4.5, $\beta o\alpha\,[\beta o\alpha(I)\,] = \beta o\alpha(I)$ we observe that, there are some ideals I in L for which $\beta o\alpha(I) = I$ (and also there are some ideals I in L for which $\beta o\alpha(I) \neq I$). This we illustrate by the following example.

Example 4.6 : Consider the 0-distributive lattice L sketched in the following diagram :

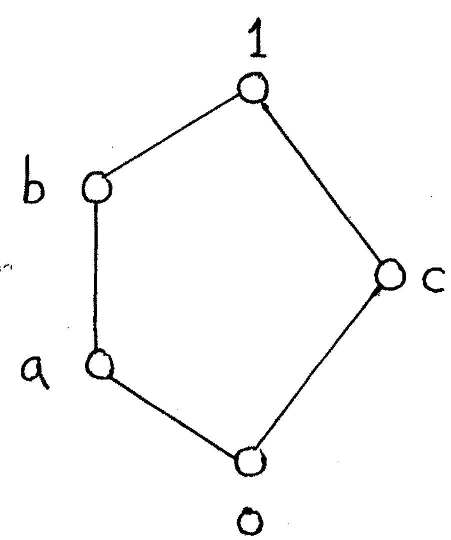

Figure-V

Consider the ideal, $I_1 = \{o, c\}$.

$\alpha(I_1) = \{(0)^*, (c)^*\}$

$= \{L, \{o, a, b\}\}$

$\beta o \alpha (I_1) = \{o, c\}$. Then $\beta o \alpha(I_1) = I_1$

Now consider the ideal, $I_2 = \{o, a\}$

$\alpha(I_2) = \{(0)^*, (a)^*\}$

$= \{L, \{o, c\}\}$

$\beta o \alpha(I_2) = \{o, a, b\}$

Here $\beta o \alpha(I_2) \neq I_2$.

Now we prove our main result :

Ideal I in L for which $\beta o \alpha$ (I) = I is nothing but an α-ideal.

<u>Theorem 4.7</u> : For an ideal I in L the following conditions are equivalent.

(1) I is an α-ideal.

(2) $\beta o \alpha$ (I) = I.

<u>Proof</u>. (1) ===> (2)

Let I be any α-ideal. By Theorem 4.5 we have $I \subseteq \beta o \alpha(I)$. Next let $z \in \beta o \alpha(I)$. Then $(z)^* \in \alpha(I)$.

And therefore $(z)^* = (y)^*$ for some $y \in I$. I being an α-ideal, this will imply $z \in I$ proving that $\beta o \alpha(I) \subseteq I$. Hence $\beta o \alpha(I) = I$.

(2) ===> (1)

Let $(x)^* = (y)^*$ and $x \in I$. Since $x \in I$ we get, $(x)^* \in \alpha(I)$. But then $(y)^* \in \alpha(I)$ which in turn implies that $y \in \beta o \alpha(I) = I$ (by assumption). Thus $(x)^* = (y)^*$ and $x \in I$ ===> $y \in I$, proving that I is an α-ideal (See Theorem 3.3). //

Combining Theorem 3.3 and Theorem 4.7 we have the following :

Corollary 4.8 : For any ideal I in L the following conditions are equivalent.

(1) I is an α-ideal.

(2) $I = \bigcup_{x \in I} (x)^{**}$.

(3) For $x, y \in L$, $(x)^* = (y)^*$ and $x \in I$ ===> $y \in I$.

(4) For $x, y \in L$, $h(x) = h(y)$ and $x \in I$ ==> $y \in I$.

(5) $\beta o \alpha\ (I) = I$.

Remark 4.9 : For any ideal I in L, $\beta o \alpha(I)$ is always an α-ideal in L (See Theorem 4.5).

Let I_1 and I_2 be any two α-ideals in L. Define,

$I_1 \textcircled{\wedge} I_2 = I_1 \cap I_2$ and

$I_1 \textcircled{\vee} I_2 = \beta o \alpha \ (I_1 \underline{\vee} I_2)$

Then obviously the set of all α-ideals in L, $I_\alpha(L)$ is a lattice under $\textcircled{\wedge}$ and $\textcircled{\vee}$.

Define the map,

$\bar{\alpha} : I_\alpha(L) \longrightarrow \mathcal{J}(A_o(L))$ by,

$\bar{\alpha}\ (I) = \{(x)^* \ / \ x \in I\}$ i.e. $\bar{\alpha}$ is the restriction of the map α to the set $I_\alpha(L)$.

Further we have,

Theorem 4.10 : The map $\bar{\alpha} : I_\alpha(L) \longrightarrow \mathcal{J}(A_o(L))$ is an isomorphism.

Proof As α and β are isotone maps, it follows that both $\bar{\alpha}$ and $\bar{\alpha}^{-1}$ are isotone maps. As for any $I \in I\alpha(L)$, $\beta o \alpha(I) = I$. We get $\bar{\alpha}^{-1}(\alpha(I)) = \beta(\alpha(I)) = (\beta o \alpha)(I) = I$ and hence $\bar{\alpha}^{-1}$ is an onto map. This proves that $\bar{\alpha}$ is an isomorphism (See Result 1.2.8).

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