C H A P T E R - I I I

SPECIAL SPACE-LIKE CONGRUENCES ON THE STREAM LINES.

Introduction :

In this Chapter we find the effect of vanishing of each of the curvature scalars on the space-like congruences associated with the stream line of a particle. It is argued that $K_3 = 0$ corresponds to the path of a classically gravitationally self interacting spin particle, while $K_2 = 0$ implies that the particle moves as a charged particle with radiation reaction but no external electromagnetic field. The case of $K_1 = 0$ yields a geodesic path. Section 2, portrays the three kinematical parameters of the space-like vector field P^a only after analysing the transport laws governing their definition. Explicit evaluation of physical components of shear and rotation is accomplished here. The transport laws of the space-like congruence P^a are expressed in terms of Ricci rotation coefficients γ_{lhk} in the next section. The generalized Serret-Frenet formulae and physical components of the shear and rotation look elegant when expressed in γ_{lhk}.

Section 1 :

Special curvatures and their significance :

Case 1 : The first curvature vanishes :

Here $K_1 = 0$, that is the magnitude of P^a vanishes. This is possible only when $\dot{u}^a = 0$. This represents the path of a free particle, in other words, it denotes trajectory of a particle upon which no force acts apart from the gravitational force. This path is also referred as the geodesic path.

Case 2 : The second curvature vanishes and the first curvature does not vanish :

The equation

$$K_2 = 0, \quad K_1 \neq 0$$

implies that

$$W^a = 0 \qquad \text{by (2.18)}$$

which means that

$$Q^a = 0$$

or

$$\frac{\ddot{u}^a}{K_1} - \frac{\dot{K}_1}{K_1^{\,2}}\,\dot{u}^a - K_1\,u^a = 0$$

or

$$\ddot{u}^a = \frac{\dot{K}_1}{K_1}\,\dot{u}^a + K_1^{\,2}\,u^a. \tag{3.1}$$

Here $\ddot{u}^a$ is a linear combination of $\dot{u}^a$ and u^a. Such a situation exists in the case of the path of a charged particle with radiation reaction but no external electromagnetic field. Specifically

$$m\dot{u}^a = \frac{e^2}{6\pi c^2}\left[\ddot{u}^a + u^a\,(-\,\dot{u}^b\dot{u}_b)\right] \tag{3.2}$$

where m is the mass and e is the charge of the particle (Barut, 1964).

<u>Remark</u> : When K_1 = constant, we have

$$\hat{K}_1 = 0.$$

So

$$\ddot{u}^a = K_1{}^2 u^a. \qquad (3.3)$$

And this equation represents the differential equation for a time-like circle. (Synge, 1960).

<u>Case 3</u> : <u>The third curvature vanishes and the first and the second do not vanish</u> :

Here

$$K_3 = 0, \quad K_2 \neq 0, \quad K_1 \neq 0. \qquad (3.4)$$

This is similar to the previous case, we get that $\dddot{u}^a$ should be a linear combination of $\ddot{u}^a$, $\dot{u}^a$, u^a that is

$$\dddot{u}^a = \lambda\, \ddot{u}^a + \mu\, \dot{u}^a + (\lambda\, \dot{u}^b \dot{u}_b - 3\, u^b \dddot{u}_b)\, u^a \qquad (3.5)$$

where λ and μ are arbitrary.

Such a situation exists in the path of a classically gravitationally self interacting spin particle with Frankel-Weyssenhoff constraints, viz.,

$$\dddot{u}^a + (3\, \dot{u}^b \ddot{u}_b)\, u^a = \left(\frac{m^2}{s^2} - \dot{u}^b \dot{u}_b \right) \dot{u}^a + \frac{8}{15} Gm$$

$$\left(\frac{m^2}{s^2} - \dot{u}^2 \right) (\dot{u}^b \dot{u}_b\, u^a + \ddot{u}^a) \qquad (3.6)$$

(Goenner <u>et al</u>. 1967)

where $S_a = S_{ab}\, u^b$ with the spin tensor S_{ab}

m = mass

G = Newtonian Gravitational constant.

<u>Remark</u> :

When $K_3 = 0$ and K_1 = constant, K_2 = constant

we have

$\dot{K}_1 = 0$ and $\dot{K}_2 = 0$

These conditions implies that the path of the particle is the time-like helix.

Section 2 :

Shear-free, irrotational space-like congruence P^a :

(1) The three parameters of the space-like congruence P^a :

The parameters of the space-like congruence P^a, relative to the time-like congruence u^a when the signature of metric is (- - - +) are cited below. We note that in Chapter I, Greenberg's formulae are for the metric signature (+ + + -).

(i) The expansion parameter is defined by

$$\underset{(1)}{\theta} \equiv \frac{1}{2} (P^a{}_{;a} - P_{a;b}\, u^a u^b) \tag{3.7}$$

here the subscript (1) below θ denotes that the parameter is for the first space-like congruence P^a.

(ii) The shear tensor field for P^a has the expression

$$\underset{(1)}{\sigma}_{ab} \equiv \perp^c_a \perp^d_b (P_{c;d} + P_{d;c}) - \perp_{ab} \underset{(1)}{\theta} \tag{3.8}$$

where $\perp_{ab}$ is the 2-dimensional projection operator defined by

$$\perp_{ab} = g_{ab} - u_a u_b + P_a P_b \tag{3.9}$$

with properties

$$\perp_{ab} = \perp_{ba}, \quad \perp^a{}_b \perp^b{}_c = \perp^a{}_c, \quad \perp^a{}_a = 2, \qquad (3.10)$$

$$\perp_{ab} P^a = 0 \quad \perp_{ab} u^a = 0.$$

(iii) The rotation tensor field for P^a is characterized by

$$\underset{(1)}{\omega}_{ab} \equiv \perp_a{}^c \perp_b{}^d (P_{c;d} - P_{d;c}). \qquad (3.11)$$

These definitions are subject to the three GREENBERG'S transport laws for P^a (after due corrections for signature):

$$u^a{}_{;b} P^b = P^a{}_{;c} u^c - u^a P_{b;c} u^b u^c + P^a P_{b;c} u^b P^c \qquad (3.12)$$

$$Q^a{}_{;b} P^b = -u^a P_{b;c} Q^b u^c + P^a P_{b;c} Q^b P^c \qquad (3.13)$$

and

$$R^a{}_{;b} P^b = -u^a P_{b;c} R^b u^c + P^a P_{b;c} R^b P^c . \qquad (3.14)$$

Since the index a ranges over 0 to 3, these are 12 equations (3 of which will be shown to be identities). The significance of these transport laws is that, they ensure the orthogonality of the tetrad (u^a, P^a, Q^a, R^a) during the parallel transport of the vector fields.

(2) <u>The expansion, the shear and the rotation of the space-like congruence P^a in terms of u^a, $\dot{u}^a$, $\ddot{u}^a$</u> :

(i) By the expressions (2.5), (3.7) becomes

$$\underset{(1)}{\theta} = \frac{1}{2} \left(\frac{\dot{u}^a{}_{;a}}{K_1} - \frac{K_{1;a}\,\dot{u}^a}{K_1{}^2} - K_1 \right) \qquad (3.15)$$

since $\ddot{u}^a u_a = K_1{}^2$ and $\dot{u}^a u_a = 0$, where K_1 is the first curvature of the world line.

(ii) Using (2.5) in (3.8) we have

$$\underset{(1)}{\sigma}{}_{ab} = \perp_a{}^c \perp_b{}^d \left(\frac{\dot{u}_{c;d}}{K_1} - \frac{K_{1;d}\,\dot{U}_c}{K_1{}^2} + \frac{\dot{u}_{d;c}}{K_1} - \frac{K_{1;c}\dot{u}_d}{K_1{}^2} \right)$$

$$- \perp_{ab}\underset{(1)}{\theta}$$

and by using (3.9) in above expression

$$\underset{(1)}{\sigma}{}_{ab} = \frac{1}{K_1} (\dot{u}_{a;b} + \dot{u}_{b;a} - u_a\ddot{u}_b - \ddot{u}_a u_b - u_a\dot{u}_{c;b}\, u^c$$
$$- u_b\dot{u}_{c;a}u^c) + \frac{\dot{K}_1}{K_1{}^2} (\dot{u}_a u_b + u_a\dot{u}_b) + 2\, K_1 u_a u_b$$
$$+ \frac{1}{K_1{}^3} (\dot{u}_a\, \dot{u}_{b;c}\, \dot{u}^c + \dot{u}_a\, \dot{u}_{c;b}\, \dot{u}^c +$$
$$\dot{u}_a\, \dot{u}_{d;c}\, \dot{u}^c\, u^d - u_a\, \dot{u}_b\, \dot{u}_{c;d}\, \dot{u}^c\, \dot{u}^d$$
$$+ \dot{u}_{a;d}\, \dot{u}_b\, \dot{u}^d + \dot{u}_{d;a}\, \dot{u}_b\, \dot{u}^d)$$
$$+ \frac{1}{K_1{}^5}\; \dot{u}_a\dot{u}_b\dot{u}^c\dot{u}^d\, (\dot{u}_{c;d} + \dot{u}_{d;c}) - \perp_{ab}\underset{(1)}{\theta} . \qquad (3.16)$$

Note : <u>On the non-zero independent components of $\underset{(1)}{\sigma}_{ab}$</u> :

Since $\perp_{ab}\, u^a = 0$ we have

$$\underset{(1)}{\sigma}_{ab}\, u^a = 0 \qquad (3.17)$$

and therefore $\underset{(1)}{\sigma}_{ab}$ is orthogonal to u^a.

Now $\perp_{ab}\; P^a = 0$

implies that

$$\perp_{ab}\, \dot{u}^a = 0$$

therefore

$$\underset{(1)}{\sigma}_{ab}\; \dot{u}^a = 0 \qquad (3.18)$$

hence $\underset{(1)}{\sigma}_{ab}$ is orthogonal to $\dot{u}^a$.

From (3.17) and (3.18) it follows that $\underset{(1)}{\sigma}_{ab}$ is in the 2-plane spanned by Q^a, R^a. Consequently there are at most 2-non-zero components of $\underset{(1)}{\sigma}_{ab}$, since

$$\left.\begin{aligned} \underset{(1)}{\sigma}_{ab} &= \underset{(1)}{\sigma}_{ba} \\ \text{and} \quad \underset{(1)}{\sigma}{}_a{}^a &= 0 \end{aligned}\right\} \qquad (3.19)$$

(iii) Now using (2.5) in (3.11) we have

$$\underset{(1)}{\omega}_{ab} = \perp_a^{\ c} \perp_b^{\ d} \left[\frac{1}{K_1} (\dot{u}_{c;d} - \dot{u}_{d;c}) + \frac{1}{K_1^{\ 2}} (K_{1;c}\, \dot{u}_d - K_{1;d}\, \dot{u}_c) \right]$$

and by using (3.9) in the above expression, we get

$$\underset{(1)}{\omega}_{ab} = \frac{1}{K_1} (\dot{u}_{a;b} - \dot{u}_a u_b - u_a \ddot{u}_b + \ddot{u}_{d;a}\, \dot{u}_b u^d - u_a \dot{u}_{c;d} u^c) + \frac{\dot{K}_1}{K_1^2} (\dot{u}_a u_b - u_a \dot{u}_b) + \frac{1}{K_1^{\ 3}} (\dot{u}_a \dot{u}^c (\dot{u}_{c;b} - \dot{u}_{b;c}) + \dot{u}_a u_b \dot{u}_{d;c}\, \dot{u}^c u^d - u_a \dot{u}_b \dot{u}_{c;d} \dot{u}^d + (\dot{u}_{a;d} - \dot{u}_{d;a}) \dot{u}_b \dot{u}^d) + \frac{1}{K_1^{\ 5}}\ \dot{u}_a \dot{u}_b \dot{u}^c \dot{u}^d (\dot{u}_{c;d} - \dot{u}_{d;c}) \qquad (3.20)$$

<u>Note</u> : Since $\perp_{ab}\, u^a = 0$ and $\perp_{ab}\, P^a = 0$,

$$\underset{(1)}{\omega}_{ab}\, u^a = 0 \qquad (3.21)$$

and

$$\underset{(1)}{\omega}_{ab}\, \dot{u}^a = 0$$

i.e. $\underset{(1)}{\omega}_{ab}$ is orthogonal to u^a and $\dot{u}^a$,

From (3.21) and (3.22) it follows that $\underset{(1)}{\omega}_{ab}$ is in the 2-plane spanned by Q^a, R^a. There is atmost one non-zero conponent of $\underset{(1)}{\omega}_{ab}$, since

$$\underset{(1)}{\omega}_{ab} = - \underset{(1)}{\omega}_{ba}. \tag{3.23}$$

The vorticity space-like congruence ω^a is given by

$$\omega^a = \frac{1}{2} \eta^{abcd} u_b \omega_{cd} . \tag{3.24}$$

(3) <u>Physical Components of a tensor</u> :

We define the physical components of a tensor A_{abcd} to be the <u>set of scalars</u>

$$A_{\alpha\beta\gamma\delta} = e^{\ a}_{(\alpha)} \ e^{\ b}_{(\beta)} \ e^{\ c}_{(\gamma)} \ e^{\ d}_{(\delta)} \ A_{abcd} \tag{3.25}$$

where Greek indices range over 0,1,2,3 and

$$e^{\ a}_{(\alpha)} = \{ u^a, P^a, Q^a, R^a \} .$$

(i) <u>Physical components of $\underset{(1)}{\sigma}_{ab}$</u> :

From (3.25), we write

$$\underset{(1)}{\sigma}_{\alpha\beta} = e^{\ a}_{(\alpha)} \ e^{\ b}_{(\beta)} \ \underset{(1)}{\sigma}_{ab}$$

We have atmost two non-zero components of $\underset{(1)}{\sigma}_{ab}$ and so we evaluate $\underset{(1)}{\sigma}_{22}$, viz.,

$$\underset{(1)}{\sigma}_{22} = e^a_{(2)} \, e^b_{(2)} \, \underset{(1)}{\sigma}_{ab}$$

$$= Q^a \, Q^b \, \underset{(1)}{\sigma}_{ab}$$

by (3.16), above expression becomes

$$\underset{(1)}{\sigma}_{22} = Q^a \, Q^b \, \frac{1}{K_1} \left(\dot{u}_{a;b} + \dot{u}_{b;a} \right) - \perp_{ab} Q^a \, Q^b \underset{(1)}{\theta}$$

since $Q^a \, u_a = 0$, $Q^a \, \dot{u}_a = 0$.

or

$$\underset{(1)}{\sigma}_{22} = \frac{2 \; \dot{u}_{a;b}}{K_1} \, Q^a \, Q^b - g_{ab} \; Q^a \, Q^b \underset{(1)}{\theta}$$

i.e. $$- \underset{(1)}{\sigma}_{33} = \underset{(1)}{\sigma}_{22} = \frac{2}{K_1} \; \dot{u}_{a;b} \, Q^a \, Q^b + \underset{(1)}{\theta} \; . \qquad (3.26)$$

Now, from (3.25)

$$\underset{(1)}{\sigma}_{23} = Q^a \, R^b \, \underset{(1)}{\sigma}_{ab}$$

by (3.16), we have

$$\underset{(1)}{\sigma}_{23} = Q^a \, R^b \; \frac{\dot{u}_{a;b} + \dot{u}_{b;a}}{R_1} - \perp_{ab} Q^a \, R^b \underset{(1)}{\theta}$$

the equation (3.9), (3.19) gives that

$$\underset{(1)}{\sigma}_{23} = + \underset{(1)}{\sigma}_{32} = \frac{1}{K_1} \left(\dot{u}_{a;b} + \dot{u}_{b;a} \right) Q^a \, R^b. \qquad (3.27)$$

We summerize these results -

$$\underset{(1)}{\sigma}_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{K_1}\dot{u}_{a;b}\, Q^a Q^b + \underset{(1)}{\theta} & \frac{1}{K_1}(\dot{u}_{a;b} + \dot{u}_{b;a})\, Q^a R^b \\ 0 & 0 & \frac{1}{K_1}(\dot{u}_{a;b} + \dot{u}_{b;a}) Q^a R^b & -\frac{2}{K_1}\dot{u}_{a;b}\, Q^a Q^b + \underset{(1)}{\theta} \end{bmatrix}.$$

<u>Note</u> : Shear free P^a is characterized by the two conditions

$\underset{(1)}{\sigma}_{22} = 0$, $\underset{(1)}{\sigma}_{23} = 0$, *these are satisfied*

when P^a is a killing vector field.

(ii) <u>Physical components of $\underset{(1)}{\omega}_{ab}$</u> :

From (3.25), we write

$$\underset{(1)}{\omega}_{\alpha\beta} = e^{a}_{(\alpha)}\; e^{b}_{(\beta)}\; \underset{(1)}{\omega}_{ab}$$

but we have only one non-zero independant component of $\underset{(1)}{\omega}_{ab}$,

i.e. $\underset{(1)}{\omega}_{23} = Q^a R^b \underset{(1)}{\omega}_{ab}$

by using (3.20) in above equation, we have -

$$\underset{(1)}{\omega}_{23} = - \underset{(1)}{\omega}_{32} = \frac{1}{K_1} \left(\dot{u}_{a;b} - \dot{u}_{b;a} \right) Q^a R^b \qquad (3.28)$$

we summerize these results

$$\underset{(1)}{\omega}_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{K_1}(\dot{u}_{a;b} - \dot{u}_{b;a}) Q^a R^b \\ 0 & 0 & -\frac{1}{K_1}(\dot{u}_{a;b} - \dot{u}_{b;a}) Q^a R^b & 0 \end{bmatrix}.$$

<u>Note</u> : Irrotational congruence P^a can be described through

$$\dot{u}_{a;b} = \dot{u}_{b;a}$$

i.e. $\dot{u}_a$ is a harmonic congruence.

Section-3 :

Serret-Frenet formulae, transport laws and physical components in terms of Ricci Rotation Coefficients :

(1) Ricci rotation coefficients (Scalars) :

The set of invariants γ_{lhk}, defined by the equations

$$\gamma_{lhk} = e_{(l)|a;b} \; e_{(h)|}{}^{a} \; e_{(k)|}{}^{b} \quad \text{(Eisenhart,1960)} \qquad \ldots \qquad (3.29)$$

where l, h, k range over (0, 1, 2, 3) and where

$$e_{(0)}{}^{a} = u^{a},\; e_{(1)}{}^{a} = P^{a},\; e_{(2)}{}^{a} = Q^{a},\; e_{(3)}{}^{a} = R^{a} \qquad (3.30)$$

is called as Ricci rotation coefficients (Scalars) with properties

$$\gamma_{lhk} + \gamma_{hlk} = 0 \qquad (3.31)$$

$$\gamma_{llk} = 0 \qquad \text{(l is not dummy)} \qquad (3.32)$$

(2) Transport laws of the space-like congruence P^a in terms of u^a, $\dot{u}^a$, $\ddot{u}^a$:

(i) By using (2.5) in (3.12) we have

$$u^{a}{}_{;b}\, P^{b} = \frac{\ddot{u}^{a}}{K_1} - \frac{\dot{K}_1}{K_1{}^{2}}\, \dot{u}^{a} - K_1\, \dot{u}^{a} + \frac{1}{k_1{}^{3}}\, \dot{u}^{a}\, \dot{u}_{b;c}\dot{u}^{c}\, u^{b}$$

or

$$u^a{}_{;b}\, P^b = K_2\, Q^a + \left(\frac{1}{K_1{}^2}\ \dot{u}_{b;c}\, \dot{u}^c\, u^b\right) P^a \qquad (3.33)$$

We now express these 4 equations in terms of γ_{lhk}.

(ii) By using (2.5) in (3.13), we have

$$Q^a{}_{;b}\, P^b = K_2\, u^a + \left(\frac{1}{K_1 K_2}\ \dot{u}_{b;c}\, \dot{u}^c\, Q^b\right) P^a . \qquad (3.34)$$

(iii) By using (2.5) in (3.14), we get

$$R^a{}_{;b}\, P^b = \frac{1}{K_1{}^5 K_2}\ \dot{u}^a\, \dot{u}_{b;c}\, \dot{u}^c\ \eta^{blmn}\, u_l\, \dot{u}_m\, \ddot{u}_n$$

$$\text{i.e., } R^a{}_{;b}\, P^b = \left(\frac{1}{K_1{}^2}\ \dot{u}_{b;c}\, \dot{u}^c\, R^b\right) P^a . \qquad (3.35)$$

(3) <u>GSF formulae in terms of Ricci rotation coefficients</u> :

By contracting expressions (2.10), (2.11), (2.12) and (2.13) with u^a, P^a, Q^a, R^a we get

$$-\gamma_{010} = \gamma_{100} = K_1$$

$$-\gamma_{120} = \gamma_{210} = K_2$$

$$-\gamma_{230} = \gamma_{320} = K_3$$

and remaining

$$\gamma_{020} = \gamma_{030} = \gamma_{130} = \gamma_{200} = \gamma_{300} = \gamma_{310} = 0 \qquad (3.36)$$

(4) Transport laws in terms of Ricci rotation coefficients :

By contracting equation (3.33) with Q^a

$$\gamma_{021} = -K_2 \qquad (3.37)$$

and equation (3.34), contracting with u^a

$$\gamma_{201} = K_2$$

and equation (3.35) gives

$$\gamma_{311} = \gamma_{131}$$

This is true identically $\gamma_{abc} = -\gamma_{bac}$.
[illegible] (3.33), (3.34), (3.35) (3.37)
[illegible] elegant relation (3.37).
It proves the efficiency of the Ricci formalism.

(5) Physical components in terms of Ricci rotation coefficients :

The equations (3.26), (3.27) and (3.28) give that

$$2\gamma_{122} = \underset{(1)}{\sigma}_{22} - \underset{(1)}{\theta} = -\underset{(1)}{\sigma}_{33} - \underset{(1)}{\theta}$$

but $\underset{(1)}{\theta} = \frac{1}{2}(\gamma_{122} + \gamma_{133})$

which implies that

$$\underset{(1)}{\sigma}_{22} = \frac{3}{2}\gamma_{122} + \frac{1}{2}\gamma_{133}$$

and $\underset{(1)}{\sigma}_{23} = \underset{(1)}{\sigma}_{32} = \gamma_{123} + \gamma_{132}$

and $\underset{(1)}{\omega}_{23} = \gamma_{123} - \gamma_{132}$. $\qquad$ (3.38)