CHAPTER - II

THE THREE NATURAL SPACE-LIKE CONGRUENCES (SC) ON THE WORLD-LINE OF A PARTICLE IN RELATIVISTIC CONTINUUM MECHANICS AND THE THREE CURVATURES.

Introduction :

Starting from the famous time-like congruence representing the velocity field of a particle in a contimuum, three space-like congruences are constructed in this chapter. The three curvatures of the stream-line are also evaluated using the generalized Serret-Frenet formulae in the 4-dimensional space-time of general relativity. In Section 2, it is shown that when the matter is pressure free there do not exist the space-like congruences. A simple illustration of the three space-like congruences as well as the three curvature scalars for the early universe is described in Section 3. In the last Section, the Frenet apparatus for the Definite Material Scheme is presented.

Section 1 :

Acceleration vector field as the natural space-like congruence :

If s is the arc-length parameter then the natural equation for the world-line of a particle in a contimuum is

$$x^a = x^a(s), \quad (a = 0,1,2,3).$$

If u^a is the tangent vector to this curve then

$$u^a = \frac{dx^a}{ds} \tag{2.1}$$

The metric relation $ds^2 = dx^a dx_a$ implies

$$u^a u_a = 1. \qquad (2.2)$$

Thus u^a is the unit tangent vector field on the world-line. It is a time-like vector field since $u^a u_a > 0$. The acceleration field (which is not equal to $\frac{du^a}{dt}$) is denoted by $\dot{u}^a$ and is defined by

$$\dot{u}^a = u^a{}_{;b}\, u^b \qquad (2.3)$$

where a semicolon denotes covariant differentiation. Henceforth an overhead dot means covariant derivative along the flow field. It should be noted that $\dot{u}^a$ is not a unit vector field. It is a space-like vector field since it is orthogonal to the time-like vector u^a, for, from (2.2)

$$(u^a\, u_a)^{\cdot} = 0$$

which implies

$$\dot{u}^a u_a = 0. \qquad (2.4)$$

This one orthogonal relation prompts us to examine whether an orthogonal tetrad can be constructed, by introducing two more space-like vector fields.

<u>The three natural space-like congruences</u> :

(I) <u>Expressions for P^a and K.</u>

Let P^a represent the unit vector field along the acceleration, then we have

$$P^a = \dot{u}^a / |\dot{u}|$$

or $P^a = \dot{u}^a / (-\dot{u}^k \dot{u}_k)^{\frac{1}{2}}$, since $\dot{u}^a$ is space-like

$\dot{u}^a \dot{u}_a < 0$.

Suppose that

$$P^a = \dot{u}^a / K_1 , \quad (2.5)$$

Here K_1 is a scalar field called as the <u>first curvature</u> of the world-line and it is given by

$$K_1 = (-\dot{u}^a \dot{u}_a)^{\frac{1}{2}}. \quad (2.6)$$

Suppose Q^a, R^a represent the two unit vector fields which are orthogonal to both u^a and P^a. Then we have the algebraic relations

$$P^a Q_a = 0 \quad (2.7)$$

$$P^a P_a = Q^a Q_a = -1 \quad (2.8)$$

and the differential relations (GSF formulae) as described in Chapter-I (1.45)

$$\begin{bmatrix} \dot{u}^a \\ \dot{P}^a \\ \dot{Q}^a \\ \dot{R}^a \end{bmatrix} = \begin{bmatrix} 0 & K_1 & 0 & 0 \\ K_1 & 0 & K_2 & 0 \\ 0 & -K_2 & 0 & K_3 \\ 0 & 0 & -K_3 & 0 \end{bmatrix} \begin{bmatrix} u^a \\ P^a \\ Q^a \\ R^a \end{bmatrix}$$

or equivalently

$$\dot{u}^a = K_1 P^a \tag{2.10}$$

$$\dot{P}^a = K_1 u^a + K_2 Q^a \tag{2.11}$$

$$\dot{Q}^a = -K_2 P^a + K_3 R^a \tag{2.12}$$

$$\dot{R}^a = -K_3 Q^a \tag{2.13}$$

where K_2, K_3 are called the second and the third curvatures of the stream-line.

(II) Expression for Q^a and K_2 :

From the expression (2.10) we write

$$\dot{P}^a = (\dot{u}^a / K_1)^{\cdot}$$

$$= (\ddot{u}^a / K_1) - (\dot{K}_1/K_1{}^2)\dot{u}^a .$$

In order to compare this with GSF formula (2.11) we add and substract $K_1 u^a$, i.e.,

$$\dot{P}^a = K_1 u^a + \frac{\ddot{u}^a}{K_1} - \frac{\dot{K}_1}{K_1^2} \dot{u}^a - K_1 u^a$$

$$\text{i.e., } \dot{P}^a = K_1 u^a + W^a, \text{ say}. \tag{2.14}$$

Accordingly we get

$$W^a W_a = (\frac{\ddot{u}^a}{K_1} - \frac{\dot{K}_1}{K_1{}^2} \dot{u}^a - K_1 u^a) (\frac{\ddot{u}_a}{K_1} - \frac{\dot{K}_1}{K_1^2} \dot{u}_a - K_1 u_a)$$

$$= \frac{\dddot{u}^a \dddot{u}_a}{K_1^2} - \frac{2\dot{K}_1}{K_1^3}\ddot{u}^a\dddot{u}_a - 2\ddot{u}^a u_a - \left(\frac{\dot{K}_1}{K_1}\right)^2 + K_1^2 \qquad (2.15)$$

on using (2.4), (2.6)

<u>Convenient expression for $\ddot{u}^a\dddot{u}_a$ and $\ddot{u}^a\dot{u}_a$</u> :

The relation

$$\dot{u}^a \dot{u}_a = -K_1^2$$

implies

$$(\dot{u}^a \dot{u}_a)^{\cdot} = (-K_1^2)^{\cdot}$$

i.e.

$$\boxed{\ddot{u}^a \dot{u}_a = -K_1\dot{K}_1} \qquad (2.16)$$

On covariantly differentiating the equation

$$\dot{u}^a u_a = 0$$

in the direction of u^a, we get

$$(\dot{u}^a u_a)^{\cdot} = 0$$

i.e. $\ddot{u}^a u_a = -\dot{u}^a\dot{u}_a$

or $\boxed{\ddot{u}^a u_a = K_1^2}$. $\qquad (2.17)$

Substituting (2.16) and (2.17) in (2.15), we obtain

$$W^a W_a = \frac{\ddot{u}^a \ddot{u}_a}{K_1^2} + \frac{\dot{K}_1^2}{K_1^2} - K_1^2$$

$$= -K_2^2 \text{ , say} \qquad (2.18)$$

in order to agree with the GSF formulae. So the relation

$$\frac{W^a}{K_2} \cdot \frac{W_a}{K_2} = -1$$

suggests that

$$Q^a = \frac{W^a}{K_2}$$

or
$$\boxed{Q^a = \frac{1}{K_2}\left(\frac{\ddot{u}^a}{K_1} - \frac{\dot{K}_1}{K_1^2}\dot{u}^a - K_1 u^a\right)} \qquad (2.19)$$

and
$$K_2 = \left(-\frac{\ddot{u}^a \ddot{u}_a}{K_1^2} - \left(\frac{\dot{K}_1}{K_1}\right)^2 + K_1^2\right)^{\frac{1}{2}}. \qquad (2.20)$$

Expression for R^a :

Since R^a is orthogonal to u^a, P^a, Q^a, it should be of the form

$$R^a = e \; \eta^{abcd} u_b P_c Q_d \; . \qquad (2.20)$$

Where η^{abcd} is the Levi Civita tensor and e is to be chosen such that $R^a R_a = -1$. With this intention, we evaluate

$$R^a R_a = e^2 \eta^{abcd} \eta_{almn} u_b P_c Q_d u^l P^m Q^n .$$

According to Stephani (1982),

$$\eta^{abcd} \eta_{almn} = - \begin{vmatrix} \delta^b_l & \delta^c_l & \delta^d_l \\ \delta^b_m & \delta^c_m & \delta^d_m \\ \delta^b_n & \delta^c_n & \delta^d_n \end{vmatrix} .$$

Hence we have

$$R^a R_a = -e^2 \{ \delta^b_l (\delta^c_m \delta^d_n - \delta^c_n \delta^d_m) + \delta^c_l (\delta^b_n \delta^d_m - \delta^b_m \delta^d_n)$$

$$+ \delta^d (\delta^b_m \delta^c_n - \delta^b_n \delta^c_m) \} u_b P_c Q_d u^l P^m Q^n,$$

on expansion

$$= -e^2 \{ (1) ((-1)(-1) - 0) + 0 + 0 \},$$

on expansion by (2.2) (2.7)(2.8).

$$R^a R_a = -e^2.$$

It follows from $R^a R_a = -1$, that $e^2 = 1$ and so

$$e = \pm 1. \qquad (2.21)$$

Substituting the expressions for Q^a and P^a in (2.20), we get

$$R^a = e \eta^{abcd} u_b \frac{\dot{u}_c}{K_1} \left(\frac{\ddot{u}_d}{K_1 K_2} - \frac{K_1}{K_1{}^2 K_2} \dot{u}_d - \frac{K_1}{K_2} u_d \right)$$

$$R^a = \frac{e}{K_1{}^2 K_2} \eta^{abcd} u_b \dot{u}_c \ddot{u}_d \qquad (2.22)$$

since the inner product $\dot{u}_c \, \dot{u}_d \; \eta^{abcd}$ vanishes due to the fact that $\dot{u}_c \cdot \dot{u}_d$ is symmetric in (c,d) and η^{abcd} is skew symmetric in (c,d). Similarly $u_b \, u_d \, \eta^{abcd}$ also vanishes.

Expression for K_3 :

By GSF formula

$$\dot{Q}^a = -K_2 \, P^a + K_3 \, R^a$$

this gives us

$$\dot{Q}^a \, R_a = 0 + K_3 \, R^a \, R_a$$

implies

$$K_3 = -\dot{Q}^a \, R_a \; . \qquad (2.23)$$

We note from (2.22)

$$u^a \, R_a = 0, \quad \dot{u}^a R_a = 0, \quad \ddot{u}^a \, R_a = 0 \; , \qquad (2.24)$$

Since $\dot{u}_a \, \dot{u}_c$ and $\ddot{u}_a \, \ddot{u}_b$ are symmetric in (a,c) and (a,b) and η^{abcd} is skew symmetric.

Now from expression (2.19)

$$\dot{Q}^a = \left(\frac{\ddot{u}^a}{K_1 K_2} - \frac{\dot{K}_1}{K_1{}^2 K_2} \, \dot{u}^a - \frac{K_1}{K_2} \, u^a \right)^{\bullet}$$

$$= \frac{\dddot{u}^a}{K_1 \; K_2} + \text{terms in } \ddot{u}^a + \text{terms in } \ddot{u}^a + \text{term in } \dot{u}^a$$

$$+ \text{ terms in } \dot{u}^a + \text{ terms in } u^a \text{ , on using Leibnitz rule.}$$

$$\dot{Q}^a R_a = \frac{\dddot{u}^a}{K_1 K_2} R_a + 0 + 0 + 0 + 0 + 0 .$$

Now (2.24) and (2.23) imply that

$$K_3 = -\frac{R_a \dddot{u}^a}{K_1 K_2} = -\frac{R^a \dddot{u}_a}{K_1 K_2} .$$

Accordingly from (2.22) we have

$$K_3 = -e \frac{1}{K_1^3 K_2^2} \eta^{abcd} u_b \dot{u}_c \ddot{u}_d \dddot{u}_a . \tag{2.25}$$

Since K_1, K_2, $K_3 > 0$, we should have to choose $e = -1$, consequently

$$R^a = -\frac{1}{K_1^2 K_2} \eta^{abcd} u_b \dot{u}_c \ddot{u}_d \tag{2.26}$$

and

$$K_3 = \frac{1}{K_1^3 K_2^2} \eta^{abcd} u_b \dot{u}_c \ddot{u}_d \dddot{u}_a . \tag{2.27}$$

Finally we obtain the natural tetrad as

$$\left\{ u^a, \frac{\dot{u}^a}{K_1}, \frac{\ddot{u}^a}{K_1 K_2} - \frac{\dot{K}_1}{K_1^2 K_2} \dot{u}^a - \frac{K_1}{K_2} u^a , -\frac{1}{K_1^2 K_2} \eta^{abcd} u_b \dot{u}_c \ddot{u}_d \right\} .$$

Note : We have the completeness relation

$$g^{ab} = u^a u^b - P^a P^b - Q^a Q^b - R^a R^b \tag{2.28}$$

Section 2 :

On the non-existence of natural space-like vector fields for the pressure-free matter :

We note that, the matter as a contimuum of dust particles (pressure-free matter) is characterized by the stress tensor

$$T^{ab} = \rho \, u^a u^b, \qquad (2.29)$$

where ρ is the density.

Hence the identity (energy-balance equations)

$$T^{ab}{}_{;b} = 0 \qquad (2.30)$$

implies that

$$(\rho \, u^a u^b)_{;b} = 0$$

i.e., $$(\dot{\rho} + \rho \, u^b{}_{;b}) \, u^a + \rho \, \dot{u}^a = 0 \; . \qquad (2.31)$$

Now consider the equation of contimuity (time component of (2.30))

$$T^{ab}{}_{;b} \, u_a = 0 \qquad (2.32)$$

which together with (2.31) gives

$$\dot{\rho} + \rho \, u^b{}_{;b} = 0.$$

Substituting this in (2.31) we get since $\rho \neq 0$, that

$$\boxed{\dot{u}^a = 0} \qquad (2.33)$$

This obviously implies that $P^a = 0$. It means that for pressure free matter, the natural tetrad does not exist, i.e., the tetrad can not be constructed, when there is no interaction between particles of the continuous medium.

The natural space-like vector fields are thus trivial for dust distribution of matter. We infer that the three natural space-like congruences exist for interacting matter only. The simplest such matter is presented in the next section.

Section 3 :

Illustrations of space-like congruences for the early universe :

The early universe is radiation dominated era and the equation of state is given by Wald (1984):

$$\rho = \frac{3}{c^2} p \tag{2.32}$$

where p is the pressure and ρ is the density of matter. This is referred as disordered radiation zone. The energy momentum tensor for such a distribution of matter is

$$T^{ab} = \rho u^a u^b - \frac{p}{c^2} (g^{ab} - u^a u^b)$$

or

$$T^{ab} = \frac{p}{c^2} (4 u^a u^b - g^{ab}). \tag{2.33}$$

From the energy balance equations

$$T^{ab}{}_{;b} = 0 ,$$

we get

$$4 (\dot{p} + p \theta) u^a + 4 p \dot{u}^a - p^{,a} = 0 \tag{2.34}$$

where $\theta = u^a{}_{;a}$. The acceleration field is

$$\dot{u}^a = \frac{-(\dot{p} + p \theta)}{p} u^a + \frac{p^{;a}}{4 p} . \tag{2.35}$$

The equation of continuity

$$T^{ab}{}_{;b} u_a = 0$$

yields

$$\dot{p} + p\,\theta = \frac{\dot{p}}{4}\,, \quad \text{since} \quad p^{,a}\,u_a = \dot{p}\,. \tag{2.36}$$

Substituting in (2.35) gives the equation of stream lines

$$\dot{u}^a = \frac{-\dot{p}\,u^a + p^{,a}}{4\,p}\,. \tag{2.37}$$

This expression gives the acceleration field of a particle in the early universe. The relation

$$K_1 = \sqrt{-\dot{u}^a\,\dot{u}_a}$$

implies

$$K_1 = \frac{1}{4p}\left(\dot{p}^2 - F\right)^{\frac{1}{2}} \tag{2.38}$$

where F represents the magnitude of pressure gradient

$$F = p_{,a}\,p^{,a}. \tag{2.39}$$

Therefore,

$$P^a = \left(\dot{p}^2 - F\right)^{-\frac{1}{2}}\left(-\dot{p}\,u^a + p^{,a}\right). \tag{2.40}$$

Now for the evaluation of Q^a, R^a, K_2 we need $\ddot{u}^a$. From (2.37) we readily have

$$\ddot{u}^a = \frac{1}{4\,p}\left(\ddot{p}\,u^a - \dot{p}\,\dot{u}^a + \dot{p}^{,a}\right) + \frac{1}{4p^2}\left(\dot{p}^2\,u^a - \dot{p}\,p^{,a}\right)$$

or

$$\ddot{u}^a = \frac{1}{4p^2}\left(p\,\ddot{p} + \dot{p}^2\right)u^a - \frac{\dot{p}}{4p}\,\dot{u}^a + \frac{\dot{p}^{,a}}{4p} - \frac{1}{4p^2}\,\dot{p}\;p^{,a}$$

... (2.41)

Remark : The inner product $\ddot{u}^a u_a \neq 0$, implies that $\ddot{u}^a$ is not space-like although $\dot{u}^a$ is space-like. Therefore,

$$K_2 = \{ \frac{1}{16\, p^2 K_1{}^2} \Big[(\ddot{p}^2 + K_1{}^2 + 2\dot{p}\, \dot{p}_{,a}\, \dot{u}^a - \dot{p}^{,a}\, \dot{p}_{,a}) - \frac{2\,\dot{p}}{p} (\dot{p}\,\ddot{p} + \dot{p}\, p_{,a}\, \dot{u}^a - \dot{p}_{.a}\, p^{,a}) - \frac{\dot{p}^2}{p^2} (F - \dot{p}^2) - \frac{1}{K_1} (\frac{1}{4\,p} (\dot{p}\,\ddot{p} + \dot{p}^2 - \dot{p}^{,a}\, p_{,a}) - \frac{\dot{p}}{4p^2} (\dot{p}^2 - F)^2 + 16\, p^2\, K_1{}^4 \Big] \}^{\frac{1}{2}} \tag{2.42}$$

and

$$Q^a = \frac{1}{K_1\, K_2} \Big[(- \frac{\ddot{p}}{4p} + \frac{\dot{p}^2}{4p^2} + K_1)\, u^a + \{ - \frac{\dot{p}}{4p} + \frac{\dot{p}}{64\, p^3\, K_1} (\ddot{p} + \dot{p} + \frac{\dot{F}}{2} - \frac{\dot{p}^2 - F}{p}) \}\, \dot{u}^a + (1 - \frac{\dot{p}^2}{4p^4})\, p^{,a} \Big] .$$

Now using (2.37) and puting in terms of $\dot{u}^a$, u^a

$$Q^a = \frac{1}{K_1\, K_2} \Big[(- \frac{\ddot{p}}{4p} + \frac{\dot{p}^2}{4p^2} + K_1 - \dot{p} + \frac{\dot{p}^3}{4p^4})\, u^a + \{- \frac{\dot{p}}{4p} + \frac{\dot{p}}{64\, p^3 K_1} (\ddot{p} + \dot{p} + \frac{\dot{F}}{2} - \frac{\dot{p}^2 - F}{p}) + 4p - \frac{\dot{p}}{p^3} \}\, \dot{u}^a \Big] \tag{2.43}$$

and -

$$R^a = \frac{-1}{K_1^2 K_2 64 p^3} \eta^{abcd} u_b p_{,c} (4 p \dot{p}_{,d} - 5\ddot{p} p_{,d}) \quad \ldots (2.44 a)$$

i.e., $R^a = \frac{-1}{K_1^2 K_2 16 p^2} \eta^{abcd} u_b p_{,c} \dot{p}_{,d}$, ... (2.44 b)

since $p_{,c} p_{,d}$ is symmetric and η^{abcd} is skew symmetric in (c,d).

Now for the expression of K_3 , we need $\dddot{u}^a$. By definition

$$\dddot{u}^a = (\ddot{u}^a)^{\cdot}$$

and by using (2.41), we get

$$\dddot{u}^a = \frac{1}{4p} \Big[(\dddot{p} u^a - \dot{p} \ddot{u}^a + \ddot{p}^{,a}) - \frac{\dot{p}}{p} (\ddot{p} u^a - \dot{p} \dot{u}^a + \dot{p}^{,a}) + \frac{\dot{p}}{p}(2 \ddot{p} u^a + \dot{p} \dot{u}^a - \frac{\ddot{p} \dot{p}^{,a}}{\dot{p}} - \dot{p}^{,a}) - \frac{2 \dot{p} \ddot{p}}{p^3} (\ddot{p} u^a - \dot{p} \dot{u}^a + \dot{p}^{,a} \Big]$$

...(2.45)

therefore,

$$K_3 = \frac{-1}{K_1^3 K_2^2} \eta^{abcd} u_b \dot{u}_c \ddot{u}_d \dddot{u}_a .$$

By (2.37), (2.41) and(2.45) above expression becomes

$$K_3 = \frac{-1}{64\, p^3\, K_1{}^3 K_2{}^2}\ \eta^{abcd}\ u_b\, p_{,c}\, \dot{p}_{,d}\, \ddot{p}_{,a} \qquad (2.46)$$

since $p_{,c}\, p_{,d}$, $\dot{p}_{,d}\, \dot{p}_{,a}$ and $u_b u_c$ are symmetric and η^{abcd} is skewsymmetric.

<u>Inferences</u> : Some sufficient criteria for the vanishing of K_3.

(i) When $\ddot{p}_{,a} = 0$,

this gives

$K_3 = 0$,

in this case $K_1 \neq 0$ and $K_2 \neq 0$.

This means that

$\ddot{p} = A$, a constant,

For a <u>comoving</u> observer $u^a = (0, 0, 0, u^o)$

$\dot{p} = A s + B$

where s is arc-length parameter and B is a constant and

$p = \frac{A}{2} s^2 + B s + C$.

(ii) When $4\dot{p}\, \dot{p}_{,d} = 5\ddot{p}\, p_{,d}$ we get $K_3 = 0$ by (2.44 a)

i.e., $\dot{p}_{,d} = \frac{5}{4} \frac{\ddot{p}}{p}\ p_{,d}$.

Section 4 :

The space-like congruences for DEFINITE MATERIAL SCHEME

Introduction :

The field of stress-energy-momentum tensor T^{ab} in a domain of space-time is known as energy scheme. If T^{ab} have time-like eigen vector then the scheme is known as normal scheme. A normal scheme with positive eigen-value is called material scheme. The stress-energy momentum tensor T^{ab} is said to be positive definite if and only if.

$$\begin{vmatrix} T^{00} \end{vmatrix} > 0, \quad \begin{vmatrix} T^{00} & T^{01} \\ T^{10} & T^{11} \end{vmatrix} > 0, \quad \begin{vmatrix} T^{00} & T^{01} & T^{02} \\ T^{10} & T^{11} & T^{12} \\ T^{20} & T^{21} & T^{22} \end{vmatrix} > 0,$$

$$\begin{vmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{vmatrix} > 0.$$

The corresponding energy scheme is known as definite Scheme.

(i) Expression for the space-like vector field $\hat{u}^a$ on the stream-lines of a Definite Material Scheme :

Following Carter and Quintana (1977) we consider (with due corrections to the signature of the metric)

$$ds^2 = c^2 \; d\,T^2 \tag{2.47}$$

and

$$u^a = \frac{dX^a}{dT} \tag{2.48}$$

where u^a are the components of the 4- velocity vector, dT is proper time differential, for which

$$u^a\,u_a = c^2. \tag{2.49}$$

The energy momentum tensor for a definite material scheme (Lichnerowicz 1955) is

$$T^{ab} = \rho\, u^a\, u^b - p^{ab} \tag{2.50}$$

where ρ is total (relativistic) energy density of a material measured by u^a, and p^{ab} are the components of the pressure tensor which satisfies

$$\left.\begin{aligned} p^{ab} &= p^{ba} \\ \text{and} \qquad u^a\, p_{ab} &= 0. \end{aligned}\right\} \tag{2.51}$$

Here $T^{ab}\,u_a = \rho\, u^b$ and so ρ is the time-like eigenvalue.

Einstein field equations for gravitating matter, whose stress-energy-momentum is characterized by the tensor T^{ab} are

$$\frac{-8\pi G}{c^4} T^{ab} = R^{ab} - \frac{1}{2} R g^{ab} \tag{2.52}$$

where G is the universal constant of gravitation. The energy balance equation

$$T^{ab}{}_{;b} = 0$$

yields

$$\dot{\rho} u^a + \rho \dot{u}^a + \rho u^a u^b{}_{;b} - p^{ab}{}_{;b} = 0$$

$$\text{i.e., } \dot{u}^a = \frac{1}{\rho} \left(f^a - (\dot{\rho} + \rho \theta) u^a \right) \tag{2.53}$$

where $\boxed{f^a = p^{ab}{}_{;b}}$ (2.54)

and $\theta = u^a{}_{;a}$.

Now by the equation of local conservation of mass

$$T^{ab}{}_{;b} u_a = 0$$

we get

$$\dot{\rho} + \rho \theta = \frac{1}{c^2} f^a u_a \tag{2.55}$$

eliminating $(\dot{\rho} + \rho \theta)$ from (2.54) by using (2.55) we have

$$\dot{u}^a = \frac{1}{\rho} f^a - \frac{1}{\rho c^2} f^b u_b u^a . \tag{2.56}$$

Equation (2.56) expresses the acceleration vector field of the definite material scheme in terms of the dynamical tensor f^a which is the divergence of the arbitrary pressure tensor p^{ab}.

Note 1 :

Role of conservation of mass : The State of the continuous matter is sometimes subject to the conservation law

$$(\mu u^a)_{;a} = 0$$

where μ is the particle rest mass density measured by an observer travelling with velocity u^a (Ellis, 1971).

Note 2 :

The 4-vector u^a is not a unit time-like vector as

$$u^a u_a = c^2.$$

To adopt this general formalism for our dissertation we put

$$U^a = \frac{1}{c} u^a \tag{2.57}$$

to get

$$U^a U_a = 1 \tag{2.58}$$

where U^a is the unit velocity vector of the world line of a particle in the definite material scheme.

(ii) A convenient expression for the acceleration field $\dot{U}^a$:

Equation (2.57) gives

$$\dot{u}^a = c\,\dot{U}^a$$

since c is constant, and therefore (2.56) becomes

$$\dot{U}^a = \frac{1}{\rho c}\,(f^a - f^k U_k U^a)$$

or

$$\dot{U}^a = \rho^{-1} c^{-1}\,(f^a - A\,U^a) \qquad (2.59)$$

where $A = f^k U_k$.

Note 3 :

This $\dot{U}^a$ is not a unit vector field. The normalization of this vector will be utilized in the construction of the tetrad.

Note 4 :

The expression for $\dot{U}^a$ is an identity

$$\dot{U}^a = \left(\frac{1}{\rho c}\right) f^a + \left(\frac{-A}{\rho c}\right) U^a$$

as it is derived from the identity

$$T^{ab}{}_{;b} = 0.$$

(iii) Expression for the space-like vector field P^a for a definite material scheme :

We recall that (U^a, P^a, Q^a, R^a) is the tetrad on the path of material point in the contimuum. Since $K_1 = \sqrt{-\dot{U}^a \dot{U}_a}$

we get

$$K_1 = \frac{1}{\rho c} (A^2 - f^a f_a)^{\frac{1}{2}} \tag{2.60}$$

and $$P^a = (A^2 - f^K f_K)^{-\frac{1}{2}} (f^a - A U^a) , \tag{2.61}$$

Remark : We have

$$f^a = (A^2 - f^K f_K)^{\frac{1}{2}} P^a + A U^a$$

which implies

$$\left. \begin{array}{l} f^a Q_a = 0 \\ \text{and} \\ f^a R_a = 0 \end{array} \right\} \tag{2.62}$$

by orthogonal relations. Thus the divergence of the general pressure tensor lies in the plane of P^a, U^a.

(iv) Expression for space-like vector field Q^a for a definite material scheme :

For expression Q^a, we require $\ddot{U}^a$, therefore,

$$\ddot{U}^a = \left[\frac{1}{\rho c} (f^a - A U^a) \right]^{\bullet}$$

gives

$$\ddot{U}^a = \rho^{-1} c^{-1} \{ \dot{f}^a - \rho^{-1} (\dot{\rho} + c^{-1} A) f^a - (\dot{A} - \rho^{-1} c^{-1} A^2 - \rho^{-1} \dot{\rho} A) U^a \} . \tag{2.63}$$

Hence, we have

$$\ddot{U}^a \ddot{U}_a = \frac{1}{\rho^2 c^2} \left[\dot{f}^a \dot{f}_a - 2 (\dot{\rho}/\rho + A/\rho c) \dot{f}^a f_a \right.$$

$$- 2 (\dot{A} - \frac{A^2}{\rho C} - \dot{\rho}/\rho \quad A) \dot{f}^a U_a + (\dot{\rho}/\rho + A/\rho C)^2 f^a f_a$$

$$+ (\dot{A}^2 - \frac{2\dot{\rho}}{\rho^2 c} \quad A^3 - \frac{A^4}{\rho^2 C^2}) \Big] \qquad (2.64)$$

and

$$\ddot{U}^a \dot{U}_a = \frac{1}{\rho^2 C^2} \Big[\dot{f}^a f_a - A \dot{f}^a U_a - (\frac{\dot{\rho}}{\rho} + \frac{A}{\rho C}) f^a f_a + A \dot{A} \Big].$$

$$\ldots (2.65)$$

Using the equations (2.64) and (2.65), we have expression of K_2 as

$$K_2 = (- \frac{\ddot{U}^a \ddot{U}_a}{K_1^{\ 2}} + \frac{(\ddot{U}^a \dot{U}_a)^2}{K_1^{\ 4}} + K_1^{\ 2})^{\frac{1}{2}}$$

i.e.

$$K_2 = \frac{1}{(A^2 - f^k f_k)^{\frac{1}{2}}} \Big[- \dot{f}^a \dot{f}_a + 2 (\dot{\rho}/\rho + A/\rho c) \dot{f}^a f_a$$

$$+ 2(\dot{A} - A^2/\rho C + \dot{\rho}/\rho \quad A) \dot{f}^a U_a - \{ (\dot{\rho}/\rho + A/\rho c)^2 + \frac{1}{\rho^2 C^2} \}$$

$$f^a f_a - (\frac{A^2}{\rho^2 C^2} (\dot{A}^2 - \frac{2\dot{\rho}}{\rho^2 C} \quad A^3 - \frac{A^4}{\rho^2 c^2}) +$$

$$\frac{1}{(A^2 - f^b f_b)} \quad B^2 \Big]^{\frac{1}{2}} \qquad (2.66)$$

where

$$B = \dot{f}^a f_a - A \dot{f}^a U_a - (\dot{\rho}/\rho + A/\rho C) f^a f_a + A\dot{A} \qquad (2.67)$$

therefore,

$$Q^a = \frac{1}{K_1 K_2} \left(U^a + \frac{\ddot{U}^b \ddot{U}_b}{K_1} \dot{U}^a - K_1^2 U^a \right)$$

becomes

$$Q^a = \frac{1}{K_1 K_2 C} \Big[\dot{f}^a - (\dot{\rho}/\rho + A/\rho C - \frac{B}{K_1}) f^a$$

$$- (\dot{A} - \frac{A^2}{\rho C} - \dot{\rho}/\rho \; A - \frac{AB}{K_1} + \rho C K_1^2) U^a \Big] \qquad (2.68)$$

(v) <u>Expression for space-like vector field R^a for a definite material scheme</u> :

We know that

$$R^a = \frac{-1}{K_1^2 K_2} \eta^{abcd} U_b \dot{U}_c \ddot{U}_d$$

by using (2.59) and(2.63), we have

$$R^a = \frac{-1}{K_1^2 K_2} \eta^{abcd} U_b \frac{1}{\rho c} (f_c - A U_c) \cdot$$

$$\frac{1}{\rho c}\Big[\dot{f}_d - (\dot{\rho}/\rho + A/\rho c) f_d - (\dot{A} - A^2/\rho c - \frac{\dot{\rho} A}{\rho}) U_d \Big]$$

i.e. $$R^a = \frac{-1}{K_1^2 K_2 \rho^2 C^2} \eta^{abcd} U_b f_c f_d \tag{2.69}$$

since $U_b U_c$, $U_b U_d$ and $f_c f_d$ are symmetric and η^{abcd} is skew symmetric.

(vi) Expression for the third curvature K_3 :

For this expression we require $\dddot{U}^a$, therefore, from expression (2.63) we have

$$\dddot{U}_a = \frac{1}{\rho C}\Big[\ddot{f}_a - (\dot{\rho}/\rho + f/\rho c)\dot{f}_a - (\dot{\rho}/\rho - \dot{\rho}^2/\rho^2 + \frac{\dot{\rho}A}{\rho^3 c^2} - A^2/\rho^2 c^2)\, f_a - (\ddot{A} - \frac{A\dot{A}}{\rho C} - \frac{\dot{\rho}\dot{A}}{\rho c} + \frac{\dot{\rho}^2 A}{\rho^2} - \frac{\dot{\rho}A^2}{\rho^3 c^2} - \frac{A^3}{\rho^2 c^2})\, U_a\Big]. \tag{2.70}$$

Since

$$K_3 = \frac{-1}{K_1^3 K_2^2} \eta^{abcd} U_b \dot{U}_c \ddot{U}_d \dddot{U}_a$$

from expressions (2.59), (2.63) and (2.70), we have

$$K_3 = \frac{-1}{K_1^3 K_2^2 \rho^3 c^3} \eta^{abcd} U_b f_c \dot{f}_d \ddot{f}_a \tag{2.71}$$

Since all other terms vanish because of symmetry in $(U_b U_d)$, $(f_c f_d)$, $(\dot{f}_d \dot{f}_a)$ and skew symmetry in η^{abcd}.

Note 5 :

In the limit when $C \rightarrow \infty$, from the equation (2.59) we get that

$$\dot{U}^a \rightarrow 0 .$$

This means that the tetrad does not exist in the classical (Newtonian) continuum mechanics. Thus the tetrad

$$\left\{ U^a, (A^2 - f^b f_b)^{-\frac{1}{2}} (f^a - AU^a), \frac{1}{K_1 K_2 \rho C} (\dot{f}^a - Df^a - E U^a), \frac{-1}{K_1^2 K_2 \rho^2 C^2} \eta^{abcd} U_b f_c \dot{f}_d \right\}$$

where $D = \frac{\dot{\rho}}{\rho} + \frac{A}{\rho C} - \frac{B}{K_1}$

and $E = \dot{A} - \frac{A^2}{\rho C} - \frac{\dot{\rho} A}{\rho} - \frac{AB}{K_1} + \rho C K_1^2$

is a special non-Newtonian feature of relativistic continuum mechanics; just as the concept of a gravitational radiation is a non-Newtonian characterstic of general relativity.

Note 6 :

The tetrad $\{U^a, P^a, Q^a, R^a\}$ described in (2.57), (2.61), (2.68), (2.69) is the most general one since it correspondes to the most general material expressed in (2.50).