CHAPTER-III

Fixed points of generalised Contraction mappings in Hilbert space

1 Introduction

A new definition of generalised contraction mapping in Hilbert space is introduced. Based upon this concept some theorems of Naimpally and Singh [32] and Johnson [20] are extended.

The well-known Banach [1] contraction principle has been extended by a number of research workers working in the field of fixed point theory in several directions to different spaces which can be formulated as follows

Let X be a Banach space and C be a closed convex subset of X, then a contraction mapping T of C into itself (i.e. $\|Tx-Ty\| \leq \alpha \|x-y\|$ for some $\alpha \in [0,1)$ and for all x and y in C) has a unique point $p \in C$ such that $TP = P$.

The definition of contraction mapping has undergone successive generalisations [37] in complete metric space by R. Kannan [25], Reich [35], Hardy and Rogers [18] and Wong [43]. Hardy and Rogers [18] considered the following more general form of contraction mapping and proved some fixed point theorems.

For each x, y in complete metric space X,

$$d(Tx-Ty) \leq a_1 d(x,y) + a_2 d(x,Ty) + a_3 d(y,Tx) + \\ + a_4 d(x,Tx) + a_5 d(y,Ty),$$

where $a_i \geqslant 0$ and $\sum_{i=1}^{5} a_i < 1$.

Khan and Imdad [24] considered the above generalised contraction in Banach space in the following form :

T be a self-map of closed convex subset of a Banach space X satisfying

$$\|Tx - Ty\| \leqslant a \|x-y\| + b (\|x-Tx\| + \|y - Ty\|) + c (\|x-Ty\| + \|y-Tx\|)$$

for every x and y in C, a, b, c $\geqslant 0$ and $0 \leqslant a + 4b + 4c < 2$.

Naimpally and Singh [32] used the two contraction conditions defined by (I-1.3.24 and 1.3.25) and proved some fixed point theorems.

Ganguly [14] in his recent paper defined a generalised nonexpansive mapping in the following way.
A self map T of a subset of a normed linear space X is said to be generalised non-expansive (for definition of non-expansive map see I-1.1.5) if,

$$\|Tx-Ty\| \leqslant \max \left\{ \|x-y\| , \|x-Tx\| , \|y-Ty\| , \|x-Ty\| , \|y-Tx\| \right\}.$$

By considering above generalisations of contraction mapping in different spaces, we have introduced the following definition of generalised contraction mapping in Hilbert space and shown that our definition includes each one of the mappings defined by (I-1.1.4, 1.1.5, 1.1.6, 1.1.7, 1.1.11 and 1.1.12).

Our definition runs as follows.

Definition 1.1. (Generalised Contraction) : Let C be a closed convex subset of a Hilbert space H. A mapping $T : C \longrightarrow C$ is said to be generalised condition if for all $x, y \in C$

$$\|Tx-Ty\|^2 \leq a_1 \|x-y\|^2 + a_2 \|x-Ty\|^2 + a_3 \|y-Tx\|^2 + a_4 \|(I-T)x-(I-T)y\|^2, \qquad \ldots (1.2)$$

$$\text{where} \quad a_i \geq 0, \quad \sum_{i=1}^{4} a_i < 1. \qquad \ldots (1.3)$$

We justify our claim of generalised contraction mapping by discussing the following special cases.

Case (i) : If we put $a_2 = a_3 = a_4 = 0$, $0 < \sqrt{a_1} = K < 1$, we obtain definition (I-1.1.4).

Case (ii) : If we put $a_2 = a_3 = a_4 = 0$, $\sqrt{a_1} = 1$, we obtain definition (I-1.1.5).

Case (iii) : If we put $a_1 = 1$, $a_2 = a_3 = 0$, $a_4 < 1$, we obtain definition (I-1.1.6).

Case (iv) : If we put $a_1 = a_4 = 1$, $a_2 = a_3 = 0$, we obtain definition (I-1.1.7).

Case (v) : If we put $a_1 + a_2 + a_3 = 1$, $a_3 + a_4 < 1$ and $y = p = TP$, we obtain definition (I-1.1.11).

Case (vi) : If we put $a_1 + a_2 + a_3 = 1$, $a_3 + a_4 = 1$ and $y = P = TP$, we obtain definition (I-1.1.12).

2. Fixed Point Theorems

In this section Theorems (I-1.3.26, 1.3.27) of Naimpally and Singh [32] have been extended for the generalised contraction mapping T defined by (1.2, 1.3). For our first result it is further assumed that T is monotone mapping i.e. $(Tx - Ty, x-y) \geq 0$ for all x and y in C. Finally the result (I-1.3.20) of Johnson [20] has been generalised.

Our first result is the following theorem:

Theorem 2.1 : Let C be a closed convex subset of a real Hilbert space H. Let $T : C \longrightarrow C$ such that it satisfies (1.2) and (1.3) with $a_2 > 0$, $0 \leq a_2 + a_4 < 1$. Further we assume that T is monotone. Suppose x_0 is any point in C and the sequence $\{x_n\}$ associated with T is defined by Ishikawa scheme (I-1.3.9, 1.3.10 and 1.3.11). Suppose further that $\{\alpha_n\}$ is bounded away from zero (i.e. $\underline{\text{Lim}}\ \alpha_n = \alpha > 0$). If the sequence $\{x_n\}$ converges to P, then P is a fixed point of T.

<u>Proof</u> : Equation (I-1.3.9) implies that $x_{n+1} - x_n = \alpha_n(Ty_n - x_n)$.

Suppose $x_n \longrightarrow P$, then $\|x_{n+1} - x_n\|^2 \longrightarrow 0$ and since $\{\alpha_n\}$ is bounded away from zero, $\|Ty_n - x_n\|^2 \longrightarrow 0$. Using triangle inequality it follows that

$$\|Ty_n - P\|^2 \leq \left\{ \|Ty_n - x_n\| + \|x_n - P\| \right\}^2 \longrightarrow 0$$

as $n \longrightarrow \infty$.

i.e. $\|Ty_n - P\|^2 \longrightarrow 0$.

Using (I-1.3.12) and (I-1.3.10), where t stands for β_n we obtain the following inequalities :

$$\|y_n - x_n\|^2 = \|\beta_n Tx_n + (1-\beta_n)x_n - x_n\|^2$$

$$= \beta_n \|Tx_n - x_n\|^2 - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2$$

$$= \beta_n^2 \|Tx_n - x_n\|^2.$$

$$\leq \|Tx_n - x_n\|^2 .$$

$$\leq \left\{ \|Tx_n - Ty_n\| + \|Ty_n - x_n\| \right\}^2,$$

using tringle inequality.

$$\leq \|Tx_n - Ty_n\|^2 + \|Ty_n - x_n\|^2 +$$

$$+ 2\|Tx_n - Ty_n\| \, \|Ty_n - x_n\| \quad \ldots(2.2)$$

$$\|y_n - Tx_n\|^2 = \|\beta_n Tx_n + (1-\beta_n)\, x_n - Tx_n\|^2.$$

$$= (1-\beta_n)\ \|x_n - Tx_n\|^2 - \beta_n(1-\beta_n)$$

$$\|Tx_n - x_n\|^2.$$

$$= (1-\beta_n)^2\ \|Tx_n - x_n\|^2.$$

$$\leq \|Tx_n - x_n\|^2.$$

$$\leq \left\{\|Tx_n - Ty_n\| + \|Ty_n - x_n\|\right\}^2,$$

using triangle inequality.

$$\leq \|Tx_n - Ty_n\|^2 + \|Ty_n - x_n\|^2 +$$

$$+ 2\,\|Tx_n - Ty_n\|\ \|Ty_n - x_n\| \quad \ldots .(2.3)$$

since T satisfies (1.2), we have

$$\|Tx_n - Ty_n\|^2 \leq a_1\ \|x_n - y_n\|^2 + a_2\ \|x_n - Ty_n\|^2 +$$

$$+ a_3\ \|y_n - Tx_n\|^2 + a_4\left\{\ \|x_n - y_n\|^2 +\right.$$

$$\left. +\ \|Tx_n - Ty_n\|^2 - 2(x_n - y_n, Tx_n - Ty_n)\right\}$$

$$\leq (a_1 + a_4)\ \|x_n - y_n\|^2 + a_2\ \|x_n - Ty_n\|^2 +$$

$$+ a_3\ \|y_n - Tx_n\|^2 + a_4\ \|Tx_n - Ty_n\|^2.$$

$$\ldots (2.4)$$

Since H is a real Hilbert space and T is monotone, $(x_n - y_n, Tx_n - Ty_n) =$ $(Tx_n - Ty_n, x_n - y_n) \geq 0.$

Using relations (2.2) and (2.3) in (2.4), we obtain

$$1-(a_1+a_3+a_4)\ \|Tx_n - Ty_n\|^2 \leq (a_1+a_2+a_3+a_4)\ \|Ty_n - x_n\|^2 + 2(a_1+a_3+a_4)\ \|Tx_n - Ty_n\| \times \|Ty_n - x_n\|.$$

$$a_2\ \|Tx_n - Ty_n\|^2 \leq \|Ty_n - x_n\|^2 + 2(a_1+a_3+a_4)\ \|Tx_n - Ty_n\|\ \|Ty_n - x_n\|. \quad \ldots(2.5)$$

since $\sum_{i=1}^{4} a_i < 1.$

Taking limit of (2.5) as $n \to \infty$, we have

$\|Tx_n - Ty_n\|^2 \to 0$ since $a_2 > 0$, $\|Ty_n - x_n\|^2 \to 0$, as $n \to \infty$.

Using triangle inequality and as $n \to \infty$, It follows that

$$\|x_n - Tx_n\|^2 \leq \left\{ \|x_n - Ty_n\| + \|Ty_n - Tx_n\| \right\}^2 \to 0. \quad \ldots (2.6)$$

and $$\|P - Tx_n\|^2 \leq \left\{ \|P - x_n\| + \|x_n - Tx_n\| \right\}^2 \to 0. \quad \ldots (2.7)$$

Now we shall show that P is a fixed point of T. Since T satisfies (1.2), we have

$$\|Tx_n - TP\|^2 \leq a_1\ \|x_n - P\|^2 + a_2\ \|x_n - TP\|^2 + a_3\ \|P - Tx_n\|^2 + a_4 \{ \|x_n - P\|^2 +$$

$$\|Tx_n - TP\|^2 - 2(x_n - P, Tx_n - TP) \Big\} .$$

$$\longrightarrow \frac{a_2}{1-a_4} \|P - TP\|^2 \qquad \ldots (2.8)$$

since as $n \to \infty$, $x_n \to P$,

and using (2.6) and (2.7).

Now, $\|P - TP\|^2 \leq \{ \|P - Tx_n\| + \|Tx_n - TP\| \}^2$ using triangle inequality.

$$\leq \frac{a_2}{1-a_4} \|P - TP\|^2, \text{ as } n \to \infty, \text{ and}$$

using (2.7) and (2.8).

$$[1-(a_2 + a_4)] = \|P - TP\|^2 \leq 0$$

which implies that

$\|P - TP\|^2 = 0$, since $0 \leq a_2 + a_4 < 1$ and

$\|P - TP\| \nless 0$,

$\Longrightarrow \|P - TP\| = 0$

$\Longrightarrow TP = P$

i.e. P is fixed point of T.

Theorem 2.8 : Let $T : C \to C$, where C is closed convex subset of a Hilbert space H. Suppose T satisfies (i) conditions (1.2), (1.3) with further assumption that $a_1 + a_2 + a_3 = 1$, $a_3 + a_4 < 1$,

(ii) Tricomi condition (I-1.3.23), Suppose set F(T) of fixed points of T in C is nonempty. Suppose $\sum \alpha_n \beta_n$ diverges and $\overline{\text{Lim}}\ \beta_n = \beta < 1$. Then $\underline{\text{Lim}}\ \|x_n - Tx_n\| = 0$ for each $x_0 \in C$, where x_{n+1} is defined by Ishikawa Scheme (I-1.3.9, 1.3.10 and 1.3.11).

Proof : Using definition of x_{n+1}(I-1.3.9) and Technique of Ishikawa (I-1.3.12), where t stands for α_n, we have for $P \in F(T)$ and each integer n,

$$
\begin{aligned}
0 \leq \|x_{n+1} - P\|^2 &= \|\alpha_n Ty_n + (1-\alpha_n)x_n - P\|^2. \\
&= \alpha_n \|Ty_n - P\|^2 + (1-\alpha_n)\|x_n - \\
&\quad - P\|^2 - \alpha_n(1-\alpha_n)\|x_n - Ty_n\|^2. \\
&\leq (1-\alpha_n)\|x_n - P\|^2 + \alpha_n \|Ty_n - \\
&\quad - P\|^2, \text{ since } 0 \leq \alpha_n \leq 1. \\
&\leq (1-\alpha_n)\|x_n - P\|^2 + \alpha_n \|y_n - \\
&\quad - P\|^2, \text{ By condition (I-1.2.23).}
\end{aligned}
\qquad \ldots (2.9)
$$

Since T satisfies (1.2, 1.3)

$$
\begin{aligned}
\|Tx_n - P\|^2 = \|Tx_n - TP\|^2 &\leq a_1 \|x_n - P\|^2 + a_2 \|x_n - \\
&\quad - P\|^2 + a_3 \|P - Tx_n\|^2 + \\
&\quad + a_4 \|x_n - Tx_n\|^2.
\end{aligned}
$$

$$(1-a_3)\ \|Tx_n-P\|^2 \leq (a_1+a_2)\ \|x_n-P\|^2 + a_4\ \|x_n - $$
$$-Tx_n\|^2 .$$

$$\|Tx_n-P\|^2 \leq \left(\frac{a_1+a_2}{1-a_3}\right) \|x_n-P\|^2 +$$
$$+\frac{a_4}{1-a_3}\ \|x_n-Tx_n\|^2.$$
$$\leq \|x_n-P\|^2 + r\ \|x_n-Tx_n\|^2, \quad \ldots (2.10)$$

since $a_1+a_2+a_3 = 1$, $r = \dfrac{a_4}{1-a_3} < 1$.

Using definition of y_n (I-1.3.10) and (2.10)

$$\|y_n-P\|^2 = \|\beta_n Tx_n + (1-\beta_n)x_n - P\|^2 .$$
$$= \beta_n\ \|Tx_n-P\|^2 + (1-\beta_n)\ \|x_n-P\|^2 -$$
$$-\beta_n(1-\beta_n)\ \|Tx_n-x_n\|^2.$$
$$\leq \beta_n\ \|x_n-P\|^2 + r\beta_n\ \|x_n-Tx_n\|^2 +$$
$$+(1-\beta_n)\ \|x_n-P\|^2 - \beta_n(1-\beta_n)\ \|Tx_n -$$
$$-x_n\|^2.$$
$$\leq \|x_n-P\|^2 - \beta_n(1-\beta_n-r)\ \|Tx_n-x_n\|^2, \quad \ldots (2.11)$$

where $r < 1-\beta_n$

In (2.9) using (2.11)

$$0 \leq \|x_{n+1} - P\|^2 \leq (1-\alpha_n)\|x_n - P\|^2 + \alpha_n\left\{\|x_n - P\|^2 - \right.$$

$$\left. - \beta_n(1-\beta_n-r)\|Tx_n - x_n\|^2\right\}.$$

$$\leq \|x_n - P\|^2 - \alpha_n\beta_n(1-\beta_n-r)\|Tx_n - x_n\|^2.$$

$$\ldots (2.12)$$

summing these inequalities over $j = 0$ to $j = n$

$$\sum_{j=0}^{n} \alpha_j \beta_j (1-\beta_j-r)\|Tx_j - x_j\|^2 \leq \sum_{j=0}^{n}\left\{\|x_j - P\|^2 - \right.$$

$$\left. - \|x_{j+1} - P\|^2\right\}.$$

$$\leq \|x_o - P\|^2.$$

which implies that

$$\sum_{n=0}^{\infty} \alpha_n \beta_n (1-\beta_n-r)\|Tx_n - x_n\|^2 \leq \|x_o - P\|^2$$

$$\ldots (2.13)$$

Since $0 \leq \beta_n \leq 1$, $\beta_n \geq \beta_n(1-\beta_n)$. Let $S = 1-\beta-r$.

Then $S > 0$ and there exists an integer N such that

$\beta_n < \beta + S/2$ for $n \geq N$. Thus $1 - \beta_n - r > 1 - \beta - r - S/2 =$
$S/2$.

Therefore $\sum \alpha_n \beta_n (1-\beta_n-r) \geq S/2 \sum \alpha_n \beta_n$. But by condition of Ishikawa Scheme (I - 1.3.11 (ii)) i.e. $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$,

$\sum \alpha_n \beta_n (1-\beta_n - r)$ diverges. Thus from (2.13) we obtain
$\text{Lim inf} \; \|x_n - Tx_n\| = 0$.

Remark 2.14 : Since $1 - \beta_n - r \longrightarrow 1 - \beta - r > 0$, there exists an integer No such that $1 - \beta_n - r > 0$ for all $n \geq N_o$. From (2.12) we obtain $\|x_{n+1} - P\|^2 \leq \|x_n - P\|^2$ which implies that $\|x_{n+1} - P\| \leq \|x_n - P\|$. Thus $\{x_n\}$ is monotonically decreasing sequence.

Theorem 2.15 : Let $T : C \longrightarrow C$, where C is compact convex subset of a Hilbert space H. T satisfies conditions (1.2) and (1.3) with further assumptions $a_3 + a_4 < 1$ and $a_1 + a_2 + a_3 = 1$. Suppose x_1 is any point in C. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges in the norm of H to a fixed point of T, where x_n is defined iteratively for each positive integer n by (I-1.3.13) and α_n satisfies $\alpha_o = 1$, $0 \leq \alpha_n < 1$.

Proof : From Schauder's fixed point theorem, T has a fixed point in C, since C is convex compact set. Let P denote one such fixed point of T and x_1 a point in C.

Using (I-1.3.12) and definition of x_{n+1} (I-1.3.13), where t stands for α_n

$$0 \leq \|x_{n+1} - P\|^2 = \|\alpha_n Tx_n + (1-\alpha_n)x_n - P\|^2$$

$$= \alpha_n \|Tx_n - P\|^2 + (1-\alpha_n) \|x_n - P\|^2 - \alpha_n(1-\alpha_n) \|Tx_n - x_n\|^2. \quad \ldots (2.16)$$

since T satisfies (1.2)

$$\|Tx_n - P\|^2 = \|Tx_n - TP\|^2 \leq a_1 \|x_n - P\|^2 + a_2 \|x_n - P\|^2 + a_3 \|Tx_n - P\|^2 + a_4 \|x_n - Tx_n\|^2.$$

$$\|Tx_n - TP\|^2 \leq \frac{a_1 + a_2}{1-a_3} \|x_n - P\|^2 + \frac{a_4}{1-a_3} \|x_n - Tx_n\|^2 \quad \ldots (2.17)$$

From (2.16) and (2.17)

$$0 \leq \|x_{n+1} - P\|^2 \leq \alpha_n \|x_n - P\|^2 + (1-\alpha_n) \|x_n - P\|^2 - \alpha_n(1-\alpha_n - \frac{a_4}{1-a_3}) \|Tx_n - x_n\|^2.$$

Since $a_1 + a_2 + a_3 = 1$.

$$\leq \|x_n - P\|^2 - \frac{1}{n+1} (1 - \frac{1}{n+1} - r) \|Tx_n - x_n\|^2.$$

Setting $\alpha_n = \frac{1}{n+1}$, $r = \frac{a_4}{1-a_3} < 1$.

$$\leq \|x_n - P\|^2 - \left\{ [(1-r)n-r]/(1+n)^2 \right\} \|Tx_n - x_n\|^2.$$

There exists a positive integer N [20] such that

$r = \dfrac{a_4}{1-a_3} < \dfrac{N}{N+1}$ and for positive integer i

$$\|x_{n+i} - P\|^2 \leq \|x_{n+i-1} - P\|^2 - \left\{ [(1-r)(N+i-1) - r] / (N+i)^2 \right\} \|Tx_{N+i-1} - x_{N+i-1}\|^2.$$

which implies

$$\sum_{t=0}^{i} \left\{ [1-r)(N+t)-r] / (N+t+1)^2 \right\} \|Tx_{N+t} - x_{N+t}\|^2 \leq \|x_{N+i} - P\|^2 - \|x_{N+i+1} - P\|^2 \quad \ldots (2.18)$$

Since C is bounded (2.18) implies that

$$\sum_{t=0}^{\infty} \left\{ [(1-r)(N+t) - r] / (N+t+1)^2 \right\} \|Tx_{N+t} - x_{N+t}\|^2 < \infty.$$

and $\|x_{N+i+1} - P\| \leq \|x_{N+i} - P\|$ for $i = 1, 2, \ldots$

Therefore $\lim_{n \to \infty} \|Tx_{N+t} - x_{N+t}\| = 0.$

By compactness of C, there is a subsequence $\{x_{N+n_j}\}_{j=1}^{\infty}$ converging to a point q in C. Now q being fixed point of T, for each positive integer i we have

$\|x_{N+i+1} - q\| \leqslant \|x_{N+i} - q\|$. This implies that $\{\|x_{N+i} - q\|\}$ is monotonically decreasing sequence. This fact along with the convergence of subsequence $\{x_{N+n_j}\}_{j=1}^{\infty}$ to a point q implies the convergence of entire sequence $\{x_n\}_{n=1}^{\infty}$ to a fixed point q of T.

Remark 2.19 : If we put $a_1 = 1$, $a_2 = a_3 = 0$ and $a_4 < 1$, we obtain the theorem of Johnson [20].