CHAPTER-I

Introduction

INTRODUCTION

We present this introductory chapter into two parts. In PART-I, we give some basic concepts, and fixed point theorems which are needed for our investigations. PART-II of this chapter deals with applications of some fixed point theorems.

PART-I

1. Basic concepts, and fixed point theorems :

1.1 Hilbert spaces :

The origin of the theory of Hilbert spaces goes back to the work of the great German mathematician of twentieth century D. Hilbert [17] on integral equations. An axiomatic basis [11] for Hilbert space was provided by the famous mathematician J.Von. Neumann in 1926. Neumann [30] employed Hilbert spaces and the spectral expansion of self adjoint operators to establish a regorous foundation of quantum mechanics. We give the following axiomatic definition of Hilbert space [13] due to Von.Neumann.

Definition 1.1.1 (Inner product space, Hillbert Space): Let H be a linear space over the field K of scalars (real or complex), and to any pair of elements $x, y \in H$, let there correspond a scalar $(x, y) \in K$, called inner product of x and y satisfying the following conditions :

(i) $(x + y, z) = (x, z) + (y, z)$;

(ii) $(\alpha x, y) = \alpha (x, y)$, $\alpha \in K$;

(iii) $(x, y) = \overline{(y, x)}$, where bar denotes complex conjugation;

(iv) $(x, x) \geq 0$;

(v) $(x, x) = 0 \Longleftrightarrow x = 0$. Then H is called an inner product space or pre-Hilbert space.

An inner product on H is a function $(;) : H \times H \longrightarrow K$ satisfying the conditions (i) to (v) and defines a norm on H given by

$$\|x\| = \sqrt{(x, x)} \qquad (\geq 0)$$

and a metric on H given by

$$d(x, y) = \|x - y\| = \sqrt{(x-y,\ x-y)}.$$

H is called a normed linear space. If H is complete with respect to the distance $\|x-y\|$ (i.e. $\|x_n - x_m\| \longrightarrow 0$, $(m, n) \longrightarrow \infty$ implies the existence of $\text{Lim}\ x_n = x$), then H is called Hilbert space. Thus the [redacted] complete inner product space is a Hilbert space. H is called real or complex Hilbert space according as K is real or complex.

The concept of Hilbert space [40] is of great importance in many branches of mathematics and theoretical physics. It is closely related to the concept of Banach spaces. In fact every Hilbert space is a Banach space.

Definition 1.1.2 (Convex Set) : A subset C of a Hilbert space H is said to be convex if $x, y \in C$ and $0 \leq \lambda \leq 1$ imply that $\lambda x + (1-\lambda) y \in C$.

Definition 1.1.3 (Fixed Point) : A fixed point of a self-mapping T of a set X is a point $x \in X$ such that

$$Tx = x,$$

i.e. the image Tx coincides with x.

Example (i) A mapping $x \longrightarrow x^3$ of R into itself has three fixed points (0, -1 and 1).

(ii) a translation has no fixed point.

(iii) a rotation of the plane has a single fixed point i.e. the centre of rotation.

The following definitions in Hilbert space are due to Browder and Petryshyn [7].

Let C be a convex subset of a real Hilbert space H and T be a nonlinear (possibly)mapping from C into H, then

Definition 1.1.4 : T is said to be strictly contractive if there exists a constant K with $0 < K < 1$ such that

$$\|Tx - Ty\| \leq K \|x - y\| \text{ for all } x, y \in C.$$

Definition 1.1.5 : T is said to be contractive (or non-expansive) if for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|x-y\| .$$

Definition 1.1.6 : T is said to be strictly pseudocontractive if there exists a constant $0 < K < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + K \|(I-T)x-(I-T)y\|^2,$$

for all $x, y \in C$.

Definition 1.1.7 : T is said to be pseudocontractive if for all $x, y \in C$,

$$\|Tx - Ty\|^2 < \|x-y\|^2 + \|(I-T)x-(I-T)y\|^2.$$

These mappings admit iterative methods for the construction of their fixed points.

Definition 1.1.8 (Reasonable wanderer map): A self-mapping of a closed convex subset C of H is said to be reasonable wanderer in C if starting at any point x_o in C, its successive steps $x_n = T^n x_o$ ($n = 1, 2, 3...$) are such that the sum of squares of their lengths is finite i.e.

$$\sum_{n=0}^{\infty} \| x_{n+1} - x_n \|^2 < \infty.$$

<u>Definition 1.1. 9</u> <u>(Asymptotically regular map)</u>: A self-mapping T of C, where C is closed convex subset of H, is called asymptotically regular at x if and only if

$$\| T^n x - T^{n+1} x \| \longrightarrow 0, \text{ as } n \to \infty.$$

It follows that every operator which is a reasonable wanderer is asymptotically regular; see [7]

<u>Definition 1.1.10</u> : A mapping $T : C \longrightarrow C$ is said to belong to Lipschitz class (Lip.) if there exists a constant $L > 0$ such that for all x and y in C

$$\| Tx - Ty \| \leq L \| x-y \| .$$

<u>Definition 1.1.11:</u> A mapping $T : C \longrightarrow C$, where C is subset of H is said to be demicontractive [16] if there exists a constant $(0 < K < 1)$ such that, for each fixed point P of T and each $x \in C$.

$$\| Tx - P \|^2 \leq \| x-P \|^2 + K \| x - Tx \|^2.$$

K is known as contraction coefficient.

<u>Definition 1.1.12</u> : A mapping $T : C \longrightarrow C$, where C is subset of H is said to be hemicontractive if for each fixed point P of T and each $x \in C$

$$\|Tx - P\|^2 \leq \|x-P\|^2 + \|x - Tx\|^2.$$

Definition 1.1.13 : A mapping $T : C \longrightarrow H$ is called demi-compact (petryshyn [34] if it has the property that whenever $\{x_n\}$ is a bounded sequence in H and $\{Tx_n - x_n\}$ is strongly convergent sequence, then there exist a subsequence $\{x_{n_i}\}$ which is strongly convergent.

1.2 Fundamental Fixed Point Theorems :

In the area of fixed point theory and its applications, Brouwer's [8] and Schauder's [39] fixed point theorems are regarded as most fundamental theorems. Though Brouwer obtained his result in 1910 [21, P.116], Poincare proved a slightly different version of it much earlier in 1886 which was subsequently rediscovered by Bhole in 1904.

A fixed point theorem [40] in general is one which states that a certain type of mapping of a set in to itself leaves at least one point fixed. Brouwer's fixed point theorem is the classical example of such a theorem.

Theorem 1.2.1 (Brouwer's Fixed Point Theorem) : Every continuous map of the closed unit ball $S = \{x: \|x\| \leq 1\}$ in R^n to itself has a fixed point.

Brouwer's theorem does not give any computational scheme [21] for obtaining a fixed point. Scarf [41] considered some additional conditions and developed a computational scheme in 1967 for computing a fixed point of a mapping. Sincethen many algorithms for the construction of fixed points have been devised.

<u>Definition 1.2.2</u> [22] : A subset C of a normed linear space X is said to be (sequentially) compact if every infinite sequence of elements of C has a subsequence which converges to an element of C.

<u>Definition 1.2.3</u> [22] : A subset C of X is said to be relatively (sequentially) compact if every sequence in C has a subsequence converging to an element of X.

<u>Definition 1.2.4</u> [26] : A linear topological space E is called locally convex, and its topology is called a locally convex topology, if and only if the family of convex neighbourhoods of zero is a local base.

Birkhoff and Kellog [9] were the first to prove fixed point theorems in infinite-dimensional spaces. They considered continuous self-maps defined on convex compact subsets of $C[0, 1]$ and $L^2[0,1]$ and established the existence of fixed points for them. Schauder [39] generalised these results.

<u>Theorem 1.2.5</u> (Schauder's Fixed Point Theorem) : Let C be

a non-empty compact convex subset of a normed linear space X. Then every continuous self-map of C has a fixed point.

Many author's have extended Schauder's theorm in different spaces. Tychonoff [42] considered a general locally convex topological vector space instead of normed linear space and extended Schauder's theorem. He has further shown that his result includes Schauder's theorem as a special case. Browder [2] established a new generalisation of Schauder and Tychonoff fixed point theorems which follows from an argument that uses the conjugate space E* of the locally convex topological vector space E.

1.3 Motivation of the Work :

Since the last three decades the mathematical community deals with great interest in fixed point theory. There are a number of eminent scholars in the field of fixed point theory who have fully devoted to study the properties of fixed points of various types of contractive mappings in Hilbert space. Browder [3, 4, 5, 6] initiated the study of fixed point theory of non-expansive mappings in Hilbert space without compactness conditions. Petryshyn [35] studied an iteration method for the actual construction of fixed points of a nonlinear contraction map T of a closed ball B_r of radius $r > 0$ in to real or complex Hilbert space H under the additional assumption that T is demicompact. Browder and Petryshyn [7] introduced the four classes of mappings(for the mappings 1.1.4, 1.1.5, 1.1.6 and 1.1.7) which admit iterative methods for the construction of their fixed points. They established the following basic existence result.

Theorem 1.3.1 [7] : Let T be a self-map of a closed bounded convex subset of a Hilbert space H such that

$\|Tx - Ty\| \leq \|x-y\|$ for all x, $y \in C$. Then T has atleast one fixed point in C.

Based upon this theorem a number of results have been proved by the authors. Before quoting a few of them, we give the following definitions :

Definition 1.3.2 (Strongly Convergence) [22] : A sequence $\{x_n\}$ in a normed space X is said to be strongly convergent (or convergent in the norm) if there is an $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

i.e. $\lim_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$.

We say that $\{x_n\}$ converges strongly to x and x is called strong limit of $\{x_n\}$.

<u>Definition 1.3.3</u> (Weak convergence) [22] : A sequence $\{x_n\}$ in a normed space X is said to be weakly convergent if there is an $x \in X$ such that for every $f \in X'$ (dual space of X)

$$\lim_{n \to \infty} f(x_n) = f(x)$$

i.e. $x_n \xrightarrow{w} x$ or $x_n \longrightarrow x$.

We say that $\{x_n\}$ converges weakly to x and x is called the weak limit of $\{x_n\}$.

<u>Theorem 1.3.4</u> : If T is contractive (non-expansive) mapping of C in to C, where C is closed convex subset of a Hilbert space H and the set F(T) of fixed points of T in C is non-empty, then the mapping defined by $T_\lambda = \lambda I + (1 - \lambda)T$ for any given λ with $0 < \lambda < 1$ is a reasonable wanderer from C into C with the same fixed points as T.

The following corollary is a consequence of above theorem.

<u>Corollary 1.3.5</u> : If T is contractive (non-expansive) mapping of C in to C with non-empty set F(T) of fixed points of T in C, then the mapping defined by $T_\lambda = \lambda I + (1-\lambda)T$ for a given λ with $0 < \lambda < 1$, (i) maps C into C, (ii) has the same fixed points as T and (iii) is asymptotically regular.

<u>Theorem 1.3.6</u> : Let T be a self-map of a bounded closed

convex subset C of a Hilbert space H. Suppose T is (i) Contractive mapping, (ii) demicompact. Then the set F(T) of fixed points of T in C is a non-empty convex set and for any given x_o in C and any fixed $\lambda > 0$ with $0 < \lambda < 1$ the sequence $x_n = \left\{ T_\lambda^n \ x_o \right\}$ determined by the process

$$x_n = \lambda \, Tx_{n-1} + (1-\lambda)x_{n-1}, \quad n = 1, 2, 3, \ldots \qquad \ldots (1.3.7)$$

converges strongly to a fixed point of T in C.

Hicks and Huffman [15] generalized theorem (1.3.4) and theorem (1.3.6) in generalized Hilbert space (see Theorem 6,7 of [15]).

Ishikawa [19] introduced a new iteration scheme (I-scheme) for the construction of fixed points of contractive type mapping and obtained the following result.

<u>Theorem 1.3.8</u> : If T is a lipschitizian pseudo-contractive self-map of C, where C is a convex compact subset of a Hilbert space H and x_1 is any point in C, then the sequence $\left\{ x_n \right\}_{n=1}^{\infty}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer n by

$$x_{n+1} = \alpha_n \, Ty_n + (1-\alpha_n)x_n \qquad \ldots (1.3.9)$$

$$y_n = \beta_n \, Tx_n + (1 - \beta_n)x_n \qquad \ldots (1.3.10)$$

where $\left\{\alpha_n\right\}_{n=1}^{\infty}$ and $\left\{\beta_n\right\}_{n=1}^{\infty}$ are sequences of positive numbers that satisfy the following three conditions :

$$\left.\begin{array}{ll} \text{(i)} & 0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1, \\ \text{(ii)} & \lim\limits_{n \to \infty} \beta_n = 0, \\ \text{(iii)} & \sum\limits_{n=1}^{\infty} \alpha_n \beta_n = \infty . \end{array}\right) \quad \ldots (1.3.11)$$

Ishikawa derived the following technique and used it to prove above theorem.

For any x, y, z in a Hilbert space H and a real number t,

$$\| tx + (1-t)y-z \|^2 = t \|x-z\|^2 + (1-t) \|y - z\|^2 - t(1-t) \|x - y\|^2 \quad \ldots (1.3.12)$$

Mann [29] gave the following iteration process. For a self-mapping T of a compact interval of the real line having a unique fixed point the iteration process

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n Tx_n \quad \ldots (1.3.13)$$

with $\alpha_n = \dfrac{1}{n+1}$, converges to the fixed point of T.

Hicks and Kubicek [16] studied Mann iteration process

(1.3.13) in Hilbert space and employing Ishikawa technique (1.3.12) established the following interesting results.

<u>Theorem 1.3.14</u> : Suppose (i) $T : C \longrightarrow C$, where C is a convex subset of a Hilbert space H, (ii) T is demicontractive (I-1.1.11) with contraction coefficient K, (iii) Set F(T) of fixed points of T in C is nonempty, (iv) $\sum \alpha_n(1-\alpha_n)$ diverges and (v) $\alpha_n \longrightarrow \alpha < 1 - K$. Then $\underline{\text{Lim}} \ \|x_n - Tx_n\| = 0$ for each $x_1 \in C$, where x_{n+1} is defined by (1.3.13).

For their second result the following lemma due to Opial [33] is employed.

Lemma [33] : Suppose H is a Hilbert space and the sequence $\{x_n\}$ is weakly convergent to x_o, then for any $x \neq x_o$, $\underline{\text{Lim}} \ \|x_n - x_o\| < \underline{\text{Lim}} \ \|x_n - x\|$.

<u>Theorem 1.3.15</u> : Suppose $T : C \longrightarrow C$, where C is a closed convex subset of H such that

(i) $F(T) \neq \emptyset$,

(ii) T is demicontractive with contraction coefficient K,

(iii) If any sequence $\{x_n\}$ converges weakly to x and $(I-T)(x_n)$ converges strongly to zero then $(I-T)\,x = 0$.

Das and Debata [12] studied the Ishikawa [19] iteration scheme which converges to a fixed point of a lipschitzian pseudo-contractive map and convergence of an iteration scheme to a fixed point of demicontractive maps in Hilbert space considered by Hicks and Kubicek [16]. They gave a generalisation of Ishikawa iteration scheme and obtained common fixed points of a family of less restrictive hemicontractive (I-1.1.12) maps. Their result statesas follows :

<u>Theorem 1.3.16</u> : Let C be a convex, compact subset of a Hilbert space H. Let $\{T_i\}$, $i = 1, 2, \ldots K$, $K > 2$ be a family of hemicontractive maps defined on C and have atleast one common fixed in C. Let the family of maps $\{T_i\}$ satisfy

$$\|T_i x - T_j y\| \leq M \|x-y\| \text{ for all } x \text{ and } y \text{ in } C.$$

and all pairs (i, j), M being a positive constant. Then the sequence $\{x_n\}$ converges to a common fixed point of the family of maps $\{T_i\}$, where x_n is defined iteratively for each positive integer n by

$$\left.\begin{aligned} & x_1 \in C \\ & x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, T_k\, u_{k-1}\,(n) \end{aligned}\right\} \quad \ldots (1.3.17)$$

$$\text{where } u_o(n) = x_n,\ u_i(n) = (1 - \beta_n) x_n + \beta_n T_i u_{i-1}(n) \quad \ldots (1.3.18)$$

for $i = 1, 2, \ldots K$ and $\{\alpha_n\}$, $\{\beta_n\}$are real sequences in $[0, 1]$ such that

$$\left.\begin{array}{ll} \text{(i)} & 0 \leq \alpha_n \leq \beta_n \leq 1, \text{ for } n = 1, 2, 3 \ldots \\ \text{(ii)} & \lim_{n \to \infty} \beta_n = 0 \\ \text{(iii)} & \sum_{n=1}^{n=\infty} \alpha_n \beta_n^{k-1} = \infty \text{ for each } k > 2 \end{array}\right\} \ldots (1.3.19)$$

The authors claimed that for $k = 2$, $T_1 = T_2$, the above theorem includes the result of Ishikawa [19] as a corollary. They further claim that the Ishikawa iteration can be extended to Lip. hemicontractive mappings.

Johnson [20] used Mann iteration scheme (1.3.13) and obtained a fixed point of a strictly pseudocontractive (I.1.1.6) mapping defined on a convex compact subset of a Hilbert space.

<u>Theorem 1.3.20</u> [20] : If $T : C \longrightarrow C$, where C is a compa^c convex subset of a Hilbert space H, is a strictly pseudo contractive mapping and x_1 is any point in C, then th$^{\text{point}}$ sequence $\{x_n\}_{n=1}^{\infty}$ converges in the norm of H to a $^{\text{ixed}}$ point of T, where for each positive integer n, x $^{\text{is}}$ defined by (1.3.13).

<u>Definition 1.3.21</u> (Quasicontraction) : A sel$^{\text{apping}}$ T of a Banach space X is said to be a quasicont$^{\text{ive}}$ if there exists a constant K, $0 \leq K < 1$ such that $^{\text{each}}$ $x, y \in X$

$$\|Tx - Ty\| \leq K \max\{ \|x-y\| , \|x-Tx\| ,$$
$$\|y - Ty\| , \|x-Ty\| , \|y - Tx\| \}.$$

Definition 1.3.22 : A mapping $T : C \longrightarrow C$, where C is nonempty subset of a Hilbert space H, is said to satisfy Tricomi condition (T) if for all $x \in C$ and $P \in F(T) = \{x \in C : Tx = x\}$,

$$\|Tx - P\| \leq \|x - P\| . \qquad \ldots (1.3.23)$$

Naimpally and Singh [32] considered Ishikawa scheme defined by (1.3.9, 1.3.10, 1.3.11) with further assumptions that (1) $0 \leq \alpha_n \leq \beta_n \leq 1$, (2) $\underline{\text{Lim}}\ \alpha_n = \alpha > 0$, (3) $\overline{\text{Lim}}\ \beta_n = \beta < 1$ and shown that if the sequence of Ishikawa iterates converges, it converges to the fixed point of T, where T is self-map of a nonempty subset of a Banach space X which satisfies either of the following two conditions :

(I) T is quasicontraction, ... (1.3.24)

(II) At least one of the following conditions holds for each $x, y \in X$

(A) $\|x - Tx\| + \|y - Ty\| \leq a \|x-y\| , 1 \leq a < 2;$

(B) $\|x-Tx\| + \|y - Ty\| \leq b[\|x-Ty\| + \|y-Tx\| + \|x-y\|] , \frac{1}{2} \leq b < \frac{2}{3};$

(C) $\|x-Tx\| + \|y-Ty\| + \|Tx-Ty\| \leq c[\|x-Ty\| + \|y-Tx\|], 1 \leq c < \frac{3}{2};$

(D) $\|Tx-Ty\| \leq K \max\{ \|x-y\| , \|x-Tx\|, \|y-Ty\| , [\|x-Ty\| + \|y - Tx\|]/2\} \; 0 \leq K < 1.$ .../·

The authors claim that these results extend the results of Rhoades [36], and Hicks and Kubicek [16] in the following way.

<u>Theorem 1.3.26</u> : Let T be a self map of a closed convex subset C of a normed linear space X and satisfies condition (I), $\{x_n\}$ be the sequence of Ishikawa scheme associated with T and such that $\{\alpha_n\}$ is bounded away from zero. If $\{x_n\}$ converges to P, then P is a fixed point of T.

<u>Theorem 1.3.27</u> : Let T : $C \longrightarrow C$, C is a convex subset of a Hilbert space H such that

(i) T satisfies tricomi condition (T)

(ii) $F(T) \neq \emptyset$.

Suppose $\sum \alpha_n \beta_n$ diverges and $\beta_n \longrightarrow \beta < 1$. Then $\underline{\text{Lim}} \ \|x_n - Tx_n\| = 0$ for each $x_o \in C$, where x_{n+1} is the sequence of Ishikawa iterates with further assumptions (1), (2) and (3).

PART-II

2. Applications of fixed point theorems :

2.1. Introduction :

The fixed point theory is applied in general to establish existence theorems for non-linear differential and integral equations and in particular to the theory of positive matrices. Many of the most important non-linear problems [7] of applied mathematics reduce to finding solutions of non-linear functional equations such as non-linear integral equations, boundary value problems for non-linear ordinary or partial differential equations etc. which can be formulated in terms of finding the fixed points of a given non-linear mapping of an infinite dimensional function space X in to itself.

The well known Banach contraction principle [1] is applied to establish existence and uniqueness theorems [11] for (i) Linear equations (ii) differential equations (Picard's existence and uniqueness theorem for ordinary differential equations) (iii) Integral equations (Fredholm and Voltera integral equations). Parron's Theorem [11] is an application of Brouwer's fixed point theorem to the theory of matrices which play an important role in many applied fields.

Schauder's theorem (1.2-5) is applied [11] to solve nonlinear integral equations of the form

$$u(x) = \int_a^b K(x, y)\, f(y, u(y))\, dy \qquad \ldots (2.1.1)$$

where K and f are given functions and u is unknown. Equation (2.1.1) is known as the Hermestein equation.

For further applications of fixed point theorems we need the following definitions and fixed point theorems.

Definition 2.1.2 : A closed, convex subset C of a real Banach space X is called a (positive) cone if the following conditions are satisfied.

(i) $x \in C$, then $\lambda x \in C$ for $\lambda \geq 0$,

(ii) if $x \in C$ and $-x \in C$, then $x = 0$.

A cone C in X induces a partial ordering $\leq$ in X by $x \leq y$ if and only if $y - x \in C$.

Definition 2.1.3 (Ordered Banach Space) : A Banach space X with a partial ordering $\leq$ induced by a cone C is called an ordered Banach space.

Definition 2.1.4 : A completely continuous map means a continuous function which takes bounded sets in to relatively compact sets.

Definition 2.1.5 : Let C be a cone of an ordered Banach space X. A map $T : C \longrightarrow C$ is called a comparison of the cone if $T(0) = 0$ and if there exists numbers $q > s > 0$ such that (i) $Tx \nleq x$ if $x \in C$, (ii) $\|x\| \leq s$ and $x \neq 0$., (iii) for all $\epsilon > 0$, $(1 + \epsilon)\, x \nleq Tx$ if $x \in C$ and $\|x\| \geq q$.

Krasnoselskii [23] proved the following result.

Theorem 2.1.6 : If T is (i) a comparison of the cone C, (ii) completely continuous on C, then T has atleast one non-zero fixed point x in C with $s \leq \|x\| \leq q$.

Rothe [38] established the following fixed point theorem.

Theorem 2.1.7 (Rothe's fixed point theorem) : If a continuous map $f : B^N \longrightarrow R^N$ (B^N the unit ball in R^N) satisfies $f(\partial B^N) \subset B^N$, then f has a fixed point.

2.2 Infectious Disease Model :

Leggett [27] considered the following integral equation

$$x(t) = Tx(t) = \int_{t-\tau}^{t} f(v, x(v))\, dv \qquad \ldots (2.2.1)$$

as a model for the spread of certain infectious disease with periodic contact rate that varies seasonally. In equation

(2.2.1) $x(t)$ represent the proportion of infectives in the population at time t, $f(t, x(t))$ is the proportion of new infectives per unit time ($f(t,0) = 0$) and τ is the duration of time an individual remains infections.

As consequence of the comparison of the cone theorem (2.1.6) Leggett [27] established the following theorem :

Theorem 2.2.2 : Let $T : C_S \longrightarrow C$ be a completely continuous operator with $T(0) = 0$. If (i) there exist a number q, $0 < q < S$ and a vector $u \in C \setminus \{0\}$, such that

$$Tx \nleq x \quad \text{if } x \in C(u) \text{ and } \|x\| = q$$

(ii) for each $\epsilon > 0$,

$$(1 + \epsilon)\, x \nleq Ax \text{ if } x \in C \text{ and } \|x\| = S, \text{ then } T$$

has a fixed point in C with $q \leq \|x\| \leq S$.

He further applied this theorem to establish the following existence theorem for solution of integral equation (2.2.1) with the assumption that τ and w are positive constants.

Theorem 2.2.3 : Suppose (i) the function $f(t, x)$ is continuous from $(-\infty, \infty) \times [0, \infty)$ in to $[0, \infty)$,

(ii) for each $t \in R$ and $x \geq 0$, $f(t, x) = f(t + w, x)$ and $f(t, 0) = 0$,

(iii) there exists $S > 0$ such that $f(t, x) \leq S/\tau$ for all

$(t, x) \in [0, w] \times [0, S]$,

(iv) for each t, $a(t) = \lim_{x \to 0} (f(t,x)/x)$, and for each $K \in (0,1)$ there exists $\epsilon_k > 0$ such that $f(t,x) \geq K\, a(t)x$, $t \in R$, $0 \leq x \leq \epsilon_k$, are satisfied and N is the smallest integer such that $w/N \leq \tau/2$.

Set $I_j = [\ ((j-1)/N)w,\ (j/N)w\ ]$ for $j = 0, 1, 2, \ldots N$.

If $\prod_{j=1}^{N} \int_{I_j} a(v)\, dv > 1$, then the equation (2.2.1) has a nonzero solution.

This result implies that even if the contact rate is zero over some short time intervals, the disease may recur periodically.

Leggett [27] further shown that the occurrence of a disease modeled by equation (2.2.1) is periodic in nature even if average contact rate is small (even zero) during some seasons. What is required is that the average effective rate should be high enough during the remaining seasons and it should offset the smaller contact rates. This means that a certain product of average seasonal contact rates should be greater than one.

2.3 Monetary Economics :

The famous mathematician J. Von Neumann [31] initiated the use of fixed point theorems to establish the existence of general equilibrium in economic systems. Chichilnisky and Kalman [10] translated mathematical assumptions in to natural economic conditions about the role of money in the economy and applied Rothe's theorem (2.1.7) to establish the following existence theorem of equilibria in monetary economics.

<u>Theorem 2.3.1</u> : Suppose the economy E satisfies the following assumptions :

(i) The utility function $u^k : R_+^N + R_+ X \prod \longrightarrow R_+$ is continuous for all K,

(ii) $u^k(. , P)$ is strictly quasiconcave and monotone increasing for every $P \in \Pi$

(iii) $u^k(x + (0, \ldots, 0, a_i, 0, \ldots, 0), m, P) <$

$$u^k(x, m + P_i a_i, P)$$

for some $\epsilon > a_i > 0$, $M > 0$ and some commodity i, $1 \leq i \leq N$ (i possibly dependent on M and ϵ) with $|P| > M$, $m > M$ and $|x| > M$.

Then this economy E has a general equilibrium P* in R_+^N with a positive exchange value of money.

Here

(i) $R_+^N \times R_+$ represents the space of all possible consumption money vectors representing bundles of goods or commodities and money;

(ii) R_+^N, R_+ represent positive cones of R^N and R respectively;

(iii) K = 1, 2, ... are traders;

(iv) P, a vector of R_+^N denotes consumption goods monetary price;

(v) $S = \{(P, 1) : P \in R_+^N\}$ represent space of money prices of commodity money bundles;

(vi) π is the projection of S in to its first N coordinates i.e. $\pi = \{P : (P, 1) \in S\} = R_+^K$

(vii) u^k = the utility function which is a real valued function representing agent's preferences;

(viii) $u^k (x^k, m^k, p)$ - K^{th} trader's utility function.

There are many more applications of fixed point theory in other branches of mathematics and in pure and social sciences. It is applied to approximation theory, boundary value problems [28] arising in chemical reactor theory, for solving problems of boundary layer theory in fluid mechanics and in solving the stability problems of hydrodynamics.