C H A P T E R - III

RICCI COLLINEATIONS AND ISOMETRIES COMPATIBLE WITH FERROFLUID

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4. The Ricci Collineations and Isometries (Coupling).

1. Introduction

The study of symmetries like groups of motions and collineations forms a major area of investigations in the general theory of relativity. Katzin, Levine (1972) enumerated sixteen types of symmetries including collineations in Riemannian Spaces. The Ricci Collineation is defined as "the point transformation

$$x^{*a} = x^a + K^a \, \delta t,$$

with t as the parameter which leaves invariant the form of the Ricci tensor", (Collinson, 1970b). Its tensorial form is

$$\underset{X}{\pounds} R_{ab} = 0, \quad \text{(Ricci Collineation).} \qquad \ldots (1.1)$$

Here $\underset{X}{\pounds}$ denotes the Lie derivative (Yano, 1955).

The conditions on eigenvalues of stress-energy tensor describing the definite magnetofluid scheme admitting the Ricci collineation are found by Shaha (1974). Oliver and Davis (1977) have studied the time-like symmetries, with special reference to conformal motions and family of contracted Ricci Collineations, for the space-times filled with perfect fluid. The perfect fluid space-times including electromagnetic fields admitting symmetry mappings belonging to the family of contracted Ricci collineations were studied by Norris, Green and Davis (1977). Nunez, Percoco and Villalba (1990) discussed the Ricci and the contracted Ricci collineations of the Robertson-Walker space-time.

The Lie derivative of second order tensor is expressible in terms of covariant derivative in the form

$$\underset{X}{\pounds} R_{ab} = R_{ab;c} X^{c} + R_{bc} X^{c}{}_{;a} + R_{ac} X^{c}{}_{;b} ,$$

Consequently the equation (1.1) yields

$$R_{ab;c} X^{c} + R_{bc}X^{c}{}_{;a} + R_{ac} X^{c}{}_{;b} = 0 . \qquad \ldots (1.2)$$

According to Trautman (1962) the conservation law like $(T^{ab} X_{a})_{;b} = 0$ exists only when $\bar{X}$ is Killing vector. It is proved in Section 2 that the Ricci collineation along $\bar{X}$ leads to a conservation law $(R^{b}_{a} X^{a})_{;b} = 0$, (Collinson, 1970) even if $\bar{X}$ is not a Killing vector. This conservation law is examined in the space-time of ferrofluid admitting the Ricci collineations along the flow vector and also along the magnetic field. Section 3 deals with the Ricci collineations and contracted Ricci collineations corresponding to the ferrofluid space-time. We have established the explicit impact of variable magnetic permeability on the kinematical and dynamical quantities like θ, w_{ab}, σ_{ab} and ρ, P, H^2 respectively. We have also proved that the Ricci collineations imply the infinitesimal isometries under the restriction that the magnetic permeability is invariant along the flow. In Section 4, Ricci collineations and isometries are studied simultaneously to prove the results like $\theta = 0$, $\dot{u}_a = 0$, $\mathcal{E} = 0$, $\dot{\mu} = 0$, $\mu_{;b} H^b = 0$, $H^b_{;b} = 0$.

2. The Local Conservation Laws

Einstein had to introduce the pseudoenergy tensor t_{ab} in order to express $T^{ab}_{;b} = 0$, in terms of an ordinary divergence for the generation of conservation laws without postulating such pseudo-tensors, conservation laws can be generated when the space-time admitts collineation or motion. If $\bar{X}$ is the Killing vector then the corresponding conservation law (Trautman, 1962) $(T^{ab} X_a)_{;b} = 0$ exists. In the similar way the conservation law based on the Ricci collineation (Collinson, 1970) when $\bar{X}$ is not Killing vector is derived. This can be obtained by using contracted Bianchi identities in the following way. The contraction of (1.2) with g^{ab} leads to the result

$$R_{;c} X^c + R^a_c X^c_{;a} + R^b_c X^c_{;b} = 0,$$

i.e., $$R_{;c} X^c + 2 R^b_c X^c_{;b} = 0,$$

or $$(R^b_c X^c)_{;b} + (\tfrac{1}{2} \delta^b_c - R^b_c)_{;b} X^c = 0 . \qquad \ldots (2.1)$$

The last term in (2.1) will vanish by virtue of the contracted Bianchi identities and so it is stated that if a space-time admits a Ricci collineation then there exists a conservation law generator of the form

$$(R^b_c\ X^c\ )_{;b} = 0. \qquad \ldots (2.2)$$

Further, it should be noted that (2.2) holds good only when both the indices in (1.1) are in co-variant position. The Ricci tensor (II.2.3) can be written as

$$R^b_a = (\ \rho + P + \mu H^2)\ u_a u^b - \frac{1}{2}\ (\ \rho - P + \mu H^2)\ \delta^b_a - \mu H_a\ H^b \qquad \ldots (2.3)$$

On replacing arbitrary vector X^c in (2.2) by flow vector in space-time of ferrofluid, one can get the conservation law to $\underset{u}{\pounds}\ R_{ab} = 0$ in the form

$$(R^b_a\ u^a)_{;b} = 0,$$

i.e. $\left[\left\{ (\ \rho + P + \mu H^2) u_a u^b - \frac{1}{2}\ (\ \rho - P + \mu H^2)\ \delta^b_a - \mu H_a H^b \right\}\ u^a\ \right]_{;b} = 0$, (vide, 2.3) ,

i.e. $\left[\ (\ \rho + 3P + \mu H^2) u^b\ \right]_{;b} = 0, \quad \because\ H^a u_a = 0,\ u^a u_a = 1,$

or $\quad (\ \rho + 3P + \mu H^2)_{;b} u^b + (\ \rho + 3P + \mu H^2) u^b_{;b} = 0. \qquad \ldots (2.4)$

Here (2.4) is considered as a consequence of the Ricci collineation along a world line. The similar consequence of the Ricci collineation along the magnetic field using (2.3) in $(R^b_a\ H^a)_{;b} = 0$ is obtained as

$$[\{(\rho + P + \mu H^2)u_a u^b - \frac{1}{2}(\rho - P + \mu H^2)\delta_a^b -$$

$$- \mu H_a H^b\} H^a]_{;b} = 0,$$

i.e. $[(\rho - P - \mu H^2)H^b]_{;b} = 0, \quad \because H^a H_a = -H^2, \; u^a H_a = 0,$

or $(\rho - P - \mu H^2)_{;b} H^b + (\rho - P - \mu H^2) H^b_{;b} = 0.$ ... (2.5)

3. The Ricci Collineations In Ferrofluid

(A) Ricci Collineation Along The Fluid Flow :

We have the Ricci tensor expression for the ferrofluid as given by (II.2.3)

$$R_{ab} = -K[Au_a u_b - Bg_{ab} - \mu H_a H_b], \qquad \text{... (3.1)}$$

where $A = \rho + P + \mu H^2$, $B = \frac{1}{2}(\rho - P + \mu H^2)$. ... (3.2)

Further if the space-time of ferrofluid admits the Ricci collineation along the fluid flow $\bar{u}$, then we write from equations (1.2) and (3.1)

$$(Au_a u_b - Bg_{ab} - \mu H_a H_b)_{;c}\, u^c + (Au_a u_c - Bg_{ac} -$$

$$- \mu H_a H_c)u^c{}_{;b} + (Au_b u_c - Bg_{bc} - \mu H_b H_c)u^c_{;a} = 0,$$

i.e. $A_{;c}\, u^c u_a u_b - B_{;c} u^c g_{ab} - \mu_{;c} u^c H_a H_b + A(u_{a;c}\, u^c u_b +$

$$+ u_a u_{b;c} u^c) - B(u_{a;b} + u_{b;a}) - \mu[H_a (H_{b;c} u^c +$$

$$+ H_c u^c{}_{;b}) + H_b (H_{a;c} u^c + H_c u^c{}_{;a})] = 0,$$

$$(\because \dot{u}_a u^a = 0).$$

This after simplification gives

$$\dot{A}\, u_a u_b - \dot{B} g_{ab} - \dot{\mu}\, H_a H_b + A(\dot{u}_a u_b + u_a \dot{u}_b) -$$

$$- B\,(u_{a;b} + u_{b;a}) - \mu\, [H_a\,(H_{b;c} u^c + H_c u^c{}_{;b}) +$$

$$+ H_b (H_{a;c} u^c + H_c u^c{}_{;a})\,] = 0. \qquad \ldots (3.3)$$

Step 1 :

By taking the inner multiplication of (3.3) with u^a and using the results $u^a H_a = 0$, $\dot{u}_a u^a = 0$ generates the equation

$$(A-B)^{\cdot} u_b + (A-B)\dot{u}_b - \mu\,[H_b(H_{a;c} u^c u^a + H_c u^c{}_{;a} u^a)] = 0.$$

If we use the result $H_{a;c}\, u^a = - u_{a;c}\, H^a$, then this equation immediately produces

$$(A-B)^{\cdot}\, u_b + (A-B)\dot{u}_b = 0,$$

i.e. $$(\rho + 3p + \mu H^2)^{\cdot}\, u_b + (\rho + 3P + \mu H^2)\, \dot{u}_b = 0.$$

(vide, 3.2) ... (3.4)

This implies the following two consequences

$$(\rho + 3P + \mu H^2)^{\cdot} = 0, \text{ and } \dot{u}_a = 0. \qquad \ldots (3.5)$$

Step II :

The equation (3.3) after transvecting with H^a and using (3.5) we obtain

$$- \dot{B} H_b + \mu H^2 H_b - BH^a (u_{a;b} + u_{b;a}) +$$

$$+ \mu H^2 (H_{b;c} u^c + H_c u^c{}_{;b}) - \mu (H_{a;c} u^c + H_c u^c{}_{;a}) H^a H_b = 0,$$

i.e.
$$- \dot{B} H_b + \dot{\mu} H^2 H_b - BH^a (u_{a;b} + u_{b;a}) +$$

$$+ \mu H^2 (H_{b;c} u^c + H_c u^c{}_{;b}) + \frac{1}{2} \mu (H^2)^{\cdot} H_b - \mu H^c u_{c;a} H^a H_b = 0.$$

$$(\text{Since } H_{a;c} H^a u^c = - \frac{1}{2} (H^2)^{\cdot}, \; H^a H_a = - H^2). \qquad \ldots \quad (3.6)$$

This equation (3.6) when transvected with H^b produces

$$\dot{B} H^2 - \dot{\mu} H^2 H^2 - 2BH^a H^b u_{a;b} + 2\mu H^2 (H_{b;c} H^b u^c +$$

$$+ u_{c;b} H^b H^c) = 0,$$

where $H^a H_a = - H^2$,

i.e.
$$\dot{B} H^2 - \dot{\mu} H^2 H^2 - 2B H^a H^b u_{a;b} - \mu H^2 (H^2)^{\cdot} +$$

$$+ 2\mu H^2 H^a H^b u_{a;b} = 0 ,$$

$$(\text{Since } H^a{}_{;c} H_a u^c = - \frac{1}{2} (H^2)^{\cdot} \text{ and } H^a H_a = - H^2),$$

i.e.
$$(B - \mu H^2)^{\cdot} H^2 - 2 (B - \mu H^2) H^a H^b u_{a;b} = 0,$$

i.e.
$$(\rho - P - \mu H^2)^{\cdot} H^2 - 2(\rho - P - \mu H^2) H^a H^b u_{a;b} = 0.$$

[vide, (3.2)] ... (3.7)

Step III :

If we contract (3.3) with g^{ab} and use the conditions $\dot{u}_a u^a = 0$, $g_{ab} u^a u^b = 1$, $g_{ab} H^a H^b = - H^2$, then we obtain

$$- \dot{A} - 4\dot{B} + \dot{\mu} H^2 - B (u^b{}_{;b} + u^a{}_{;a}) -$$

$$- \mu[H^b(H_{b;c}u^c + H_c u^c{}_{;b}) + H^a(H_{a;c}u^c + H_c u^c{}_{;a})] = 0.$$

Further since $H^b H_{b;c} u^c = - \frac{1}{2} (H^2)_{;c} u^c = - \frac{1}{2} (H^2)^{\cdot}$,

the above equation becomes

$$\dot{A} - 4\dot{B} + \dot{\mu} H^2 - 2B\theta + \mu (H^2)^{\cdot} - 2\mu H^a H^b u_{a;b} = 0,$$

i.e. $\quad (A - 4B + \mu H^2)^{\cdot} - 2B\,\theta - 2\mu H^a H^b u_{a;b} = 0.$

By using (3.2) in this equation we have

$$(\rho - 3p)^{\cdot} + (\rho - P + \mu H^2)\,\theta + 2\mu H^a H^b u_{a;b} = 0 \,. \qquad \ldots (3.8)$$

<u>Lemma (1)</u> : <u>(Kinematical Conditions)</u> :

If the space-time of ferrofluid admits the Ricci collineation along flow lines then the flow is geodesic and expansion free.

<u>Proof</u> : We infer from the equation (3.5) that

$$\dot{u}_a = 0. \qquad \ldots (3.9)$$

Further the conservation law (2.4) produces the result with the help of (3.5)

$$(\rho + 3P + \mu H^2)\,\theta = 0.$$

As $\overline{u}$ is time-like congruence $\rho + 3P + \mu H^2 \neq 0$, hence this equation yields

$$\theta = 0 . \qquad \ldots (3.10)$$

Thus we have proved the required conditions (3.9) and (3.10)

Remark : These conditions direct that the flow lines are geodesic and expansion free.

Lemma (2) : For ferrofluid

$$\underset{u}{\pounds} R_{ab} = 0 \implies \text{i) } H^b{}_{;b} = 0 \iff \mu_{;b}H^b = 0,$$

$$\text{ii) } P_{;b}H^b = 0 \iff \mu_{;b}H^b = 0.$$

Proof : When the equation (3.5) is applied in the Maxwell's equation (II.2.6) then these will give

$$\mu H^b_{;b} + \mu_{;b} H^b = 0,$$

i.e. $$H^b_{;b} = 0 \iff \mu_{;b}H^b = 0 . \qquad \ldots (3.11)$$

Again if we use the result (3.5) in the continuity equation (II.3.7) we get

$$P_{;b} H^b + \frac{1}{2} H^2\mu_{;b}H^b = 0,$$

i.e $$P_{;b}H^b = 0 \iff \mu_{;b}H^b = 0. \qquad \ldots (3.12)$$

Hence (3.11) and (3.12) are the required results.

Interpretation : If the space-time of ferrofluid admits the Ricci collineation along the flow vector $\bar{u}$ then,

i) the magnetic lines are divergence free iff the magnetic permeability is preserved along the magnetic lines [vide, (3.11)].

ii) the isotropic pressure is constant along the magnetic lines iff the magnetic permeability is preserved along these lines [vide, (3.12)].

<u>Lemma (3)</u> : <u>(Dynamical Conditions)</u> :

For the space-time of ferrofluid admitting the Ricci collineation with respect to flow lines

$$\dot{\rho} = \dot{P} = (H^2)^{\cdot} = 0 \iff \dot{\mu} = 0, \text{ when } \rho \neq P + 5/3\, \mu H^2.$$

<u>Proof</u> : Applying (3.10) in the continuity equation (II.3.3) we get the result

$$\dot{\rho} - \frac{1}{2}\,\dot{\mu}\, H^2 = 0,$$

i.e. $\dot{\rho} = 0 \iff \dot{\mu} = 0.$ ... (3.13)

If we use this in the equation (3.5), we get

$$3\dot{P} + \frac{3}{2}\,\dot{\mu}\, H^2 + \mu (H^2)^{\cdot} = 0. \qquad \text{... (3.14)}$$

[vide, (3.13)]

Further by using equations (3.10) and (3.13) in the equation (3.8) we find the result

$$2\mu H^a H^b u_{a;b} = -\frac{1}{2}\dot{\mu} H^2 + 3\dot{P} \qquad \ldots (3.15)$$

Also the equations (3.7), (3.10) and (3.15) produce

$$\dot{P}(3\rho - 3P - 2\mu H^2) + \frac{1}{2}\dot{\mu} H^2 (-\rho + P + 2\mu H^2) + \\ + \mu^2 H^2 (H^2)^{\cdot} = 0 . \qquad \ldots (3.16)$$

Now by eliminating $(H^2)^{\cdot}$ from (3.14) and (3.16) we get

$$3(\rho - P - 5/3\, \mu H^2)\dot{P} - \frac{1}{2} H^2 (\rho - P + \mu H^2)\dot{\mu} = 0.$$

If $\rho \neq P + 5/3\, \mu H^2$, then

$$\dot{P} = 0 \iff \dot{\mu} = 0 , \qquad \ldots (3.17)$$
$$\text{since } \rho - P + \mu H^2 \neq 0.$$

In similar way by eliminating $\dot{P}$ from equations (3.14) and (3.16) and supposing $\rho \neq P + 5/3\, \mu H^2$, we get

$$(H^2)^{\cdot} = 0 \iff \dot{\mu} = 0 . \qquad \ldots (3.18)$$

Thus (3.13), (3.17) and (3.18) are the required results.

Interpretations : If the space-time of ferrofluid admits the Ricci collineation along $\overline{u}$ and when $\rho \neq P + 5/3\, \mu H^2$ then (i) the energy density is conserved along the flow lines iff the magnetic permeability is constant along the flow [vide, (3.13)].

(ii) the necessary and sufficient condition for the isotropic pressure to be constant along the flow lines is that the magnetic permeability is preserved along the flow [vide, (3.17)].

(iii) if the magnetic permeability is conserved along the flow then the magnitude of the magnetic lines is constant along the flow and conversely [vide, (3.18)].

Theorem (1) : For ferrofluid with $\rho \neq P + \mu H^2$ and $3\rho \neq 3P + 5\mu H^2$, if $\dot{\mu} = 0$, then

$$\underset{u}{\mathcal{L}} R_{ab} = 0 \iff \underset{u}{\mathcal{L}} g_{ab} = 0.$$

Proof : It is clear from the definition of Ricci tensor that

$$\underset{u}{\mathcal{L}} g_{ab} = 0 \implies \underset{u}{\mathcal{L}} R_{ab} = 0. \qquad \ldots (3.19)$$

Now from the definition of Ricci collineation (1.2) we write

$$\underset{u}{\mathcal{L}} R_{ab} = R_{ab;c}u^c + R_{ac}u^c{}_{;b} + R_{bc}u^c{}_{;a}.$$

For the expression (II.2.3) of R_{ab} we write from the above equation under the imposed condition $\dot{\mu} = 0$

$$\underset{u}{\mathcal{L}} R_{ab} = \dot{A}\, u_a u_b - \dot{B}\, g_{ab} + A(\dot{u}_a u_b + u_a \dot{u}_b) -$$

$$- B\,(u_{a;b} + u_{b;a}) - \mu[H_a(H_{b;c}u^c + H_c u^c{}_{;b}) +$$

$$+ H_b(H_{a;c}u^c + H_c u^c{}_{;a})]$$

$$= T_1 + T_2 + T_3 + T_4 \text{ (say)},$$

where $T_1 = \dot{A}\, u_a u_b - \dot{B}\, g_{ab}$,

$T_2 = A(\dot{u}_a u_b + u_a \dot{u}_b)$,

$T_3 = -B(u_{a;b} + u_{b;a})$,

$$T_4 = -\mu\,[H_a(H_{b;c}u^c + H_c u^c{}_{;b}) + H_b(H_{a;c}u^c + H_c u^c{}_{;a})].$$

Thus to prove $\underset{u}{\pounds}\, R_{ab} = 0 \Longrightarrow \underset{u}{\pounds}\, g_{ab} = 0$, we start with

$$\underset{u}{\pounds}\, R_{ab} = 0,$$

i.e. $T_1 + T_2 + T_3 + T_4 = 0.$ ... (3.20)

According to Lemma (3) we can easily verify that

$$\dot{\rho} = \dot{P} = (H^2)^{\cdot} = 0, \quad \text{for } \dot{\mu} = 0,$$

i.e. $T_1 = 0.$ ... (3.21)

Also for the condition $\dot{\mu} = 0$, Lemma (1) implies

$$\dot{u}_a = 0\,,$$

$\Longrightarrow$ $T_2 = 0.$ ... (3.22)

Hence from the equations (3.20) to (3.22) we get

$$T_3 + T_4 = 0.$$

i.e. $B(u_{a;b} + u_{b;a}) + \mu\,[H_a(H_{b;c}u^c + H_c u^c{}_{;b}) +$

$+ H_b(H_{a;c}u^c + H_c u^c{}_{;a})\,] = 0,$

i.e. $BH^a\,(u_{a;b} + u_{b;a}) - \mu H^2(H_{b;c}u^c + H_c u^c{}_{;b}) -$

$- \frac{1}{2}\,\mu (H^2)^{\cdot} H_b + \mu H_b H_c u^c{}_{;a}\, H^a = 0,$

i.e. $BH^a(u_{a;b} + u_{b;a}) - \mu H^2(H_{b;c}u^c + H_c u^c{}_{;b}) = 0.$ ...(3.23)

using Maxwell (II.2.4).

Applying (3.9) and (3.6) with the understanding that $\dot{\mu} = 0$, $\dot{\rho} = 0$, $\dot{P} = 0$ and $(H^2)^{\cdot} = 0$ we obtain

$$(\rho - P - \mu H^2)(u_{a;b} + u_{b;a}) H^a = 0.$$

[vide, (3.2)].

Hence for $\rho \neq P + \mu H^2$, we write

$$(u_{a;b} + u_{b;a}) H^a = 0.$$

Consequently (3.23) implies

$$H_{b;c} u^c + H_c u^c{}_{;b} = 0. \qquad \ldots (3.24)$$

So finally we get

$$T_4 = 0. \qquad \ldots (3.25)$$

Thus by using (3.21), (3.22) and (3.25) in (3.20) we get

$$T_4 = u_{a;b} + u_{b;a} = 0,$$

i.e. $\underset{u}{\pounds}\, g_{ab} = 0.$

Here the proof is complete.

(B) <u>Ricci Collineation Along The Magnetic Field Lines</u> :

If the space-time of ferrofluid admits the Ricci collineation along the magnetic lines $\bar{H}$ then we write from the equations (1.2) and (3.1)

$$(Au_a u_b - Bg_{ab} - \mu H_a H_b)_{;c} H^c + (Au_a u_c - Bg_{ac} - \mu H_a H_c) H^c_{;b} + (Au_b u_c - Bg_{bc} - \mu H_b H_c) H^c_{;a} = 0,$$

i.e.
$$A_{;c} H^c u_a u_b - B_{;c} H^c g_{ab} - \mu_{;c} H^c H_a H_b - B(H_{a;b} + H_{b;a}) + A[u_a(u_c H^c_{;b} + u_{b;c} H^c) + u_b(u_{a;c} H^c + u_c H^c_{;a})] + \mu[H_a(H_{b;c} H^c + H_c H^c_{;b}) + H_b (H_{a;c} H^c + H_c H^c_{;a})] = 0. \quad \ldots (3.26)$$

<u>Step I</u> : The deductions of $(\underset{H}{\pounds} R_{ab}) H^a = 0$, $(\underset{H}{\pounds} R_{ab}) H^a H^b = 0$, $(\underset{H}{\pounds} R_{ab}) H^a u^b = 0$:

By contracting (3.26) with H^a we obtain

$$- B_{;c} H^c H_b + \mu_{;c} H^c H^2 H_b - BH^a(H_{a;b} + H_{b;a}) + A(H^a u_b u_{a;c} H^c + H^a u_b u_c H^c_{;a}) + \mu H^2 (H_{b;c} H^c + H^c H_{c;b}) - \mu H^a H_b (H_{a;c} H^c + H_c H^c_{;a}) = 0,$$

$$\because \quad u_a H^a = 0, \quad g_{ab} H^a H^b = - H^2.$$

Now changing dummy suffix a $\longleftrightarrow$ c whenever necessary and using the results $H^c H_{c;a} = - \frac{1}{2} (H^2)_{;a}$ and $u^c H_{c;a} = - u^c_{;a} H_c$ we get

$$- B_{;a} H^a H_b + \mu_{;a} H^a H_b H^2 - BH^a (H_{a;b} + H_{b;a}) + \mu H^2 (H_{a;b} + H_{b;a}) H^a + \mu (H^2)_{;a} H^a H_b = 0.$$

Further by employing (3.2) and simplifying we deduce

$$(\rho - P - \mu H^2)_{;a} H^a H_b - (\rho - P - \mu H^2) \times$$

$$\times (H_{a;b} + H_{b;a}) H^a = 0. \qquad \ldots (3.27)$$

Further contracting (3.27) with H^b and using the results $H^b H_b = - H^2$ and $H^a H_{a;b} = - \frac{1}{2} (H^2)_{;b}$ we write

$$(\rho - P - \mu H^2)_{;b} H^b H^2 + (\rho - P - \mu H^2)(H^2)_{;b} H^b = 0.$$

Also by contracting (3.27) with u^b and using $u^b H_b = 0$ provides

$$(\rho - P - \mu H^2)(H_{a;b} + H_{b;a}) H^a u^b = 0.$$

But the conservation Law (2.5) ==> $\rho - P - \mu H^2 \neq 0$, so we write

$$(H_{a;b} + H_{b;a}) H^a u^b = 0. \qquad \ldots (3.29)$$

<u>Step II</u> : The deduction of $g^{ab} (\underset{H}{\mathcal{L}} R_{ab}) = 0$:

By inner multiplying (3.26) with g^{ab} we infer

$$A_{;c} H^c - 4B_{;c} H^c + \mu_{;c} H^c H^2 - B(H^b_{;b} + H^a_{;a}) +$$

$$+ A [u^b u_a H^a{}_{;b} + u^a u_b H^b{}_{;a}] - \mu (H^b H_{b;a} H^a +$$

$$+ H^b H_a H^a{}_{;b} + H^a H_{a;b} H^b + H^a H_b H^b{}_{;a}] = 0.$$

where $g^{ab} u_a u_b = 1$, $g^{ab} H_a H_b = - H^2$, $u^b u_{b;c} = 0$ and changing dummy suffixes in the last term.

Further by virtue of condition $H^b H_{b;a} = -\frac{1}{2}(H^2)_{;a}$, we get

$$(A-4B)_{;b}H^b + \mu_{;b}H^bH^2 - 2B\, H^b_{\ ;b} + 2AH_{a;b}u^au^b +$$

$$+ 2\mu(H^2)_{;b}H^b = 0.$$

On using (3.2) in this relation we show

$$(\rho - 3P)_{;b}H^b + (\rho - P + \mu H^2)\, H^b_{;b} +$$

$$+ 2(\rho + P + \mu H^2)\dot{u}_b H^b - \mu(H^2)_{;b}H^b = 0. \quad \ldots (3.30)$$

<u>Step III</u> : The deductions of $u^au^b\,(\underset{H}{\pounds}\, R_{ab}) = 0$:

By transvecting (3.26) with u^au^b and using $g_{ab}u^au^b = 1$ and $u_{b;a}u^b = 0$ we produce

$$A_{;c}H^c - B_{;c}H^c - B\, u^au^b\,(H_{a;b} + H_{b;a}) +$$

$$+ A(u^bu_cH^c_{\ ;b} + u^au_cH^c_{;a}) = 0.$$

On using the condition $H_{a;b}u^b = \dot{H}_a$ in this equation and simplifying we write

$$(A-B)_{;b}H^b + 2(A-B)\,\dot{H}_b\, u^b = 0\,,$$

i.e. $(\rho + 3P + \mu H^2)_{;b}H^b + 2(\rho + 3P + \mu H^2)\,\dot{H}_bu^b = 0\,.$

... (3.31)

[vide, (3.2)]

Lemma 4 : For the space-time of ferrofluid admitting the Ricci collineation along the magnetic lines implies that the flow is expansion free and energy density is constant along the flow iff the magnetic permeability is conserved along the flow.

Proof : The Maxwell equation (II.2.5) can be expressed as

$$\mu H^2\theta + \dot{\mu} H^2 = \mu (H_{a;b} + H_{b;a}) H^a u^b.$$

By using (3.29) in this equation we get

$$\mu H^2 \theta + \dot{\mu} H^2 = 0,$$

i.e. $\theta = 0 \Longleftrightarrow \dot{\mu} = 0.$ ... (3.32)

Applying (3.32) to the continuity equation (II.3.3) we get

$$\mu \dot{\rho} - (\rho + P + \frac{1}{2} \mu H^2) \dot{\mu} = 0,$$

This with $\rho + P + \mu H^2 \neq 0$, implies

$$\dot{\rho} = 0 \Longleftrightarrow \dot{\mu} = 0 . \qquad \text{... (3.33)}$$

The results (3.32) and (3.33) justify the Lemma.

Lemma (5) : If the space-time of ferrofluid admits the Ricci collineation along the magnetic lines then

$$H^b{}_{;b} = 0 \Longleftrightarrow \mu_{;b}H^b = 0,$$

$$\dot{u}_b H^b = 0 \Longleftrightarrow \mu_{;b}H^b = 0,$$

provided $\rho \neq P + 5/3\, \mu H^2$.

<u>Proof</u> : The conservation law (2.5) and the equation (3.28) together imply

$$(\rho - P - \mu H^2)[H^b_{;b}H^2 - (H^2)_{;b}H^b] = 0.$$

$$\Longrightarrow \quad H^b_{;b}H^2 = (H^2)_{;b}H^b \qquad \because \quad \rho \neq P + \mu H^2. \qquad \ldots (3.34)$$

After simplifying (3.28) and using (3.34) we obtain

$$\rho_{;b}H^b - P_{;b}H^b - \mu_{;b}H^bH^2 + (\rho - P - 2\mu H^2) H^b_{;b} = 0. \qquad \ldots (3.35)$$

If the equation of continuity (II.3.7) and the Maxwell equation (II.2.6) are combined together then they lead to the result

$$\mu(\rho + P)H^b_{;b} + \mu P_{;b}H^b + (\rho + P + \tfrac{1}{2} \mu H^2)\mu_{;b}H^b = 0. \qquad \ldots (3.36)$$

By eliminating $P_{;b}H^b$ between (3.35) and (3.36) we get

$$\mu \rho_{;b} H^b + 2\mu (\rho - \mu H^2)H^b_{;b} + (\rho + P - \tfrac{1}{2} \mu H^2) \mu_{;b}H^b = 0 . \qquad \ldots (3.37)$$

Now we apply the Maxwell equation (II.2.6) in (3.31) in order to obtain

$$\mu \rho_{;b}H^b + 3\mu P_{;b}H^b + (2\rho + 6P + 3\mu H^2)\mu_{;b} H^b + \mu(2 \rho + 6P + 3\mu H^2) H^b_{;b} = 0.$$

Further using (3.36) in this equation we derive

$$\mu \rho_{;b} H^b + (- \rho + 3P + \frac{3}{2} \mu H^2) \mu_{;b}H^b +$$

$$+ (- \rho + 3P + 3\mu H^2) H^b_{;b} = 0. \qquad \ldots (3.38)$$

On subtracting (3.37) from (3.38), we lead to the result

$$2(\rho - P - \mu H^2)\mu_{;b}H^b + 3\mu(\rho - P - 5/3\mu H^2)H^b_{;b} = 0.$$

$$\ldots (3.39)$$

If $\rho \neq P + 5/3\ \mu H^2$ and since $\rho \neq P + \mu H^2$ then (3.39) gives

$$H^b_{;b} = 0 \iff \mu_{;b}H^b = 0. \qquad \ldots (3.40)$$

Further we use the Maxwell equation (II.2.6) in (3.39) to get the result

$$2(\rho - P - \mu H^2)\mu_{;b}H^b + 3\mu(\rho - P - 5/3\mu H^2) \times$$

$$\times (\mu \dot{H}_b u^b - \mu_{;b}H^b) = 0,$$

i.e. $(\rho + P + 3\mu H^2)\mu_{;b}H^b - 3\mu (\rho - P - 5/3\ \mu H^2)\dot{u}_b H^b = 0.$

If $\rho \neq P + 5/3\ \mu H^2$, then this result implies

$$\dot{u}_b H^b = 0 \iff \mu_{;b}H^b = 0. \qquad \ldots (3.41)$$

Hence the proof of the Lemma is complete.

<u>Interpretations</u> : If the space-time of ferrofluid admits the Ricci collineation along the magnetic lines then

i) the magnetic lines are divergence free iff the magnetic permeability is constant along the magnetic lines [vide (3.40)].

ii) if the flow is geodesic along the magnetic lines then the magnetic permeability is preserved along the magnetic lines and vice-versa. [vide (3.42)]

Lemma (6) : (Dynamical conditions) : For the space-time of ferrofluid admitting the Ricci collineation if $\varrho \neq P + 5/3\, \mu H^2$ then

$$(H^2)_{;b}H^b = 0, \quad \varrho_{;b}H^b = 0, \quad P_{;b}H^b = 0 \iff \mu_{;b}H^b = 0.$$

Proof : The equations (3.34) and (3.39) imply

$$2H^2 (\varrho - P - \mu H^2)\mu_{;b}H^b + 3\mu (\varrho - P - 5/3\, \mu H^2) \times$$
$$\times (H^2)_{;b} H^b = 0.$$

If $\varrho \neq P + 5/3\, \mu H^2$ then this result gives

$$(H^2)_{;b} H^b = 0 \iff \mu_{;b} H^b = 0. \qquad \ldots (3.42)$$

$$\because \varrho \neq P + \mu H^2.$$

The equations (3.36) and (3.39) will lead to the result

$$P_{;b} H^b =0 \iff \mu_{;b} H^b = 0. \qquad \ldots (3.43)$$

Further the equations (3.37) and (3.39) imply the result

$$\varrho_{;b}H^b = 0 \iff \mu_{;b} H^b = 0. \qquad \ldots (3.44)$$

So the proof of the Lemma is complete.

Interpretations : If the space-time of ferrofluid admits the Ricci collineation along the magnetic lines and if the magnetic permeability is conserved along the magnetic lines then (i) the magnitude of magnetic lines, (ii) isotropic pressure and (iii) the energy density are conserved along the magnetic lines and conversely [vide, (3.42), (3.43), (3.44)].

Theorem (2) : For the space-time of ferrofluid with $\rho \neq P + 5/3\, \mu H^2$ if $\mu_{;b} H^b = 0$ then

$$\underset{H}{\pounds} R_{ab} = 0 \iff \underset{H}{\pounds} g_{ab} = 0.$$

Proof : By the definition of Ricci tensor it is clear that

$$\underset{H}{\pounds} g_{ab} = 0 \implies \underset{H}{\pounds} R_{ab} = 0. \qquad \ldots (3.45)$$

To prove the converse part we consider the Lie derivative of the Ricci tensor by using (1.2), along the magnetic lines is given by

$$\underset{H}{\pounds} R_{ab} = R_{ab;c}H^c + R_{ac}H^c{}_{;b} + R_{bc}H^c{}_{;a},$$

i.e. $$\underset{H}{\pounds} R_{ab} = A_{;c}H^c u_a u_b - B_{;c}H^c g_{ab} - \mu_{;c}H^c H_a H_b -$$
$$- B(H_{a;b} + H_{b;a}) + A[u_a(u_c H^c{}_{;b} + u_{b;c}H^c) +$$
$$+ u_b(u_{a;c}H^c + u_c H^c{}_{;a})] - \mu [H_a(H_{b;c}H^c +$$
$$+ H_c H^c{}_{;b}) + H_b(H_{a;c}H^c + H_c H^c{}_{;a})], \qquad \ldots (3.46)$$

$$= T_1 + T_2 + T_3 + T_4 \quad \text{(say)} .$$

where $T_1 = A_{,c}H^c u_a u_b - B_{;c}H^c g_{ab} - \mu_{;c}H^c H_a H_b$,

$T_2 = - B(H_{a;b} + H_{b;a})$,

$T_3 = A\,[u_a(u_c H^c{}_{;b} + u_{b;c}H^c) + u_b(u_{a;c}H^c + u_c H^c{}_{;a})]$,

$T_4 = - \mu[H_a(H_{b;c}H^c + H_c H^c{}_{;b}) + H_b(H_{a;c}H^c + H_c H^c{}_{;a})]$.

Here $\underset{H}{\pounds} R_{ab} = 0 \Longrightarrow T_1 + T_2 + T_3 + T_4 = 0.$ ... (3.47)

<u>To show $T_1 = 0$:</u>

From Lemma (6) we have if $\mu_{;b}H^b = 0$ then

$$\rho_{;b}H^b = 0, \quad P_{;b}H^b = 0 \text{ and } (H^2)_{;b}H^b = 0.$$

Consequently, $T_1 = 0$, ... (3.48)

[vide, (3.2)].

<u>To show $T_4 = 0$:</u>

The equation (3.27) implies that

$$[\rho_{;a}H^a - P_{;a}H^a - \mu(H^2)_{;a}H^a]\, H_b - \mu_{;a}H^a H_b H^2 +$$

$$+ (\rho - P - \mu H^2)\,(H_{a;b} + H_{b;a})\, H^a = 0.$$

Applying Lemma (6) to this equation and taking $\mu_{;b}H^b = 0$ and since $\rho \neq P + \mu H^2$, we get

$$(H_{a;b} + H_{b;a})\, H^a = 0.$$

Consequently $\quad T_4 = 0.$ $\quad \ldots$ (3.49)

[vide, Maxwell (II.2.4)]

<u>To show $T_3 = 0$:</u>

On contracting (3.46) with u^a and using the conditions $u^a H_a = 0$, $u^a u_a = 1$ we have

$$A_{;c}H^c u_b - B_{;c}H^c u_b - B\,(H_{a;b} + H_{b;a})\, u^a +$$

$$+ A(u_c H^c{}_{;b} + u_{b;c}H^c + u_c H^c{}_{;a} u^a u_b) -$$

$$- \mu(H_{a;c}H^c + H^c H_{c;a}) u^a H_b = 0,$$

i.e.

$$(A-B)_{;c}H^c u_b - B(H_{a;b} + H_{b;a})\, u^a +$$

$$+ A(H_{a;b}u^a + u_{b;a}H^a) + A\,\dot{H}_a u^a u_b -$$

$$- \mu H_b\,(\mu H^2\theta + \dot{\mu}\, H^2) = 0. \qquad \ldots (3.50)$$

where $H_{a;c}u^c = \dot{H}_a$ and [vide, Maxwell equation (II.2.5)].

Now we use (3.2) and Lemmas (4), (5) and (6) to get

$$(\rho + 3P + \mu H^2)(H_{a;b} + H_{b;a})\, u^a = 0, \qquad \ldots (3.51)$$

$$\because \quad \mu_{;b}\, H^b = 0.$$

$$\Longrightarrow (H_{a;b} + H_{b;a})\, u^a = 0.$$

Consequently by Maxwell equation (II.25) this equation yields

$$T_3 = 0. \qquad \ldots (3.52)$$

Thus we get from the equations (3.47), (3.48), (3.49) and (3.52) that

$$T_2 = 0,$$

i.e. $\underset{H}{\pounds}\, g_{ab} = 0.$

Hence we have established the Theorem.

4. Ricci Collineation And Isometries (Coupling) :

In this Section we study the consequencies of the simultaneous occurrence of Ricci collineation and isometry in the space-time of ferrofluid. As the claims stated below follow directly from Lemmas and Theorems proved in Section 3, we give only statements of the claims.

Claim 1 : In the space-time of ferrofluid

$$\underset{u}{\pounds}\, g_{ab} = 0 \text{ and } \underset{H}{\pounds}\, R_{ab} = 0 \quad \text{imply}$$

i) Kinematical conditions :

$$\theta = 0, \quad \dot{u}_a = 0, \quad \sigma_{ab} = 0, \quad H^b{}_{;b} = 0, \quad \mathcal{E} \neq 0.$$

ii) Dynamical conditions :

$$\dot{\rho} = 0, \quad (H^2)^{\cdot} = 0, \quad \dot{\mu} = 0, \quad \rho_{;b} H^b = 0,$$

$$p_{;b}H^b = 0, \quad (H^2)_{;b}H^b = 0, \quad \mu_{;b}H^b = 0.$$

<u>Claim 2</u> : In the space-time of ferrofluid

$$\underset{H}{\pounds}\, g_{ab} = 0 \text{ and } \underset{u}{\pounds}\, R_{ab} = 0 \text{ imply}$$

i) Kinematical conditions :

$$\theta = 0, \quad \dot{u}_a = 0; \quad H^b{}_{;b} = 0, \quad \mathcal{E} \neq 0, \quad \mathcal{J}_{ab} \neq 0.$$

ii) Dynamical conditions :

$$\dot{\mu} = 0, \quad \dot{\rho} = 0, \quad (H^2)^{\cdot} = 0, \quad \dot{p} = 0,$$

$$p_{;b}H^b = 0, \quad (H^2)_{;b} H^b = 0, \quad \mu_{;b} H^b = 0.$$

<u>Claim 3</u> : In the space-time of ferrofluid

$$\underset{u}{\pounds}\, g_{ab} = 0, \quad \underset{H}{\pounds}\, g_{ab} = 0 \text{ and } \underset{u}{\pounds}\, R_{ab} = 0 \text{ imply}$$

i) Kinematical Conditions :

$$\dot{u}_b = 0, \quad \theta = 0, \quad \sigma_{ab} = 0,$$

$$H^b{}_{;b} = 0, \quad \mathcal{E} \neq 0, \quad \mathcal{J}_{ab} \neq 0.$$

ii) Dynamical conditions :

$$\dot{u} = 0, \quad \dot{\rho} = 0, \quad (H^2)^{\cdot} = 0, \quad \dot{P} = 0,$$

$$\mu_{;b} H^b = 0, \quad (H^2)_{;b} H^b = 0, \quad P_{;b} H^b = 0.$$

Claim 4 : In the space-time of ferrofluid

$$\underset{u}{\pounds}\, g_{ab} = 0, \quad \underset{H}{\pounds}\, g_{ab} = 0 \text{ and } \underset{H}{\pounds}\, R_{ab} = 0 \text{ imply}$$

i) Kinematical Conditions :

$$\dot{u}_b = 0, \quad \theta = 0, \quad \sigma_{ab} = 0,$$

$$H^b{}_{;b} = 0, \quad \mathcal{E} \neq 0, \quad \mathcal{J}_{ab} \neq 0.$$

ii) Dynamical conditions :

$$\dot{u} = 0, \quad (H^2)^{\cdot} = 0, \quad \dot{\rho} = 0, \quad P_{;b} H^b = 0,$$

$$u_{;b} H^b = 0, \quad \rho_{;b} H^b = 0, \quad (H^2)_{;b} H^b = 0.$$

By using earlier results one can easily verify the following statement :

Statement : If the matter energy density is preserved along Killing magnetic lines and the isotropic pressure is conserved along the Killing flow then

$$\underset{H}{\pounds}\, R_{ab} = 0 \iff \underset{u}{\pounds}\, R_{ab} = 0.$$