CHAPTER - I

FUNDAMENTAL NOTIONS (GEOMETRICAL AND DYNAMICAL)

1. Historical Contemplation.

2. Space-time Congruences and Corresponding Parameters.

3. The Stress-Energy Tensor for Ferrofluid and Energy Conditions.

1. Historical Contemplation

According to a general article on physics of ferrofluid by Mehta (1989), ferrofluid is defined as the magnetically soft fluid. He has presented a systematic study of the properties of ferrofluid as a function of field and a physical variable like temperature. A classical theory of ferrofluid with its applications in physically interesting magnetic materials is examined by Neuringer and Rosenweig (1964). In 1978 Cissoko has investigated the general relativistic equations characterizing the ferrofluid. Following the magnetohydrodynamical system as formulated by Lichnerowicz (1967), Yodzis (1971) and Mason (1976) have discussed the rate of growth of magnetic energy density during the gravitational collapse. This work has prompted Ray and Banerji (1980) for investigating such a growth rate in ferrofluid. Accordingly a result like variation of the magnetic permeability accelerates the growth of magnetic energy density is established. Here ferrofluid means an infinitely conducting relativistic charged fluid with ~~the~~ variable magnetic permeability. Thus Lichnerowicz's formalism deals with the constant magnetic permeability whereas the ferrofluid deals with variable magnetic permeability.

The purpose of this dissertation is to investigate the geometrical and dynamical properties of the space-time associated with ferrofluid. This target is accomplished under the plausible geometrical restrictions like motions and collineations and dynamical restrictions like Killing and harmonic physical vector fields. The corresponding results are presented in Chapter II and Chapter III.

Section 2 includes the preliminary ideas regarding space-time congruences and corresponding parameters. The formulation of stress-energy tensor characterizing ferrofluid is presented in Section 3. The energy conditions regarding T_{ab} of ferrofluid are stated.

<u>Precursory Notions</u> :

We mainly deal with four-dimensional space-time manifold V_4 with the Locally Lorentzian metric of signature (−, −, −, +). The various symbols used are as follows :

- ; : Covariant derivative,
- , : Partial derivative,
- $\dot{X}$: Covariant derivative of X with respect to time-like vector,
- ( ) : Symmetrization bracket,
- [ ] : Antisymmetrization bracket,
- $\pounds_{X}$: Lie derivative along the vector $\bar{X}$.

2. Space-Time Congruences And Corresponding Parameters

"A congruence of ~~the~~ curves is an uncountable family of non-intersecting space-filling curves, one through each point of space or a given manifold. The first comprehensive treatment of Congruences in Riemannian geometry is due to Eisenhart (1926)". These congruences can be separated in to three categories as (a) Time-like congruence, (b) Space-like congruence and (c) Null-~~like~~ congruence, according to the nature of the tangent vector fields at every point of the curve of these congruences. Thus a congruence of space-time curves is called as time-like, space-like and null ~~like~~ if tangents drawn at every point of these curves are time-like, space-like and null-~~like~~ respectively. The space-time geometry can well be described with the help of these congruences and their associated ~~scalars.~~ Parameters

The various types of ~~scalars~~ Parameters corresponding to different types of congruences that are available in the literature may be summarised as follows :

A) Time-Like Congruence : (Greenberg, 1970 a)

The time-like congruence is characterised by its tangent vector $\bar{u}$, here taken as the unit flow vector. The ~~scalar~~ Parameters associated with this time-like flow vector $\bar{u}$ are defined and discussed by Greenberg (1970a). According to him the expansion scalar θ, the acceleration vector $\dot{u}_a$, the shear ~~tensor~~ Scalar σ_{ab} and the rotation tensor w_{ab} are defined through the following expressions :

Expansion : $\theta = u^b_{;b}$, ... (2.1)

Acceleration : $\dot{u}_a = u_{a;b}\, u^b$, ... (2.2)

Shear tensor : $\sigma_{ab} = u_{(a;b)} - \dot{u}_{(a} u_{b)} - \frac{1}{3}\,\theta\, h_{ab}$, ...(2.3)

Rotation tensor : $w_{ab} = u_{[a;b]} - \dot{u}_{[a} u_{b]}$ . ... (2.4)

Here the 3-projection operator h_{ab} is defined as

$$h_{ab} = g_{ab} - u_a u_b \; . \qquad \dots (2.5)$$

This immediately gives the properties

$$h_{ab}u^b = 0, \quad h^a_a = 3 \text{ and } h^a_b h^b_c = h^a_c \; . \qquad \dots (2.6)$$

It follows from the equation (2.2)

$$\dot{u}_a u^a = 0, \qquad \because \; u^a u_a = 1. \qquad \dots (2.7)$$

This implies that the acceleration vector is normal to the time-like flow vector and hence it is space-like. Also from the equations (2.3) and (2.4) we have

$$\sigma_{ab}\, u^b = 0 = w_{ab}u^b \; , \; (\bar{u} \text{ -orthogonality}) \qquad \dots (2.8)$$

$$\sigma^a_a = 0 = w^a_a \; , \; (\text{trace-free}) \qquad \dots (2.9)$$

$$\sigma_{ab}\,\sigma^{ab} = 2\,\sigma^{2}, \quad w_{ab}\,w^{ab} = 2w^{2}, \text{ (defined quantities)}. \qquad \ldots(2.10)$$

Thus by utilising the expressions (2.1) to (2.4) the gradient of the flow vector $\bar{u}$ can be written in the form

$$u_{a;b} = \sigma_{ab} + w_{ab} + \dot{u}_a u_b + \frac{1}{3}\,\theta\, h_{ab}\,. \qquad \ldots (2.11)$$

These parameters defined above will describe the behaviour of time-like congruence in the space-time.

B) <u>Space-Like Congruence</u> : (Greenberg, 1970 b).

The parameters corresponding to space-like congruence $\{\bar{m}\}$ as introduced by Greenberg (1970b) are presented below :

Expansion Scalar : $\mathcal{E} = \frac{1}{2}\,(m^{a}_{;a} - m_{a;b}\,u^{a}u^{b})$, $\quad\ldots (2.12)$

Shear tensor : $\mathcal{J}_{ab} = \frac{1}{2}\; P^{c}_{a}P^{d}_{b}\;(m_{c;d} + m_{d;c}) - \mathcal{E}\,P_{ab}$,

$\ldots (2.13)$

Rotation tensor : $\bar{R}_{ab} = \frac{1}{2}\,P^{c}_{a}P^{d}_{b}\,(m_{c;d} - m_{d;c})$.

$\ldots (2.14)$

Here the 2-space projection operator P_{ab} is defined as

$$P_{ab} = g_{ab} - u_a u_b + m_a m_b\,. \qquad \ldots (2.15)$$

So also the time like flow vector $\bar{u}$, the space-like vector

$\bar{m}$ satisfy the conditions

$$u^a u_a = 1, \; m^a m_a = -1, \; m^a u_a = 0. \qquad \ldots (2.16)$$

Also from (2.15) we get

$$P_{ab} = P_{ba}, \; P_{ab}u^b = 0 = P_{ab}\, m^b, \; P^a_c P^c_b = P^a_b \; . \ldots (2.17)$$

Further the space-like congruence $\{\bar{m}\}$ has to satisfy natural transport laws in the form

$$u_{a;b}m^b = m_{a;b}u^b - u_a m_{b;c}\, u^c u^b + m_a m_{b;c}\, m^c u^b. \qquad \ldots (2.18)$$

C) <u>Null-Like Congruence</u> : (Pirani, 1964).

The optical scalars associated with the null-like congruence $\{\bar{n}\}$ are given by Pirani (1964). Their defining expressions are listed below :

$$\text{Expansion scalar : } \theta^* = \frac{1}{2}\, n^a_{;a} , \qquad \ldots (2.19)$$

$$\text{Shear scalar : } |\sigma^*| = [\, \frac{1}{2}\, n_{(a;b)}\, n^{a;b} - \theta^{*2}\,]^{1/2}, \qquad \ldots (2.20)$$

$$\text{Twist Scalar : } w^* = (\, \frac{1}{2}\, n_{[a;b]}\, n^{a;b}\,)^{1/2}, \qquad \ldots (2.21)$$

where the null vector n^a satisfies the nulity condition $n^a n_a = 0$.

D) Congruences In-General : (Lukacevic, 1982).

According to his process, u^a is a field of 4-velocities and ξ^a be any other field, which is in the sense that it may be tangent to a time-like, a space-like or a null congruence of lines. So that

$$g_{ab}\, u^a u^b = 1, \quad g_{ab}\, \xi^a \xi^b = \pm f^2. \qquad \ldots (2.22)$$

Accordingly he has introduced the relative kinematical quantities known as the shear X_{ab}, rotation Ψ_{ab} and the scalar expansion ξ in terms of the following defined expressions:

$$\xi = \xi^c{}_{;c} - u^c u^d\, \xi_{d;c}\,, \qquad \ldots (2.23)$$

$$X_{ab} = \frac{1}{2}\left[\xi_{a;b} + \xi_{b;a} + u_a\, \xi^c\, u_{b;c} + u_b\, \xi^c\, u_{a;c} + \xi_{c;a} u_b u^c + u_a u^c\, \xi_{c;b}\right] - \frac{1}{3}\left(\xi^c{}_{;c} - u^c u^d\, \xi_{d;c}\right)(g_{ab} - u_a u_b), \qquad \ldots (2.24)$$

$$\Psi_{ab} = \frac{1}{2}\left[\xi_{a;b} - \xi_{b;a} + u_b\, \xi^c\, u_{a;c} - u_a\, \xi^c\, u_{b;c} + u_a u^c\, \xi_{c;b} - u_b u^c\, \xi_{c;a}\right]. \qquad \ldots (2.25)$$

The new space operator defined here is given by

$$l_{ab} = g_{ab} - \frac{1}{\lambda^2 \pm f^2} \left(\pm f^2 \xi_a \xi_b + \lambda u_a \xi_b + \right.$$

$$\left. + \lambda u_b \xi_a \pm f^2 u_a u_b \right), \qquad \ldots (2.26)$$

with $\lambda = u^c \xi_c$.

It follows from the equations (2.23) to (2.25)

$$\xi_{a;b} = X_{ab} + \Psi_{ab} - \frac{1}{3} \vartheta h_{ab} + u_b \xi^c u_{a;c} -$$

$$- u_a u^c \xi_{c;b} . \qquad \ldots (2.27)$$

<u>Observations</u> :

a) If we take $f^2 = +1$, then (2.22) implies that ξ_a as time-like unit vector. Hence in this case parameters given by (2.23) to (2.25) coincide with the parameters (2.1), (2.3) and (2.4).

b) If we take $f^2 = -1$ then (2.23) directs that ξ_a is space-like unit vector then the parameters given by (2.23) to (2.25) will match with parameters (2.12) to (2.14) of the space-like congruence $\{ \bar{m} \}$.

c) The projection operator (2.26) becomes ~~3-space operator when $f^2 = 1$ and~~ 2-space operator when $f^2 = \pm 1$.

E) Special Type Of Congruence Due To Stachel (1980):

His process of defining new types of parameters includes the following aspects. Let v^a be an arbitrary contravariant and V_a be the vector parallel to covariant components v_a. His choice gives

$$V_a v^a = 1, \quad v^a = \rho\, u^a, \quad V_a = \frac{1}{\rho} u_a, \quad \text{with } u^a u_a = 1,$$

ρ is any constant. ... (2.28)

He further defines the terms

$$C^a_b = v^a V_b \, , \quad B^a_b = \delta^a_b - C^a_b \, , \qquad \text{... (2.29)}$$

$$B^{cd}_{ab} = B^c_a \, B^d_b \, . \qquad \text{... (2.30)}$$

Accordingly he has introduced new types of parameters with respect to $\overline{V}$ as follows :

Shear (Including expansion): $H_{ab} = \frac{1}{2} B^{cd}_{ab} \left(V_{c;d} + V_{d;c} \right),$

... (2.31)

Rotation : $\Omega_{ab} = \frac{1}{2} B^{cd}_{ab} \left(V_{c;d} - V_{d;c} \right).$ ... (2.32)

Remark : For $\rho = 1$ these parameters coincide with the parameters for time-like unit congruence.

3. Stress-Energy Tensor For Ferrofluid

In 1967, Lichnerowicz has formulated a Relativistic magnetohydrodynamical Scheme consisting of a space-time filled with infinitely conducting charged fluid having infinite electrical conductivity and constant magnetic permeability (μ^*). It is characterised by the Stress-energy tensor

$$T_{ab} = (\rho + P + \mu^* H^2) u_a u_b - (P + \frac{1}{2} \mu^* H^2) g_{ab} -$$

$$- \mu^* H_a H_b \quad \ldots (3.1)$$

According to the new Scheme introduced by Cissoko (1978) and studied by Ray and Banarji (1990), the constraint on constant magnetic permeability is relaxed and it is allowed to vary. Thus this new scheme composed of infinitely conducting charged fluid with variable magnetic permeability. This type of fluid is designated as ferrofluid and described by the stress-energy tensor.

$$T_{ab} = (\rho + P + \mu H^2) u_a u_b - (P + \frac{1}{2} \mu H^2) g_{ab} -$$

$$- \mu H_a H_b \quad \ldots (3.2)$$

(μ is a variable magnetic permeability).

The terms involved in (3.2) have the meanings

i) μ is variable magnetic permeability,

ii) ρ is the matter energy density,

iii) P is the isotropic pressure,

iv) the time-like vector $\bar{u}$ and space-like vector $\bar{H}$ are such that

$$u^a u_a = 1, \; H^a H_a = - H^2, \; u^a H_a = 0. \quad \ldots (3.3)$$

Eigen values of T_{ab} :

We have from (3.2) and (3.3)

$$T_{ab}\, u^a = \left(\rho + \frac{1}{2}\, \mu H^2 \right) u_b \, , \qquad \ldots (3.4)$$

$$T_{ab}\, u^a u^b = \rho + \frac{1}{2}\, \mu\, H^2 \, , \qquad \ldots (3.5)$$

$$T_{ab}\, H^a = - \left(P - \frac{1}{2}\, \mu\, H^2\right) H_b \, , \qquad \ldots (3.6)$$

$$T_{ab}\, H^a H^b = \left(P - \frac{1}{2}\, \mu H^2 \right) H^2 . \qquad \ldots (3.7)$$

It follows from these relations that $\bar{u}$ is time-like eigen vector of T_{ab} with time-like eigen value given by

$$e_1 = \rho + \frac{1}{2}\, \mu H^2 , \qquad \ldots (3.8)$$

and $\bar{H}$ is the space-like eigen vector of T_{ab} with space-like eigen value given by

$$e_2 = \left(P - \frac{1}{2}\, \mu H^2\right) H^2 . \qquad \ldots (3.9)$$

So also the trace of the (3.2) is given by

$$T = T_{ab}\, g^{ab} = \rho - 3P . \qquad \ldots (3.10)$$

Energy-Conditions :

The well-known Energy-conditions Hawking and Ellis (1968) stress-energy tensor corresponding to ferrofluid are given by i) the weak energy condition: ~~is stated through~~

$$T^{ab} u_a u_b \geqslant 0,$$

$$\Longrightarrow \quad \rho + \frac{1}{2} \mu H^2 \geqslant 0.$$

ii) the strong energy condition: ~~is given by the inequality~~

$$T^{ab} u_a u_b - \frac{1}{2} T \geqslant 0,$$

$$\Longrightarrow \quad \rho + 3P + \mu H^2 \geqslant 0.$$