CHAPTER - II

KILLING AND HARMONIC FIELDS IN FERROFLUID SPACE-TIME

1. Introduction.

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4. Killing and Harmonic Vector Fields.

1. Introduction

The stress-energy tensor characterising ferrofluid encountered in Chapter I involves time-like flow vector $\bar{u}$ and space-like magnetic field vector $\bar{H}$. The geometrical restrictions called as infinitesimal isometries play a vital role in solving the system of differential equations governing the space-time of ferrofluid. In particular fluid flow $\bar{u}$ and magnetic field vector $\bar{H}$ are considered as a infinitesimal isometry generator. Our goal here is to workout some interesting features about the dynamical variables of space-time of ferrofluid under the restrictions of killing and harmonic fields coupled with the Ricci collineation.

The field equations governing ferrofluid are presented systematically in Section 2. This also includes some interesting consequences of Maxwell equations. It is proved that the magnetic permeability is conserved along the divergence free magnetic lines iff the acceleration is normal to these lines. In Section 3 equations of motions for ferrofluid including equation of continuity and stream lines are derived. It is proved that (i) for the expansion free flow of ferrofluid the matter energy density is preserved along the flow if and only if the magnetic permeability is preserved along the flow. (ii) the isotropic pressure is conserved along the magnetic lines normal to 4-acceleration iff the magnetic permeability is preserved along these lines. The consequences regarding the killing and harmonic flow lines and the magnetic lines admitted by space-time of ferrofluid are discussed in Section 4. The associated claims emerging out of Bianchi identities are also included.

2. Field Equations Governing Ferrofluid

We study the field equations that are necessary for describing the geometrical and dynamical features of ferrofluid. These are mainly the gravitational field equations and Maxwell field equations. The coupled Einstein Maxwell field equations are stated below :
The field equations for gravitation are

$$R_{ab} - \frac{1}{2} R g_{ab} = \left(- \frac{8 \pi G}{C^4} \right) T_{ab} = - K T_{ab} , \quad \ldots (2.1)$$

where the stress-energy tensor T_{ab} is given by (I.3.2). These are 10 independent non-linear differential equations of second order. The dynamical expression for Ricci tensor compatible with ferrofluid exhibited by (I.3.2) can be written with the use of (2.1), in the following manner :

$$R_{ab} = - K \left(T_{ab} - \frac{1}{2K} R g_{ab} \right), \quad \ldots (2.2)$$

i.e. $R_{ab} = -K \left[(\rho + P + \mu H^2)\, u_a u_b - \frac{1}{2} (\rho - P + \mu H^2) g_{ab} - \right.$

$$\left. - \mu H_a H_b \right] . \qquad \ldots (2.3)$$

Further the electromagnetic field has to satisfy the Maxwell equations that are valid for the condition of infinite conductivity which are in the form

$$[\mu (H^a u^b - u^a H^b)]_{;b} = 0,$$

i.e. $\mu(H^a u^b_{;b} + H^a_{;b} u^b - u^a_{;b} H^b - u^a H^b_{;b}) +$

$$+ \mu_{;b} (H^a u^b - u^a H^b) = 0. \qquad \ldots (2.4)$$

If we contract this (2.4) with H_a and use the notations $u^a_{;a} = \theta$, $H^a H_a = -H^2$ then the othogonality condition $u^a H_a = 0$, produces a result

$$\mu \left[H^2\theta + \frac{1}{2} (H^2)^{\bullet} + u_{a;b} H^a H^b \right] + \dot{\mu} H^2 = 0 \ldots. \quad (2.5)$$

Here we have denoted $\dot{\mu} = \mu_{;b} u^b$ and $(H^2)^{\bullet} = (H^2)_{;b} u^b$.

So also the transvection of (2.4) with u_a and using the orthogonality condition we obtain

$$\mu (\dot{u}_b H^b + H^b_{;b}) + \mu_{;b} H^b = 0. \qquad \ldots (2.6)$$

Conclusion : The equation (2.6) implies that the magnetic permeability is conserved along the divergence free magnetic lines iff the 4-acceleration is normal to magnetic lines.

3. Equations Of Motion For Ferrofluid.

The well-known contracted Bianchi identities provide the local conservation laws through the conservation equation $T^{ab}_{;b} = 0$. These equations for T^{ab} expression equation (I.3.2) yields the following equation :

$$(\rho + P + \mu H^2)_{;b} u^a u^b + (\rho + P + \mu H^2)(u^a_{;b} u^b + u^a u^b_{;b}) -$$
$$- (P + \frac{1}{2}\mu H^2)_{;b} g^{ab} - \mu_{;b} H^a H^b - \mu(H^a_{;b} H^b + H^a H^b_{;b}) = 0. \qquad \ldots (3.1)$$

On contracting (3.1) with u_a and using the conditions that $u^a u_a = 1$, $u^a H_a = 0$ and simplifying we get

$$(\rho + \frac{1}{2}\mu H^2)_{;b} u^b + (\rho + P + \mu H^2)\,\theta -$$
$$- \mu H^a_{;b} u_a H^b = 0, \qquad \ldots (3.2)$$

i.e. $$\dot{\rho} + (\rho + P)\,\theta - \frac{1}{2}\dot{\mu} H^2 = 0, \text{ (vide 2.5)}. \qquad \ldots (3.3)$$

This is the continuity equation for ferrofluid.

Note 1 : If the flow lines are expansion free ($\theta = 0$) then the equation (3.3) implies that

$$\dot{\rho} = 0 \iff \dot{\mu} = 0 .$$

Hence we conclude that the matter energy density is conserved along the expansion free flow lines iff the magnetic permeability is kept invariant along these lines. Further by rearranging the terms in (3.1) and then using the continuity equation (3.3) we get the equation for stream lines in the form

$$(\rho + P + \mu H^2)\,\dot{u}^a - (P + \tfrac{1}{2}\mu H^2)_{;b}\, h^{ab} - (\mu H^b)_{;b}\, H^a = 0 \qquad \ldots (3.4)$$

It reveals from (3.4) that path lines are deviated from geodesic path. Further by contracting (3.1) with H_a and using that $u^a H_a = 0$, $H^a H_a = -H^2$ we deduce

$$(\rho + P + \mu H^2) u^a{}_{;b} u^b H_a - P_{;b} H^b + \tfrac{1}{2}\mu_{;b} H^2 H^b + \mu H^2 H^b{}_{;b} = 0 \qquad \ldots (3.5)$$

Rearranging the terms and using notation $u^a_{;b}\, u^b = \dot{u}^a$ the equation (3.5) becomes

$$(\rho + P)\,\dot{u}^a H_a - P_{;b} H^b + \left\{\mu(\dot{u}^a H_a + H^b_{;b}) + \tfrac{1}{2}\mu_{;b}\, H^b\right\} H^2 = 0. \qquad \ldots (3.6)$$

Applying (2.6) in (3.6) and simplifying we get

$$(\rho + P)\,\dot{u}_b H^b = (P_{;b} + \tfrac{1}{2}\mu_{;b}\, H^2) H^b \qquad \ldots (3.7)$$

Note 2 :If $\dot{u}_b H^b = 0$, then we write from (3.7)

$$P_{;b}H^b = 0 \iff \mu_{;b}H^b = 0$$

This implies that the isotropic pressure remains invariant along magnetic lines iff the magnetic permeability is preserved along these lines.

Raychaudhary's Equation In Ferrofluid :

In case of 4-velocity vector field $\bar{u}$, Ricci identities imply (Ellis, 1971).

$$R_{abcd}u^a u^d = u_{b;cd}u^d - \dot{u}_{b;c} + (u_{b;d})(u^d{}_{;c}), \quad \ldots (3.8)$$

where R_{abcd} is the Riemann curvature tensor.

On contracting (3.8) with g^{ab} and taking $d = b$ we get

$$-R_{ab}u^a u^b = (u^c{}_{;c})_{;b}u^b - \dot{u}^c{}_{;c} + (u^c{}_{;b})(u^b{}_{;c}). \quad \ldots (3.9)$$

Then by using (I.2.11) and symmetry properties of σ_{ab}, w_{ab}, h_{ab} and orthogonality conditions, (3.9) provide

$$-R_{ab}u^a u^b = \dot{\theta} - \dot{u}^c{}_{;c} + \sigma^2 - w^2 + 1/3\ \theta^2 .$$

If we write $\dot{\theta} = \frac{d\theta}{ds}$ and $\dot{u}^c = \bar{w}^c$, then the above equation becomes

$$\frac{d\theta}{ds} = -R_{ab}u^a u^b + w^2 - \sigma^2 - \frac{1}{3}\theta^2 + \bar{w}^a{}_{;a}. \quad \ldots (3.10)$$

The relation (3.10) is called Raychaudhary (1955) equation in general relativity. If we put the value of Ricci tensor from (2.3) then we get the Raychaudhary's equation for the space-time of ferrofluid as

$$\frac{d\theta}{ds} = \frac{K}{2}(\rho + 3p + \mu H^2) + w^2 - \sigma^2 - \frac{1}{3}\theta^2 + w^a_{;a} . \qquad \ldots (3.11)$$

The equation (3.11) exhibits the effect of kinematical entities on the dynamical quantities of ferrofluid.

<u>Remark</u> : In terms of parameters (I.2.23) to (I.2.27) defined by Lukacevic (1982), the equation (3.10) is written as

$$\frac{d\tilde{\theta}}{d\tilde{s}} = \psi^2 - x^2 - \frac{1}{3}\eta^2 + (\eta^b \eta^a{}_{;b})_{;a} - R_{ab}\eta^a\eta^b + 2u^a\eta^b{}_{;a}u^c\eta_{c;b} + (u^a u^b \eta_{b;a})^2, \qquad \ldots (3.12)$$

where $\tilde{\theta} = \eta^a{}_{;a}$ and $\frac{d}{d\tilde{s}} = \eta^a \partial_a$.

4. Killing And Harmonic Vector Fields.

<u>Definition 1</u> : Killing vector field (Yano, 1965) :

The vector $\bar{K}$ is said to be Killing vector if

$$K_{a;b} + K_{b;a} = 0. \qquad \ldots (4.1)$$

Note 1 : If the flow vector $\bar{u}$ is Killing then we write by using (4.1)

$$u_{a;b} + u_{b;a} = 0. \qquad \ldots (4.2)$$

Note 2 : If the magnetic field vector $\bar{H}$ is Killing then we write by using (4.1)

$$H_{a;b} + H_{b;a} = 0. \qquad \ldots (4.3)$$

Definition 2 : Harmonic vector field (Yano, 1965) :

The vector $\bar{K}$ is said to be harmonic vector if

$$K_{a;b} - K_{b;a} = 0, \quad K^{a}{}_{;a} = 0. \qquad \ldots (4.4)$$

Note 3 : If the flow vector $\bar{u}$ is harmonic then we write by (4.4)

$$u_{a;b} - u_{b;a} = 0, \quad u^{a}{}_{;a} = 0 . \qquad \ldots (4.5)$$

Note 4 : If the magnetic field vector $\bar{H}$ is harmonic then by (4.4) we write

$$H_{a;b} - H_{b;a} = 0, \quad H^{a}{}_{;a} = 0. \qquad \ldots (4.6)$$

Propagation of Vector-Field With Fundamental Velocity :

If H^a represent the Killing magnetic lines which propagate with the fundamental velocity then $R_{ab} H^a = 0$, (Khade, 1973). Accordingly by using the Ricci tensor (2.3) for ferrofluid we obtain

$$[(\rho + P+\mu H^2)\, u_a u_b - \frac{1}{2}(\rho - P + \mu H^2)\, g_{ab} -$$

$$- \mu\, H_a H_b]\, H^a = 0,$$

i.e. $(\rho - P - \mu H^2)\, H_b = 0.$

where $u_a H^a = 0$, $H^a H_a = - H^2$.

This implies that

$$(\rho - P - \mu H^2) = 0. \qquad \ldots (4.7)$$

Hence if the magnetic line in ferrofluid propagate with fundamental velocity then (4.7) gives

$$\rho = P + \mu H^2.$$

Remark (1) : On the same line we can prove that when H^a represent the harmonic magnetic lines which propagate with the fundamental velocity then also we get the equation of state, ($\rho = P + \mu H^2$).

Remark (2) : By the same procedure we can prove that if u^a is killing (or harmonic) flow vector then it does not propagate with the fundamental velocity. Since $R_{ab}u^a = 0 \implies \rho + 3p + \mu H^2 = 0$, which is physically impossible.

(A) Flow Vector $\bar{u}$ Is Killing :

Theorem 1 : For the Killing flow of ferrofluid

i) $\rho_{;b} u^b = 0, \quad (H^2)_{;b} u^b = 0 \iff \mu_{;b} u^b = 0,$

ii) $H^b_{;b} = 0, \quad P_{;b} H^b = 0 \iff \mu_{;b} H^b = 0.$

Proof : If u^a is killing flow vector then contracting (4.2) with g^{ab}, u^b, $H^a H^b$ and using (I.2.1) to (I.2.3) we obtain

$$\theta = 0, \quad \sigma_{ab} = 0, \quad \dot{u}_a = 0, \quad u_{a;b} H^a H^b = 0. \qquad \ldots (4.8)$$

Applying (3.3) and (4.8) together, we get

$$\rho_{;b} u^b - \frac{1}{2} \mu_{;b} u^b H^2 = 0. \qquad \ldots (4.9)$$

This leads to the result

$$\rho_{;b} u^b = 0 \iff \mu_{;b} u^b = 0. \qquad \ldots (4.10)$$

Also by making use of (4.8) in (2.5), it is deduced that

$$\frac{1}{2} \mu (H^2)_{;b} u^b + \mu_{;b} u^b H^2 = 0. \qquad \ldots (4.11)$$

This implies the condition

$$(H^2)_{;b} u^b = 0 \iff \mu_{;b} u^b = 0. \qquad \ldots (4.12)$$

On substituting (4.8) in (2.6) we get the equation

$$\mu H^{b}_{;b} + \mu_{;b} H^{b} = 0,$$

i.e. $H^{b}_{;b} = 0 \Longleftrightarrow \mu_{;b} H^{b} = 0.$ ... (4.13)

Applying (4.8) in (3.5) we produce

$$p_{;b} H^{b} + \frac{1}{2} H^{2} \mu_{;b} H^{b} = 0,$$

i.e. $p_{;b} H^{b} = 0 \Longleftrightarrow \mu_{;b} H^{b} = 0.$ ... (4.14)

Hence (4.10), (4.12) to (4.14) are required conditions of the Theorem.

Remark (1) : The Killing flow can be characterized as geodesic, expansion free and shear free (or as an essentially rotating flow), [vide, (4.8)].

For the ferrofluid admitting an essentially rotational flow, we deduce that

i) The energy density and magnitude of magnetic lines are preserved along the flow vector iff the magnetic permeability is conserved along the flow lines [vide, (4.10) and (4.12)].

ii) The magnetic lines are divergence free and the isotropic pressure of the fluid is conserved along the magnetic lines iff the magnetic permeability is preserved along the magnetic lines [vide, (4.13) and (4.14)].

Deduction (1) : The equations (4.9) and (4.11) give rise to

$$4\,\rho_{;b}u^b + \mu\,(H^2)_{;b}\,u^b = 0,$$

i.e. $\rho_{;b}\,u^b = 0 \iff (H^2)_{;b}\,u^b = 0.$

This shows that the necessary and sufficient condition for the conservation of the energy density along the flow lines is that the magnitude of the magnetic field is preserved along the flow vector.

Deduction (2) : The equations (4.13) and (4.14) together will imply

$$P_{;b}\,H^b - \frac{1}{2}\,\mu H^2\,H^b_{;b} = 0,$$

i.e. $P_{;b}H^b = 0 \iff H^b{}_{;b} = 0.$

From this equation we observe that if the isotropic pressure is constant along the magnetic lines then the magnetic lines are divergence free and vice versa.

B. The Magnetic Field Vector $\bar{H}$ is Killing :

Theorem (2) : The necessary conditions that the ferrofluid admits Killing magnetic lines are

i) $u^b{}_{;b} = 0,\quad \rho_{;b}\,u^b = 0 \iff \mu_{;b}\,u^b = 0,$

ii) $P_{;b}H^b = 0,\quad (H^2)_{;b}\,H^b = 0 \iff \mu_{;b}\,H^b = 0.$

Proof : If H^a is Killing magnetic vector field then contracting (4.3) with g^{ab}, $u^a u^b$ and $H^a H^b$ we get

$$H^{b}{}_{;b} = 0, \quad \dot{u}^{b}H_{b} = 0, \quad (H^{2})_{;b} H^{b} = 0. \qquad \ldots (4.15)$$

Further if we inner multiply (4.3) with $u^{a} H^{b}$, then it provides

$$(H^{2})_{;b} u^{b} + 2 H^{a} H^{b} u_{a;b} = 0. \qquad \ldots (4.16)$$

By using (4.15) in (I.2.12)and (I.2.13) we show that

$$\mathcal{E} \neq 0 \text{ and } \zeta_{ab} \neq 0. \qquad \ldots (4.17)$$

Applying (4.16) in Maxwell equation (2.5) we derive

$$\mu \theta + \mu_{;b} u^{b} = 0. \qquad \ldots (4.18)$$

This implies that

$$\theta = 0 \Longleftrightarrow \dot{\mu} = 0. \qquad \ldots (4.19)$$

Further, using (4.18) in the continuity equation (3.3) we get

$$\mu \dot{\rho} - (\rho + P + \tfrac{1}{2} \mu H^{2}) \dot{\mu} = 0.$$

This implies

$$\dot{\rho} = 0 \Longleftrightarrow \dot{\mu} = 0, \quad \because \rho + P + \tfrac{1}{2} \mu H^{2} \neq 0. \qquad \ldots (4.20)$$

By utilizing (4.15) in (2.6) yields the result

$$\mu_{;b} H^{b} = 0. \qquad \ldots (4.21)$$

Finally we write by (3.5), (4.15) and (4.21)

$$P_{;b}H^{b} = 0. \qquad \ldots (4.22)$$

Hence (4.15), (4.19) to (4.22) are the necessary conditions

Remark (2) : The Killing magnetic lines are characterized as expansion free, shear free and divergence free. Moreover the magnitude of the magnetic field is constant along the magnetic lines [vide, (4.8),(4.10)].

Interpretations : i) The 4-acceleration is orthogonal to the magnetic field [vide, (4.15)]. ii)The ferrofluid admitting the killing magnetic field we observe that

(a) the flow lines are expansion free and the matter energy density is conserved along the flow lines iff magnetic permeability is constant along the flow lines [vide, (4.19), (4.20)].

(b) the magnetic permeability is preserved along the magnetic lines [vide, (4.21)].

(c) the isotropic pressure is conserved along the magnetic lines [vide, (4.22)].

Deduction 3 : If we eliminate $\dot{\mu}$ between (3.3) and (4.18) we have

$$\dot{\rho} + (\rho + P + \frac{1}{2} \mu H^2) \theta = 0,$$

i.e., $\dot{\rho} = 0 \iff \theta = 0.$

This shows that the Killing magnetic lines imply that the flow is expansion free iff the matter energy density is constant along the flow.

C. The Flow Vector $\bar{u}$ is Harmonic :

Theorem (3) : If the flow of the ferrofluid is harmonic then

i) $\rho_{;b} u^b = 0, \iff \mu_{;b}u^b = 0,$

ii) $P_{;b}H^b = 0, \quad H^b_{;b} = 0 \iff \mu_{;b}H^b = 0.$

Proof : If u_a is harmonic flow then contracting (4.5) with $u^a u^b$ we obtain

$$\theta = 0 \quad \text{and } \dot{u}_a = 0. \qquad \ldots (4.23)$$

Applying(4.23) and definition (4.5) to (I.2.4) we get

$$w_{ab} = 0. \qquad \ldots (4.24)$$

Also by making use of (4.23) in the continuity equation (3.3) it is obtained that

$$\dot{\rho} - \frac{1}{2} \dot{\mu} H^2 = 0,$$

i.e. $\dot{\rho} = 0 \iff \dot{\mu} = 0.$ ... (4.25)

The second condition in (4.23) is applied to (2.6) and (3.7) in order to get

$$\mu H^{b}{}_{;b} + \mu_{;b}H^{b} = 0 \text{ and } P_{;b}H^{b} + \frac{1}{2} H^{2} \mu_{;b}H^{b} = 0, \quad \ldots (4.26)$$

i.e. $H^{b}_{;b} = 0 \iff \mu_{;b}H^{b} = 0$ and $P_{;b}H^{b} = 0 \iff \mu_{;b}H^{b} = 0.$

... (4.27)

Hence (4.25) and (4.27) are the required conditions.

Remark (3) : The harmonic flow can be characterized as geodesic, expansion free and rotation free [or as an essentially shearing flow], [vide, (4.23), (4.24)].

Interpretations : The ferrofluid when admits harmonic flow, then we conclude that

i) the proper energy density is preserved along the flow iff the magnetic permeability is also preserved along the flow, [vide (4.25)].

ii) the magnetic lines are divergence free and the isotropic pressure is constant along the magnetic lines then the magnetic permeability is conserved along the magnetic lines and conversely [vide, (4.27)].

Corollary : If we mix up both the results of (4.26) we produce

$$P_{;b}H^b - \frac{1}{2}\mu H^2 H^b{}_{;b} = 0,$$

i.e. $P_{;b}H^b = 0 \iff H^b{}_{;b} = 0.$

This result shows that for harmonic flow the isotropic pressure is constant along the magnetic lines is the necessary and sufficient condition for the magnetic lines to be divergence free.

D. The Magnetic Field Vector Is Harmonic :

Theorem (4) : For ferrofluid if H^a are harmonic magnetic lines and the 4-acceleration is orthogonal to magnetic lines and the isotropic pressure is constant along the magnetic lines iff the magnetic permeability is preserved along the divergence free magnetic lines.

Proof : If H^a is harmonic magnetic field then by definition (4.6) and (I.2.14) we have

$$H^b{}_{;b} = 0 \text{ and } \overline{R}_{ab} = 0. \qquad \ldots (4.28)$$

Applying (4.28) in Maxwell equation (2.6) we get

$$\mu\, \dot{u}_b H^b + \mu_{;b} H^b = 0, \qquad \ldots (4.29)$$

i.e. $\dot{u}_b H^b = 0 \iff \mu_{;b} H^b = 0.$ ... (4.30)

The equations (3.5) and (3.6) yield with the use of (4.29)

$$(\rho + P + \frac{1}{2}\mu H^2)\, \mu_{;b}H^b + \mu P_{;b}H^b = 0,$$

i.e. $\mu_{;b}H^b = 0 \iff P_{;b}H^b = 0, \quad \therefore \rho + P + \frac{1}{2}\mu H^2 \neq 0$ ... (4.31)

The results (4.30) and (4.31) prove the theorem.

Remark (4) : The result (4.28) shows that the harmonic magnetic lines are divergence free and also rotation free.

E. The Flow Vector $\bar{u}$ and Magnetic Field Vector $\bar{H}$ both are Killing :

In this case we can prove

1) $\theta = 0, \quad \sigma_{ab} = 0, \quad \dot{u}_a = 0, \quad (H^2)^{\cdot} = 0, \quad \dot{\rho} = 0, \quad \dot{\mu} = 0,$

2) $H^b_{;b} = 0, \quad \mathcal{E} \neq 0, \quad \mathcal{J}_{ab} \neq 0, \quad P_{;b}H^b = 0$

$(H^2)_{;b}\, H^b = 0, \quad \mu_{;b}\, H^b = 0.$

F. The Flow Vector $\bar{u}$ And Magnetic Field Vector $\bar{H}$ both are Harmonic :

In this case we prove

1) $\theta = 0, \quad w_{ab} = 0, \quad \dot{u}_a = 0,$ and $(H^2)^{\cdot} = 0,$

$\dot{\rho} = 0 \iff \dot{\mu} = 0.$

2) $H^b_{;b} = 0, \quad \mathcal{E} \neq 0, \quad \bar{R}_{ab} = 0, \quad P_{;b}\, H^b = 0, \quad \mu_{;b}H^b = 0.$

G. If Flow Vector $\bar{u}$ is Killing And The Magnetic Field Vector $\bar{H}$ is Harmonic Then We have the following results,

$$\theta = 0, \quad \sigma_{ab} = 0, \quad \dot{u}_a = 0, \quad (H^2)^{\cdot} = 0, \quad \dot{\rho} = 0, \quad \dot{\mu} = 0,$$

$$H^b_{;b} = 0, \quad \mathcal{E} \neq 0, \quad \bar{R}_{ab} = 0, \quad P_{;b}H^b = 0, \quad \mu_{;b}H^b = 0.$$

H. If the Flow vector $\bar{u}$ is Harmonic and the Magnetic Field Vector $\bar{H}$ is Killing then we have the following results

$$\theta = 0, \quad w_{ab} = 0, \quad \dot{u}_a = 0, \quad \dot{\rho} = 0, \quad \dot{\mu} = 0,$$

$$H^b_{;b} = 0, \quad \mathcal{E} \neq 0, \quad J_{ab} \neq 0, \quad P_{;b}H^b = 0,$$

$$\mu_{;b}\, H^b = 0.$$

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