CHAPTER - II

RAY ANALYSIS OF STATIONARY SHEARING DUST

1. INTRODUCTION :

a) SHEAR TENSOR :

The Shear tensor q_{ab} is defined as

$$q_{ab} = u_{(a;b)} - \frac{1}{c^2}\dot{u}_{(a}u_{b)} - \frac{1}{3}\gamma_{ab}\theta \qquad (2.1)$$

where

$$\gamma_{ab} = g_{ab} - \frac{1}{c^2}u_a u_b \;, \quad \theta = u^a{}_{;a} .$$

The signature in this thesis is (-, -, -, +). Therefore, from the definition of magnitude of q_{ab} -

$$2q^2 = q^{ab}q_{ab} \,; \quad q^2 \leqslant 0, \text{ we get } \; q = 0 \;\text{ iff }\; q_{ab} = 0.$$

The scalar q^2 enters into the Raychaudhari's equation for the propagation of θ (Hawking and Ellis 1973).

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2q^2 + 2w^2 - \frac{1}{2}K\rho^* \qquad (2.2)$$

where ρ^* is the density of a dust distribution.

$$2w^2 \equiv w^{ab}w_{ab} \;; \quad \dot{\theta} \equiv \theta_{,a}u^a .$$

Since q^2 appears with - ve sign in (2.2) we infer that shear induces contraction.

In 1971, Ellis gave the propagation equation of shear tensor :

$$q_{ab;c}u^c = w_{ca}w^c{}_b - \theta^c{}_a\theta_{bc} - C_{acbd}u^c u^d + \frac{1}{3}\gamma_{ab}\left(\frac{1}{3}\theta^2 + 2q^2 - 2w^2\right). \qquad (2.3)$$

This equation is independent of the density for dust distribution.

b) JAUMANN DERIVATIVE :

In Classical Continuum Mechanics Jaumann derivative is defined as

$$\underset{u}{J}\, T_{ab} = \dot{T}_{ab} - T_{ak}\, \partial_{[k} u_{b]} - T_{bk}\, \partial_{[k} u_{a]},$$

where $u^a = \frac{dx^a}{dt}$, $\dot{T}_{ab} = \frac{dT_{ab}}{dt}$. (2.4)

and Tab is the stress tensor of the continuum. In Relativistic Continuum Mechanics Jaumann derivative is obtained on replacing the partial derivatives by the covariant derivative in (2.4). Hence the Jaumann derivative of a tensor field with respect to $u^a = \frac{dx^a}{ds}$

$$\underset{u}{J}\, T^{a_1 \cdots a_k}_{b_1 \cdots b_s} = T^{a_1 \cdots a_k}_{b_1 \cdots b_s ; c}\, u^c + T^{a_1 \cdots c}_{b_1 \cdots b_s}\, W^{a_k}_{c} + \cdots + T^{c \cdots a_k}_{b_1 \cdots b_s}\, W^{a_1}_{c} - T^{a_1 \cdots a_k}_{c \cdots b_s}\, W^{c}_{b_1} - \cdots - T^{a_1 \cdots a_k}_{b_1 \cdots c}\, W^{c}_{b_s},$$

where (2.5)

$$W_{ab} = u_{[a;b]} - \frac{1}{c^2}\, \dot{u}_{[a} u_{b]}$$

is the rotation tensor of u^a. The vector field u^a may be spacelike or timelike or null.

c) STATIONARY SHEAR :

The concept of stationary shear has been introduced by Eringen (1962) in his Nonlinear Theory of continuous media. We extend this concept to Relativity (by invoking 'Cuckoo Process' of Synge) in the following way. (Vide (32a), p-62)

Definition :

The shear tensor field q_{ab} is said to be stationary if and only if the Jaumann stress rate of q_{ab} vanishes i.e.,

$$\underset{u}{J}\, q_{ab} = 0, \qquad (2.6)$$

In Sec. 2 dynamical equation of shear for a general distribution of matter is derived following Carter and Quintanna (1977). Sec. 3 contains the tensor version for stationary shearing dust. Ray concomitants of q_{ab}, ω_{ab} are given in Sec. 4 for the choice of

$$u^a = (2)^{-1/2} (l^a + n^a).$$

The cumbersome equations for shearing dust are analysed for Petrov type D spacetimes in Sec. 5. The necessary and sufficient conditions for stationary shear are expressed in spin coefficients.

2) DYNAMICAL EQUATION OF SHEAR :

Here we follow the treatment of Carter and Quintana (1977) to derive the kinematical equation and the dynamical equation for shear tensor.

KINEMATICAL EQUATION OF SHEAR

Carter and Quintana (1977) expressed the Lie derivative of a material tensor Z_{ab} (i.e. u - orthonormal tensor) with respect to u^a as

$$\underset{u}{\pounds} Z_{ab} = \dot{Z}_{ab} + Z_{cb}\Upsilon^c{}_a + Z_{ac}\Upsilon^c{}_b , \qquad (2.7)$$

where
$$\dot{Z}_{ab} \equiv \gamma_a{}^c \gamma_b{}^d Z_{cd;e} u^e ,$$
$$\Upsilon_{ab} \equiv \Theta_{ab} + W_{ab} . \qquad (2.8)$$

Here the expansion tensor Θ_{ab} is defined :

$$\Theta_{ab} = q_{ab} + \frac{1}{3}\theta \gamma_{ab} \qquad (2.9)$$

which has the following relation with the projective tensor γ_{ab}:

$$\underset{u}{\pounds} \gamma_{ab} = 2\,\Theta_{ab} . \qquad (2.10)$$

Since q_{ab} is a material tensor, we use (2.7) and (2.8) to get

$$\underset{u}{\pounds} q_{ab} = \tilde{q}_{ab} + q_{ak}\Upsilon^k{}_b + q_{kb}\Upsilon^k{}_a , \qquad (2.11)$$

$$= \frac{1}{2}\gamma_a^c \gamma_b{}^d [u_{c;d} + u_{d;c}]_{;k} u^k + \Theta_{ak}\Upsilon^k_b +$$
$$+ \Theta_{kb}\Upsilon^k{}_a - \frac{1}{3}(\dot{\theta}\gamma_{ab} + 2\theta\,\Theta_{ab}), \qquad (2.12)$$

Ellis introduced another u-orthogonal tensor Υ_{ab} through

$$u_{a;b} = \Upsilon_{ab} + \frac{1}{c^2}\dot{u}_a u_b . \qquad (2.13)$$

Expanding the first term on the right hand side of (2.12) we get

$$\underset{u}{\pounds} q_{ab} = \frac{1}{2}\gamma_a{}^c \gamma_b{}^d (u_{c;d;k}u^k + u_{d;c;k}u^k + \qquad (2.14)$$
$$+ \Theta_{ak}\Upsilon^k{}_b + \Theta_{kb}\Upsilon^k{}_a - \frac{1}{3}(\dot{\theta}\gamma_{ab} + 2\theta\,\Theta_{ab}).$$

Using Ricci identities

$$u_{c;d;k} + u_{d;c;k} = u^q R_{qcdk} \tag{2.15}$$

in (2.14) we obtain

$$\underset{u}{\pounds}\, q_{ab} = W_{ca} W^c_{\ b} + 2W^c_{\ (a}\theta_{b)c} + g^{cd}\theta_{ac}\theta_{bd} - u^c u^d R_{acbd} - \tfrac{1}{3}(\dot{\theta}\gamma_{ab} + 2\theta\theta_{ab}). \tag{2.16}$$

The curvature tensor R_{abcd} is related to the conformal curvature tensor through

$$R_{abcd} = C_{abcd} - \tfrac{1}{2}(g_{ac}R_{bd} - g_{ad}R_{bc} + g_{bd}R_{ac} - g_{bc}R_{ad}) - \tfrac{R}{6}(g_{ad}g_{bc} - g_{ac}g_{bd}).$$

Using this in (2.16) we get kinematical equation for shear tensor

$$\underset{u}{\pounds}\, q_{ab} = W_{ca} W^c_{\ b} + 2W^c_{\ (a}\theta_{b)c} + g^{cd}\theta_{ac}\theta_{bd} - u^c u^d C_{acbd} + \tfrac{1}{2}(R_{cd}u^c u^d - \tfrac{1}{3}R)\gamma_{ab} + \tfrac{1}{2}R_{ab}. \tag{2.17}$$

Einstein's field equations

$$R_{ab} - \tfrac{1}{2}R g_{ab} = -\frac{8\pi G}{c^4} T_{ab}$$

in the form

$$R_{ab} = -\frac{8\pi G}{c^4}\left(T_{ab} - \tfrac{1}{2}T g_{ab}\right).$$

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can be used in the last three terms of (2.17) for eliminating R_{ab} and introducing the prime dynamical variable T_{ab} the stress tensor. Consequently (2.17) gives on putting

$$T_{ab} = \rho^{*} u_a u_b + p^{ab}$$

the dynamical equation for shear as

$$
\begin{aligned}
\underset{u}{\pounds}\, q_{ab} = {} & W_{ka} W^{k}{}_{b} + 2 W^{k}{}_{(a} \theta_{b)k} + g^{km} \theta_{ak} \theta_{bm} - \\
& - C_{ab} - \frac{1}{3} \gamma_{ab} (4\pi G \rho^{*} + \dot{\theta}) - \frac{2}{3} \theta \theta_{ab} + \\
& + \frac{1}{(\rho^{*} c^{2} + p)^{2}} \gamma_{k(a} \gamma_{b)}{}^{m} (\rho^{*} c^{2} + p)_{;m} (\dot{p} u^{k} - c^{2} p_{,d} g^{kd}) - \\
& - \frac{c^{2}}{(\rho^{*} c^{2} + p)^{2}} \left[\gamma_{a}^{c} \gamma_{b}^{d} \left\{ p_{,c} p_{,d} - (\rho^{*} c^{2} + p) p_{,c;d} \right\} + \frac{1}{c^{2}} \dot{p} (\rho^{*} c^{2} + p) \theta_{ab} \right] - \\
& - \frac{4\pi G}{c^{2}} p \gamma_{ab} .
\end{aligned}
\qquad (2.18)
$$

For dust we put $p^{ab} = 0$ and so the evolution of the shear tensor for dust :

$$
\begin{aligned}
\underset{u}{\pounds}\, q_{ab} = {} & W_{ka} W^{k}{}_{b} + 2 W^{k}{}_{(a} \theta_{b)k} + g^{km} \theta_{ak} \theta_{bm} - \\
& - C_{ab} - \frac{1}{3} \gamma_{ab} (4\pi G \rho^{*} + \dot{\theta}) - \frac{2}{3} \theta \theta_{ab} .
\end{aligned}
\qquad (2.19)
$$

3) TENSOR VERSION OF STATIONARY SHEARING DUST :

From the definition of Lie derivative we have

$$\underset{u}{\pounds}\, q_{ab} = q_{ab;k}u^k + q_{ak}u^k{}_{;b} + q_{kb}u^k{}_{;a} \qquad (2.20)$$

and from the definition of Jaumann derivative we get

$$\underset{u}{J}\, q_{ab} = q_{ab;k}u^k - q_{ak}W^k{}_b - q_{kb}W^k{}_a\,, \qquad (2.21)$$

Eliminating the material derivative term

$$\underset{u}{\pounds}\, q_{ab} - \underset{u}{J}\, q_{ab} = q_{ak}u^k{}_{;b} + q_{ak}u^k{}_{;a} + $$
$$+ q_{ak}W^k{}_b + q_{kb}W^k{}_a \qquad (2.22)$$

when the shear is stationary, we impose

$$\boxed{\underset{u}{J}\, q_{ab} = 0}\,. \qquad (**)$$

Therefore, (2.22) becomes

$$\underset{u}{\pounds}\, q_{ab} - q_{ak}u^k{}_{;b} - q_{kb}u^k{}_{;a} - q_{ak}W^k{}_b - q_{kb}W^k{}_a = 0\,. \qquad (2.23)$$

Using dynamical equation (2.19) for dust and eliminating $\underset{u}{\pounds}\, q_{ab}$, we get

$$_{ab} = W_{ka}W^k{}_b + 2W^k{}_{(a}\theta_{b)k} + g^{km}\theta_{ak}\theta_{bm} - $$
$$-\frac{4\pi G\rho^*}{3}\gamma_{ab} - \frac{1}{3}\gamma_{ab}\dot{\theta} - \frac{2}{3}\theta\,\theta_{ab} - q_{ak}u^k{}_{;b} - $$
$$- q_{kb}u^k{}_{;a} - q_{ak}W^k{}_b - q_{kb}W^k{}_a\,. \qquad (2.24)$$

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4) RAY EQUIVALENTS OF g_{ab}, w_{ab}, $\dot{u}_a$, Θ FOR THE CHOICE OF u_a.

Let $u_a = (2)^{-1/2}(l_a+n_a)$, (2.25)

Therefore, the NP version of $u_{a;b}$ and $\dot{u}_a u_b$ can be given as

$$
\begin{aligned}
u_{a;b} &= (2)^{-1/2}(l_{a;b}+n_{a;b}) \\
&= (2)^{-1/2}\{(\gamma+\bar{\gamma})l_a l_b-(\alpha+\bar{\beta})l_a m_b- \\
&\quad -(\bar{\alpha}+\beta)l_a\bar{m}_b+(\epsilon+\bar{\epsilon})l_a n_b-(\nu-\bar{\tau})m_a l_b+ \\
&\quad +(\bar{\sigma}-\lambda)m_a m_b+(\bar{\rho}-\mu)m_a\bar{m}_b+(\pi-\bar{\kappa})m_a n_b+ \\
&\quad +(\bar{\nu}-\tau)\bar{m}_a l_b+(\rho-\bar{\mu})\bar{m}_a m_b+(\sigma-\bar{\lambda})\bar{m}_a\bar{m}_b+ \\
&\quad +(\bar{\pi}-\kappa)\bar{m}_a n_b-(\gamma+\bar{\gamma})n_a l_b+(\alpha+\bar{\beta})n_a m_b+ \\
&\quad +(\bar{\alpha}+\beta)n_a\bar{m}_b-(\epsilon+\bar{\epsilon})n_a n_b\}.
\end{aligned}
\tag{I}
$$

$$
\begin{aligned}
\dot{u}_a u_b &= u_{a;k}u^k u_b \\
&= (2)^{3/2}\{(\epsilon+\bar{\epsilon})l_a l_b-\bar{\kappa}m_a l_b-\kappa\bar{m}_a l_b+(\epsilon+\bar{\epsilon})l_a n_b- \\
&\quad -\bar{\kappa}m_a n_b-\kappa\bar{m}_a n_b+(\gamma+\bar{\gamma})l_a l_b-\bar{\tau}m_a l_b- \\
&\quad -\tau\bar{m}_a l_b+(\gamma+\bar{\gamma})l_a n_b-\bar{\tau}m_a n_b-\tau\bar{m}_a n_b+ \\
&\quad +\pi m_a l_b+\bar{\pi}\bar{m}_a l_b-(\epsilon+\bar{\epsilon})n_a l_b+\pi m_a n_b+ \\
&\quad +\bar{\pi}\bar{m}_a n_b-(\epsilon+\bar{\epsilon})n_a n_b+\nu m_a l_b+\bar{\nu}\bar{m}_a l_b- \\
&\quad -(\gamma+\bar{\gamma})n_a l_b+\nu m_a n_b+\bar{\nu}\bar{m}_a n_b-(\gamma+\bar{\gamma})n_a n_b\}.
\end{aligned}
\tag{II}
$$

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<u>The shear tensor</u> q_{ab}:

$$q_{ab} = u_{(a;b)} - \dot{u}_{(a}u_{b)} - \tfrac{1}{3}\gamma_{ab}\theta .$$

Using the equations (I) & (II) the NP version of is:

$$q_{ab} = A[V_a V_b - m_{(a}\bar{m}_{b)}] - B V_{(a}m_{b)} - (c.c.)$$
$$+ C m_a m_b + (c.c.),$$
$$A = -\frac{1}{3\sqrt{2}}[(\rho+\bar{\rho}) - (\mu+\bar{\mu}) + 2(\epsilon+\bar{\epsilon}) - 2(\gamma+\bar{\gamma})],$$
$$B = \frac{1}{2}[(\bar{\tau}+\pi) + 2(\alpha+\bar{\beta}) - (\bar{\kappa}-\nu)],$$
$$C = \frac{1}{\sqrt{2}}(\bar{\sigma}-\lambda), \quad V_a = \frac{1}{\sqrt{2}}(l_a - n_a). \tag{2.26}$$

<u>The vorticity tensor</u> w_{ab} :

$$w_{ab} = u_{[a;b]} - \dot{u}_{[a}u_{b]}$$

Using the equations (I) and (II) the NP version of :

$$w_{ab} = P V_{[a}m_{b]} + (c.c.) + Q m_{[a}\bar{m}_{b]},$$
$$P = \frac{1}{2}[(\bar{\tau}+\pi) - 2(\alpha+\bar{\beta}) - (\bar{\kappa}+\nu)],$$
$$Q = (2)^{-1/2}[(\bar{\rho}-\rho) - (\mu-\bar{\mu})]. \tag{2.27}$$

<u>The acceleration vector</u> $\dot{u}_a$:

$$\dot{u}_a = (2)^{-1/2}[(\epsilon+\bar{\epsilon}) + (\gamma+\bar{\gamma})]V_a + \frac{1}{2}[(\pi-\bar{\kappa}) - (\bar{\tau}-\nu)]m_a + (c.c.). \tag{2.28}$$

<u>The expansion</u> θ :

$$\theta = (2)^{-1/2}[(\epsilon+\bar{\epsilon}) - (\gamma+\bar{\gamma}) - (\rho+\bar{\rho}) + (\mu+\bar{\mu})]. \tag{2.29}$$

Here the symbol (c.c.) denotes the complex conjugate of the preceeding term.

Using (2.25) to (2.29), we get the NP version of eleven terms in (2.24) in Appendix-I of this chapter.

Using freedom condition on

$$\kappa = \pi = \epsilon = 0 \tag{2.30}$$

For dust

$$\dot{u}_a = 0$$

therefore, equation (2.28) with (2.30) implies

$$\gamma = -\bar{\gamma} \qquad \tau = \nu \tag{2.31}$$

Substituting the eleven terms of appendix-I in (2.24) we get an equation of the type.

$$\begin{aligned} J q_{ab} \equiv\ & (\ldots) l_a l_b + \cdots + (\ldots) l_a n_b + \\ & + (\ldots) m_a l_b + \cdots + (\ldots) m_a n_b + \\ & + (\ldots) \bar{m}_a l_b + \cdots + (\ldots) \bar{m}_a n_b + \\ & + (\ldots) n_a l_b + \cdots + (\ldots) n_a n_b = 0 \end{aligned}$$

Using the technique given in the (page ( ) chapter I we get seven equations which are enumerated later.

Kerr Newman Black hole is of Petrov type D and this prompts us to study at first Petrov type D gravitational field.

5. PETROV TYPE D STATIONARY SHEARING DUST :

Dynamical equation (2.24) yields :

$$\Psi_2+\bar{\Psi}_2=-\frac{4\pi G\rho^*}{3c^2}-\frac{1}{6c^2}(\Delta+D)(\mu+\bar{\mu}-\rho-\bar{\rho})-$$
$$-3\tau\bar{\tau}-3\bar{\tau}(\bar{\alpha}+\beta)-\frac{9}{2}\tau(\alpha+\bar{\beta})-$$
$$-\frac{5}{2}(\alpha+\bar{\beta})(\bar{\alpha}+\beta), \qquad (2.26)$$

$$\bar{\Psi}_1-\Psi_3=\frac{3}{4}(\bar{\mu}-\rho)+\frac{1}{4}(\bar{\rho}-\mu)(\alpha+\bar{\beta})+$$
$$+2(\bar{\alpha}+\beta)(\bar{\sigma}-\lambda)+\frac{9}{4}(\bar{\mu}-\rho)\bar{\tau}$$
$$-\frac{1}{2}(\bar{\sigma}-\lambda)\tau+\frac{1}{4}\bar{\tau}(\bar{\rho}-\mu), \qquad (2.27)$$

$$\Psi_2+\bar{\Psi}_2=-\frac{8\pi G\rho^*}{3c^2}\left(1-\frac{1}{2c^2}\right)+\left(-\frac{1}{3}+\frac{1}{6c^2}\right)(\Delta+D)\times$$
$$\times(\mu+\bar{\mu}-\rho-\bar{\rho})+\frac{5}{2}(\alpha+\bar{\beta})(\bar{\alpha}+\beta)-$$
$$-2\tau(\alpha+\bar{\beta})-2\bar{\tau}(\alpha+\bar{\beta}+\bar{\alpha}+\beta), \qquad (2.28)$$

$$\bar{\Psi}_0+\bar{\Psi}_4=2(\alpha+\bar{\beta})^2+\tau(\alpha+\bar{\beta})-\frac{7}{2}\bar{\tau}(\alpha+\bar{\beta})+$$
$$+(\bar{\sigma}-\lambda)(\rho-\bar{\rho}-\mu-\bar{\mu}), \qquad (2.29)$$

$$\Psi_2+\bar{\Psi}_2=\frac{8\pi G\rho^*}{3}+\frac{1}{3}(\Delta+D)(\mu+\bar{\mu}-\rho-\bar{\rho})+$$
$$+2\left[\bar{\tau}(\alpha+\bar{\beta})+\alpha\bar{\beta}\right]+\left[\bar{\tau}(\alpha-\bar{\tau})+\bar{\beta}\bar{\tau}+\right.$$
$$\left.+\bar{\tau}^2+\alpha^2+\bar{\beta}^2+\frac{1}{4}\left[\bar{\tau}(\bar{\alpha}-\tau)\right]+\right.$$

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$$+\frac{3}{4}\left[\bar{\tau}(\tau-\bar{\alpha}-\beta)-\tau(\alpha+\bar{\beta})\right]+$$
$$+\frac{5}{4}\bar{\tau}\beta+\frac{9}{4}(\bar{\alpha}+\beta)(\tau-\alpha-\bar{\beta})+\frac{3}{2}\rho^2+$$
$$+\frac{7}{6}\rho(\mu-\bar{\rho})+\frac{23}{6}\bar{\mu}(\bar{\rho}-\mu)-\frac{8}{3}\mu\bar{\rho}-$$
$$-\frac{1}{3}\bar{\mu}\rho+5\bar{\sigma}(\bar{\lambda}-\sigma)+3\lambda(\sigma-\bar{\lambda})+$$
$$+\frac{4}{3}(\bar{\rho}^2+\mu^2), \tag{2.30}$$

$$\Psi_3-\bar{\Psi}_1=\frac{5}{4}\bar{\tau}(\rho+\bar{\rho}-\mu-\bar{\mu})+\frac{1}{2}\tau(\bar{\sigma}-\lambda)+$$
$$+\frac{1}{2}\bar{\tau}(\bar{\mu}-\rho)+\frac{3}{2}\tau(\mu-\bar{\rho})+\frac{1}{4}(\alpha+\bar{\beta})\times$$
$$\times(\bar{\rho}-\mu)+\frac{5}{4}(\alpha+\bar{\beta})(\rho-\bar{\mu})+$$
$$+3(\bar{\alpha}+\beta)(\bar{\sigma}-\lambda), \tag{2.31}$$

$$\Psi_2+\bar{\Psi}_2=-\frac{4\pi G\rho^*}{3c^2}-\frac{1}{6c^2}(\Delta+D)(\mu+\bar{\mu}-\rho-\bar{\rho})$$
$$+\frac{\bar{\tau}(\bar{\alpha}+\beta)}{2}-\frac{\tau(\alpha+\bar{\beta})}{2}-\frac{5}{2}(\alpha+\bar{\beta})(\bar{\alpha}+\beta). \tag{2.32}$$

6. NECESSARY AND SUFFICIENT CONDITION FOR SSD :

Using NP equations NP-1, NP8, NP16, NP-17, in Appendix-2 we obtain

$$(D+\Delta)(\mu+\bar{\mu}-\rho-\bar{\rho})=2\left[\rho\bar{\mu}+\bar{\rho}\mu-(\bar{\sigma}-\lambda)(\sigma-\bar{\lambda})\right]$$
$$-(\rho^2+\bar{\rho}^2+\mu^2+\bar{\mu}^2)+2(\alpha+\bar{\beta})\tau+2(\Psi_2+\bar{\Psi}_2)$$
$$-\frac{8\pi G\rho^*}{9c^2}. \tag{2.33}$$

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For simplicity sake we assume $\mathrm{Im}\,\Psi_2=0$. Consequently using Bianchi identities (BI4) and (BI5) of Appendix-2 we get

$$\tau = 0. \tag{2.34}$$

Substituting (2.33) and (2.34) in the equations (2.26) to (2.32) the equations will reduce to

$$\Psi_2+\bar{\Psi}_2 = -\frac{4\pi G\rho^{\gamma}}{3c^2} - \frac{1}{6c^2}\Big\{2\big[\rho\bar{\mu}+\bar{\rho}\mu-(\bar{\sigma}-\lambda)(\sigma-\bar{\lambda})\big] - (\rho^2+\bar{\rho}^2+\mu^2+\bar{\mu}^2)+2(\Psi_2+\bar{\Psi}_2)-\frac{8\pi G\rho^{\gamma}}{9c^2}\Big\} + \frac{5}{2}(\alpha+\bar{\beta})(\bar{\alpha}+\beta), \tag{2.35}$$

$$(\alpha+\bar{\beta})\big[9(\bar{\mu}-\rho)-(\mu-\bar{\rho})\big]+8(\bar{\alpha}+\beta)(\bar{\sigma}-\lambda)=0, \tag{2.36}$$

$$\Psi_2+\bar{\Psi}_2 = -\frac{8\pi G\rho^{\gamma}}{3c^2}\Big(1-\frac{1}{2c^2}\Big)+\Big(-\frac{1}{3}+\frac{1}{6c^2}\Big)\Big\{2(\rho\bar{\mu}+\bar{\rho}\mu-(\bar{\sigma}-\lambda)(\sigma-\bar{\lambda})-(\rho^2+\bar{\rho}^2+\mu^2+\bar{\mu}^2)+2(\Psi_2+\bar{\Psi}_2)-\frac{8\pi G\rho^{\gamma}}{9c^2}\Big\}+\frac{5}{2}(\alpha+\bar{\beta})(\bar{\alpha}+\beta), \tag{2.37}$$

$$(\bar{\sigma}-\lambda)(-\bar{\rho}+\rho+\mu-\bar{\mu})-2(\alpha+\bar{\beta})^2=0, \tag{2.38}$$

$$\Psi_2+\bar{\Psi}_2 = -4(\rho\bar{\rho}+\mu\bar{\mu}-\rho\mu-\bar{\rho}\bar{\mu})-6(\bar{\rho}\mu+\rho\bar{\mu})+3(\bar{\rho}^2+\rho^2+\mu^2+\bar{\mu}^2)-\frac{5}{4}(\alpha+\bar{\beta})(\bar{\alpha}+\beta)-\frac{17}{3}(\bar{\sigma}-\lambda)(\sigma-\bar{\lambda})-\frac{4\pi G\rho^{\gamma}}{3}. \tag{2.39}$$

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$$(\alpha+\bar{\beta})(\bar{\rho}-\mu+5\rho-5\bar{\mu})+12(\bar{\alpha}+\beta)(\bar{\sigma}-\lambda)=0 \tag{2.40}$$

Solving (2.35) and (2.40) we get

$$\alpha+\bar{\beta}=0 \tag{2.41}$$

and putting (2.41) in (2.38) we get

$$(\bar{\sigma}-\lambda)=0 \quad \text{or} \quad \rho-\bar{\rho}+\mu-\bar{\mu}=0 \tag{2.42}$$

Using (2.41), Bianchi identity (BI1) in Appendix-2 gives

$$\delta\rho^{*}=0$$

and also by using (BI2) in Appendix-2

as $\Psi_2 \neq 0$ we infer that $\bar{\sigma}-\lambda \neq 0$

It follows that (by 2.42)

$$\rho-\bar{\rho}+\mu-\bar{\mu}=0.$$

Hence the necessary and sufficient conditions for the type D gravitational field of dust whose shear is stationary are obtained in terms of ray scalars as

(1) $\alpha+\bar{\beta}=0$

(2) $(\rho-\bar{\rho}+\mu-\bar{\mu})=0$

. . . . . .

APPENDIX - I

The NP version of eleven terms in (2.24) are given below :

I^{st} term:- $W_{ka}W^{k}_{\ b}$

$$= -\frac{P\bar{P}}{4}l_a l_b - \frac{\bar{P}Q}{4\sqrt{2}} l_a m_b + \frac{\bar{P}\bar{Q}}{4\sqrt{2}} l_a \bar{m}_b +$$

$$+ \frac{P\bar{P}}{4} l_a n_b - \frac{\bar{P}Q}{4\sqrt{2}} m_a l_b - \frac{\bar{P}^2}{4} m_a m_b +$$

$$+ (Q^2 - \frac{P\bar{P}}{8}) m_a \bar{m}_b + \frac{PQ}{4\sqrt{2}} m_a n_b + \frac{Q\bar{P}}{4\sqrt{2}} \bar{m}_a l_b +$$

$$+ (-\frac{P\bar{P}}{8} + Q^2) \bar{m}_a m_b - \frac{\bar{P}^2}{4} \bar{m}_a \bar{m}_b + \frac{\bar{P}Q}{4\sqrt{2}} \bar{m}_a n_b +$$

$$+ \frac{P\bar{P}}{4} n_a l_b + \frac{PQ}{4\sqrt{2}} n_a m_b - \frac{P\bar{Q}}{4\sqrt{2}} n_a \bar{m}_b -$$

$$- \frac{P\bar{P}}{4} n_a n_b .$$

II^{nd} term:- $2W^{k}_{\ (a}\Theta_{b)k}$

$$= l_a l_b \Big\{ \frac{\bar{P}}{4}\Big[(-\alpha - \bar{\beta} - \bar{\tau} + \nu) + \frac{1}{2c^2}(\bar{\kappa} + \tau - \pi + \nu)\Big] +$$

$$+ \frac{P}{4}\Big[(-\bar{\alpha} - \beta - \tau + \bar{\nu}) + \frac{1}{2c^2}(\kappa + \tau - \bar{\pi} - \bar{\nu})\Big]\Big\}$$

$$+(l_a m_b + m_a l_b)\Big\{\frac{\bar{P}}{8}(2\bar{\sigma}-2\lambda)+\frac{P}{8}(\rho+\bar{\rho}-\mu-\bar{\mu})+$$

$$+\frac{P}{8}\Big[-2(\gamma+\bar{\gamma})+\frac{1}{c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})+(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma})\Big]+$$

$$+\frac{Q}{4\sqrt{2}}\Big[(-\alpha-\bar{\beta}-\bar{\tau}+\nu)+\frac{1}{2c^2}(\bar{\kappa}+\bar{\tau}-\pi-\nu)\Big]\Big\}+$$

$$+(l_a\bar{m}_b+\bar{m}_a l_b)\Big\{\frac{\bar{P}}{8}(\rho+\bar{\rho}-\mu-\bar{\mu})+\frac{P}{8}(2\sigma-2\bar{\lambda})+$$

$$+\frac{\bar{P}}{8}\Big[-2(\gamma+\bar{\gamma})+\frac{1}{c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})+(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma})\Big]+$$

$$+\frac{Q}{4\sqrt{2}}\Big[(\bar{\alpha}+\beta+\tau-\bar{\nu})+\frac{1}{2c^2}(-\kappa-\tau+\bar{\pi}+\bar{\nu})\Big]\Big\}+$$

$$+(l_a n_b+n_a l_b)\Big\{\frac{\bar{P}}{8}\big[-\bar{\kappa}+\pi+2\alpha+2\bar{\beta}+\bar{\tau}-\nu\big]+$$

$$+\frac{P}{8}\big[-\kappa+\bar{\pi}+2\bar{\alpha}+2\beta+\tau-\bar{\nu}\big]\Big\}+$$

$$+m_a m_b\Big\{\frac{P}{4}(2\alpha+2\bar{\beta}+\bar{\tau}-\nu-\bar{\kappa}+\pi)+$$

$$+\frac{Q}{2\sqrt{2}}(2\bar{\sigma}-2\lambda)\Big\}+$$

$$+(m_a\bar{m}_b+\bar{m}_a m_b)\Big\{\frac{P}{8}(2\bar{\alpha}+2\beta+\tau-\bar{\nu}-\kappa+\bar{\pi})+$$

$$+\frac{\bar{P}}{8}\big[2\alpha+2\bar{\beta}+\bar{\tau}-\nu-\bar{\kappa}+\pi\big]\Big\}+$$

$$+\bar{m}_a\bar{m}_b\Big\{\frac{\bar{P}}{4}(2\bar{\alpha}+2\beta+\tau-\bar{\nu}-\kappa+\bar{\pi})-$$

$$-\frac{Q}{2\sqrt{2}}(2\sigma-2\bar{\lambda})\Big\}+$$

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$$+(m_a n_b + n_a m_b)\Big\{ \frac{\bar{P}}{8}(-2\bar{\sigma}+2\lambda) + \frac{P}{8}(-\rho-\bar{\rho}+\mu+\bar{\mu}) +$$
$$+\frac{P}{8}\Big[-(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma}) - 2(\epsilon+\bar{\epsilon}) + \frac{1}{c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})\Big] +$$
$$+\frac{Q}{4\sqrt{2}}\Big[(-\bar{\kappa}+\pi+\alpha+\bar{\beta}) + \frac{1}{2c^2}(\bar{\kappa}+\bar{\tau}-\pi-\nu)\Big]\Big\} +$$
$$+(\bar{m}_a n_b + n_a \bar{m}_b)\Big\{ \frac{P}{8}(-\rho-\bar{\rho}+\mu+\bar{\mu}) + \frac{P}{8}(-2\sigma+2\bar{\lambda}) +$$
$$+\frac{\bar{P}}{8}\Big[-(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma}) - 2(\epsilon+\bar{\epsilon}) + \frac{1}{c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})\Big] +$$
$$+\frac{Q}{4\sqrt{2}}\Big[(-\kappa+\bar{\pi}+\bar{\alpha}+\beta) + \frac{1}{2c^2}(\kappa+\tau-\bar{\pi}-\bar{\nu})\Big]\Big\} +$$
$$+n_a n_b\Big\{ \frac{\bar{P}}{4}\Big[(\bar{\kappa}-\pi-\alpha-\bar{\beta}) + \frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\Big] +$$
$$+\frac{P}{4}\Big[(\kappa-\bar{\pi}-\bar{\alpha}-\beta) + \frac{1}{2c^2}(-\kappa-\tau+\bar{\pi}+\bar{\nu})\Big]\Big\}.$$

<u>IIIrd term</u>:- $g^{km}\Theta_{ak}\Theta_{bm}$

$$= l_a l_b \Big\{ 2(\gamma+\bar{\gamma})(\epsilon+\bar{\epsilon}) - 2(\gamma+\bar{\gamma})^2 - \frac{1}{2}(\alpha+\bar{\beta}+\bar{\tau}-\nu)\times$$
$$\times\Big[\bar{\alpha}+\beta+\tau-\bar{\nu} + \frac{1}{2c^2}(-\kappa-\tau+\bar{\pi}+\bar{\nu})\Big] -$$
$$-\frac{1}{2}(\bar{\alpha}+\beta+\tau-\nu)\Big[\alpha+\bar{\beta}+\bar{\tau}-\nu + \frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\Big] -$$
$$-\frac{1}{2c^2}(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma})(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma}) - \frac{1}{2c^2}(\epsilon+\bar{\epsilon})\times$$
$$\times(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma}) - \frac{1}{c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})(\gamma+\bar{\gamma}) -$$

$$-\frac{1}{4c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\left[\bar{\alpha}+\beta+\tau-\bar{\nu}+\frac{1}{2c^2}(-\kappa-\tau+\bar{\pi}+\bar{\nu})\right]-$$

$$-\frac{1}{4c^2}(-\kappa-\tau+\bar{\pi}+\bar{\nu})\left[\alpha+\bar{\beta}+\bar{\tau}-\nu+\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\right]\Big\}+$$

$$+(l_a m_b+m_a l_b)\Big\{(\gamma+\bar{\gamma})\left[(-\bar{\kappa}+\pi+\alpha+\bar{\beta})-\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\right]+\frac{1}{2}(\alpha+\bar{\beta}+\bar{\tau}-\nu)(\rho+\bar{\rho}-\mu+\bar{\mu})+$$

$$+\frac{1}{2}(\bar{\alpha}+\beta+\tau-\bar{\nu})(2\bar{\sigma}-2\lambda)+\frac{1}{2}(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma})\left[-(\kappa+\bar{\beta}+\bar{\tau})+\right.$$

$$\left.+\bar{\nu}-\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\right]+\frac{1}{2c^2}(\epsilon+\bar{\epsilon})(\pi+\alpha+\bar{\beta}-\bar{\kappa})+$$

$$+\frac{1}{4c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)(\epsilon+\bar{\epsilon}+\rho+\bar{\rho}-\mu-\bar{\mu})+$$

$$+\frac{1}{4c^2}(-\kappa-\tau+\bar{\pi}+\bar{\nu})(2\bar{\sigma}-2\lambda)\Big\}+$$

$$+(c.c.)+$$

$$+(l_a n_b+n_a l_b)\Big\{(\gamma+\bar{\gamma})\left[-2(\epsilon+\bar{\epsilon})+\frac{1}{c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})\right]+$$

$$+\frac{1}{2}(\alpha+\bar{\beta}+\bar{\tau}-\nu)\left[(-\kappa+\bar{\pi}+\bar{\alpha}+\beta)-\frac{1}{2c^2}(-\kappa-\tau+\bar{\pi}+\bar{\nu})\right]+$$

$$+\frac{1}{2}(\bar{\alpha}+\beta+\tau-\bar{\nu})\left[(-\bar{\kappa}+\pi+\alpha+\bar{\beta})-\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\right]$$

$$+\frac{1}{2}(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma})(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma})-$$

$$-\frac{1}{4c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})\left[-2(\epsilon+\bar{\epsilon})+\frac{1}{c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})\right]+$$

$$+\frac{1}{4c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\left[(-\kappa+\bar{\pi}+\bar{\alpha}+\beta)-\frac{1}{2c^2}(-\kappa-\tau+\bar{\pi}+\bar{\nu})\right]+$$

$$+\frac{1}{4c^2}(-\kappa-\tau+\bar{\pi}+\bar{\nu})\left[(-\bar{\kappa}+\pi+\alpha+\bar{\beta})-\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\right]\Big\}+$$

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$$+n_a n_b\Big\{(\epsilon+\bar\epsilon)\Big[-(\epsilon+\bar\epsilon)+\frac{1}{2c^2}(\epsilon+\bar\epsilon+\gamma+\bar\gamma)\Big]-$$

$$-\frac{1}{2}(\bar\kappa+\pi+\alpha+\bar\beta)\Big[(-\kappa+\bar\pi+\bar\alpha+\beta)-\frac{1}{2c^2}(-\kappa-$$

$$-\tau+\bar\pi+\bar\nu)\Big]-\frac{1}{2}(-\kappa+\bar\pi+\bar\alpha+\beta)\Big[(-\bar\kappa+\pi+$$

$$+\alpha+\bar\beta)-\frac{1}{2c^2}(-\bar\kappa-\bar\tau+\pi+\nu)\Big]-\frac{1}{2}(\gamma+\bar\gamma)\times$$

$$\times\Big[-2(\epsilon+\bar\epsilon)+\frac{1}{c^2}(\epsilon+\bar\epsilon+\gamma+\bar\gamma)\Big]-(\epsilon+\bar\epsilon)\times$$

$$\times(\epsilon+\bar\epsilon-\gamma-\bar\gamma)+\frac{1}{2c^2}(-\bar\kappa-\bar\tau+\pi+\nu)\Big[(-\kappa+$$

$$+\bar\pi+\bar\alpha+\beta)-\frac{1}{4c^2}(-\kappa-\tau+\bar\pi+\nu)\Big]+\frac{1}{2c^2}(-\kappa-$$

$$-\tau+\bar\pi+\bar\nu)\Big[(-\bar\kappa+\pi+\alpha+\bar\beta)-\frac{1}{4c^2}(-\bar\kappa-\bar\tau+\pi+\nu)\Big]+$$

$$+\frac{1}{2c^2}(\epsilon+\bar\epsilon+\gamma+\bar\gamma)(\epsilon+\bar\epsilon-\gamma-\bar\gamma)\Big\}+$$

$$+(m_a\bar m_b+\bar m_a m_b)\Big\{\frac{1}{2}(-\alpha-\bar\beta-\bar\tau+\nu)\Big[(-\kappa+\bar\pi+$$

$$+\bar\alpha+\beta)-\frac{1}{2c^2}(-\kappa-\tau+\bar\pi+\bar\nu)\Big]+2\bar\sigma(\bar\lambda-\sigma)+$$

$$+\frac{1}{2}(\rho+\bar\rho-\mu-\bar\mu)(\mu+\bar\mu-\rho-\bar\rho)-2\lambda(\bar\lambda-\sigma)-$$

$$+\frac{1}{2}(-\bar\kappa+\pi+\alpha+\bar\beta)\Big[(-\bar\alpha-\beta-\tau+\nu)-\frac{1}{2c^2}(-\kappa-\tau+\bar\pi+\bar\nu)\Big]-$$

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$$-\frac{1}{4c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\Big[(-\kappa+\bar{\pi}+\bar{\alpha}+\beta)-\frac{1}{2c^2}(-\kappa-$$

$$-\tau+\bar{\pi}+\bar{\nu})\Big]-\frac{1}{4c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\Big[(-\bar{\alpha}-\beta-\tau+\bar{\nu})-$$

$$-\frac{1}{2c^2}(-\kappa-\tau+\bar{\pi}+\bar{\nu})\Big]\Big\}+$$

$$+(m_a n_b+n_a m_b)\Big\{\frac{1}{2}\Big[(-\alpha-\bar{\beta}-\bar{\tau}+\nu)-\frac{1}{2c^2}(-\bar{\kappa}-$$

$$-\bar{\tau}+\pi+\nu)\Big]\Big[-2(\epsilon+\bar{\epsilon})+\frac{1}{2c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma})\Big]-$$

$$-(\bar{\sigma}-\lambda)\Big[(-\kappa+\bar{\pi}+\bar{\alpha}+\beta)-\frac{1}{2c^2}(-\kappa-\tau+\bar{\pi}+\nu)\Big]-$$

$$-\frac{1}{2}(\rho+\bar{\rho}-\mu-\bar{\mu})\Big[-\bar{\kappa}+\pi+\alpha+\bar{\beta})-\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\Big]+$$

$$+\frac{1}{2}(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma})\Big[(-\bar{\kappa}+\pi+\alpha+\bar{\beta})-\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\Big]\Big\}+$$

$$+(c.c.)+$$

$$+m_a m_b\Big\{\frac{1}{2}\Big[-\alpha-\bar{\beta}-\bar{\tau}+\nu-\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\Big]\times$$

$$\times\Big[(-\bar{\kappa}+\pi+\alpha+\bar{\beta})-\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\Big]+$$

$$+2(\bar{\sigma}-\lambda)(\mu+\bar{\mu}-\rho-\bar{\rho})+\frac{1}{2}\Big[(-\bar{\kappa}+\pi+\alpha+\bar{\beta})-$$

$$-\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\Big]\Big[(-\alpha-\bar{\beta}-\bar{\tau}+\nu)-\frac{1}{2c^2}(-\bar{\kappa}-\bar{\tau}+\pi+\nu)\Big]\Big\}+$$

$$+(c.c.).$$

$\underline{\underline{IV^{th}}}$ $\underline{\text{term}}$:- C_{ab}

$$= -(\Psi_2+\bar{\Psi}_2)\left[V_aV_b - m_{(a}\bar{m}_{b)}\right] + (\bar{\Psi}_1-\Psi_3)\times$$
$$\times\left[l_{(a}m_{b)} - m_{(a}n_{b)}\right] + (\Psi_1-\bar{\Psi}_3)\left[l_{(a}\bar{m}_{b)} -\right.$$
$$\left. - \bar{m}_{a}n_{b)}\right] - \frac{1}{2}(\bar{\Psi}_0+\Psi_4)\, m_am_b -$$
$$- \frac{1}{2}(\Psi_0+\bar{\Psi}_4)\,\bar{m}_a\bar{m}_b.$$

$$= -\frac{(\Psi_2+\bar{\Psi}_2)}{2}\left[l_al_b + n_an_b\right] +$$
$$+ \frac{(\Psi_2+\bar{\Psi}_2)}{2}\left[l_an_b + n_al_b\right] +$$
$$+ \frac{(\bar{\Psi}_1-\Psi_3)}{2}\left[l_am_b + m_al_b\right] +$$
$$+ \frac{(\Psi_1-\bar{\Psi}_3)}{2}\left[l_a\bar{m}_b + \bar{m}_al_b\right] +$$
$$+ \frac{(-\bar{\Psi}_1+\Psi_3)}{2}\left[n_am_b + m_an_b\right] +$$
$$+ \left(\frac{-\Psi_1+\bar{\Psi}_3}{2}\right)\left[n_a\bar{m}_b + \bar{m}_an_b\right] +$$
$$+ \left(\frac{-\bar{\Psi}_0-\Psi_4}{2}\right) m_am_b + \left(\frac{-\Psi_0-\bar{\Psi}_4}{2}\right)\bar{m}_a\bar{m}_b$$
$$+ \left(\frac{\Psi_2+\bar{\Psi}_2}{2}\right)(m_a\bar{m}_b + \bar{m}_am_b).$$

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<u>V^{th} term:</u>– $\frac{4\pi G\rho^*}{3}\gamma_{ab}$

$$= +\frac{4\pi G\rho^*}{3}\Big\{\left(\frac{-1}{2c^2}\right) l_a l_b + \left(1-\frac{1}{2c^2}\right) l_a n_b + \left(1-\frac{1}{2c^2}\right) n_a l_b - \left(\frac{1}{2c^2}\right) n_a n_b - m_a\bar{m}_b - \bar{m}_a m_b .$$

<u>VI^{th} term:</u> – $\frac{1}{3}\gamma_{ab}\dot{\Theta}$

[Putting $X = \epsilon + \bar{\epsilon} - \bar{\rho} - \rho + \mu + \bar{\mu} - (\gamma + \bar{\gamma})$]

$$= \frac{1}{12c^2}(\Delta X + DX)\,[l_a l_b + n_a n_b] + $$

$$+ \left(-\frac{1}{6} + \frac{1}{12c^2}\right)(\Delta X + DX)\,\{l_a n_b + n_a l_b\} +$$

$$+ \frac{1}{6}(\Delta X + DX)\,[m_a\bar{m}_b + \bar{m}_a m_b].$$

<u>VII^{th} term:</u>– $\frac{2}{3}\Theta\Theta_{ab}$.

$$= \frac{2}{3}(\epsilon + \bar{\epsilon} - \bar{\rho} - \rho + \mu + \bar{\mu} - \gamma - \bar{\gamma})\times$$

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$$\times \Big\{ l_a l_b \left[2(\gamma+\bar{\gamma}) - \frac{1}{\sqrt{2}c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma}) \right] +$$

$$+ n_a n_b \left[-2(\epsilon+\bar{\epsilon}) + \frac{1}{\sqrt{2}c^2}(\epsilon+\bar{\epsilon}+\gamma+\bar{\gamma} \right] +$$

$$+ (l_a m_b + m_a l_b) \left[-\alpha-\bar{\beta}-\bar{\tau}+\nu - \frac{1}{2\sqrt{2}c^2} \times \right.$$

$$\left. \times \left[-\bar{\kappa}-\bar{\tau}+\pi+\nu \right] \right] +$$

$$+ (l_a \bar{m}_b + \bar{m}_a l_b) \left[-\bar{\alpha}-\beta-\tau+\bar{\nu} - \frac{1}{2\sqrt{2}c^2} \times \right.$$

$$\left. \times \left[-\kappa-\tau+\bar{\pi}+\bar{\nu} \right] \right] +$$

$$+ (l_a n_b + n_a l_b)(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma}) +$$

$$+ 2 m_a m_b (\bar{\sigma}-\lambda) + 2\bar{m}_a \bar{m}_b (\sigma-\bar{\lambda}) +$$

$$+ (m_a \bar{m}_b + \bar{m}_a m_b)(\bar{\rho}+\rho-\mu-\bar{\mu}) +$$

$$+ (m_a n_b + n_a m_b) \left[-\bar{\kappa}+\pi+\alpha+\bar{\beta} - \frac{1}{2\sqrt{2}c^2}(-\bar{\kappa}- \right.$$

$$\left. -\bar{\tau}+\pi+\nu) \right] +$$

$$+ (\bar{m}_a n_b + n_a \bar{m}_b) \left[-\kappa+\bar{\pi}+\bar{\alpha}+\beta - \frac{1}{2\sqrt{2}c^2}(-\kappa- \right.$$

$$\left. -\tau+\bar{\pi}+\bar{\nu}) \right] \Big\}.$$

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$\underline{\mathrm{VIII}^{th} + \mathrm{IX}^{th} \text{ term}}$: $-\left(g_{ak} U^k{}_{;b} + g_{kb} U^k{}_{;a}\right)$

$$= l_a l_b \left[-\sqrt{2} A (\gamma + \bar{\gamma}) - \frac{B}{2} (\tau + \bar{\nu}) - \frac{\bar{B}}{2} (\bar{\tau} + \nu) \right] +$$

$$+ (l_a m_b + m_a l_b) \left[\frac{A}{\sqrt{2}} (\alpha + \bar{\beta}) - \frac{A}{2\sqrt{2}} (\bar{\tau} - \nu) + \right.$$

$$+ \frac{B}{4} (\rho - \bar{\mu}) + \frac{B}{2} (\gamma + \bar{\gamma}) + \frac{\bar{B}}{4} (\bar{\sigma} - \lambda) +$$

$$\left. + \frac{C}{\sqrt{2}} (\tau - \bar{\nu}) \right] +$$

$$+ (l_a \bar{m}_b + \bar{m}_a l_b) \left[\frac{A}{\sqrt{2}} (\bar{\alpha} + \beta) - \frac{A}{2\sqrt{2}} (\tau - \bar{\nu}) + \right.$$

$$+ \frac{B}{4} (\sigma - \bar{\lambda}) + \frac{B}{2} (\gamma + \bar{\gamma}) + \frac{\bar{B}}{4} (\bar{\rho} - \mu) +$$

$$\left. + \frac{\bar{C}}{\sqrt{2}} (\bar{\tau} - \nu) \right] +$$

$$+ (l_a n_b + n_a l_b) \left[\frac{A}{\sqrt{2}} (\gamma + \bar{\gamma} - \epsilon - \bar{\epsilon}) + \frac{B}{4} (\tau - \bar{\nu} - \right.$$

$$\left. - \kappa + \bar{\pi}) + \frac{\bar{B}}{4} (\bar{\tau} - \nu - \bar{\kappa} + \pi) \right] +$$

$$+ m_a m_b \left[\frac{A}{\sqrt{2}} (\bar{\sigma} - \lambda) - B (\alpha + \bar{\beta}) + \sqrt{2} C (\bar{\mu} - \rho) \right] +$$

$$+ (C.C.) +$$

$$+ (m_a n_b + n_a m_b) \left[\frac{A}{2\sqrt{2}} (\pi - \bar{\kappa} - 2\alpha - 2\bar{\beta}) - \frac{B}{2} (2\epsilon + \right.$$

$$\left. + 2\bar{\epsilon}) - \frac{\bar{B}}{4} (\bar{\sigma} - \lambda) + \frac{C}{\sqrt{2}} (\kappa - \bar{\pi}) \right] +$$

$$+ (C.C.)$$

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$$+n_a n_b\left[\sqrt{2}A(\epsilon+\bar{\epsilon})+\frac{B}{2}(\kappa-\bar{\pi})+\frac{\bar{B}}{2}(\bar{\kappa}-\pi)\right]+$$

$$+(\bar{m}_a m_b+m_a\bar{m}_b)\left[\frac{A}{2\sqrt{2}}(\rho-\bar{\mu})+\frac{\bar{B}}{2}(\kappa+\bar{\beta})+\right.$$

$$\left.+\frac{\bar{C}}{\sqrt{2}}(\lambda-\bar{\sigma})\right].$$

<u>$X^{th}+XI^{th}$ term</u>: $-(\varrho_{ak}W^k_b+\varrho_{kb}W^k_a)$

$$=-\frac{(B\bar{P}+\bar{B}P)}{4}(l_a l_b+n_a n_b)+\left(\frac{B\bar{P}+\bar{B}P}{4}\right)\times$$

$$\times(l_a n_b+n_a l_b)+m_a m_b\left(\frac{BP}{4}+\frac{CQ}{2}\right)+$$

$$+(l_a m_b+m_a l_b)\left[-\frac{AP}{2\sqrt{2}}-\frac{AP}{4\sqrt{2}}-\frac{BQ}{4\sqrt{2}}+\frac{C\bar{P}}{2\sqrt{2}}\right]+$$

$$+(l_a\bar{m}_b+\bar{m}_a l_b)\left[\frac{-A\bar{P}}{2\sqrt{2}}-\frac{A\bar{P}}{4\sqrt{2}}+\frac{\bar{B}Q}{4\sqrt{2}}+\frac{\bar{C}P}{2\sqrt{2}}\right]+$$

$$+(m_a\bar{m}_b+\bar{m}_a m_b)\left[\frac{B\bar{P}}{4}+\frac{\bar{B}P}{4}\right]+\bar{m}_a\bar{m}_b\left(\frac{\bar{B}\bar{P}}{4}-\frac{\bar{C}Q}{2}\right)+$$

$$+(m_a n_b+n_a m_b)\left[\frac{AP}{2\sqrt{2}}+\frac{AP}{4\sqrt{2}}+\frac{BQ}{4\sqrt{2}}-\frac{C\bar{P}}{2\sqrt{2}}\right]+$$

$$+(\bar{m}_a n_b+n_a\bar{m}_b)\left[\frac{A\bar{P}}{2\sqrt{2}}+\frac{A\bar{P}}{4\sqrt{2}}-\frac{\bar{B}Q}{4\sqrt{2}}-\frac{\bar{C}P}{2\sqrt{2}}\right].$$

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APPENDIX - II

Ray Analysis of Ricci Einstein Identity for Dust :

The NP equations (Ch.I Sec.4) can be reduced to the following conditions using (2·30 , 2·31);) :

NP 1 $$D\rho = \rho^2 + \sigma\bar{\sigma} + \frac{2\pi G}{c^2}\rho^*,$$

NP 2 $$D\sigma = (\rho + \bar{\rho})\sigma,$$

NP 3 $$D\tau = \tau\rho + \bar{\tau}\sigma,$$

NP 4 $$D\alpha = \rho\alpha + \beta\bar{\sigma}$$

NP 5 $$D\gamma = \tau\alpha + \bar{\tau}\beta + \Psi_2 - \frac{2\pi G\rho^*}{c^2},$$

NP 6 $$D\beta = \alpha\sigma + \bar{\rho}\beta,$$

NP 7 $$D\lambda = \rho\lambda + \bar{\sigma}\mu,$$

NP 8 $$D\mu = \bar{\rho}\mu + \sigma\lambda + \Psi_2 + \frac{6\pi G\rho^*}{c^2},$$

NP 9 $$D\nu = \bar{\tau}\mu + \tau\lambda,$$

NP 10 $$\Delta\lambda - \bar{\delta}\bar{\tau} = -(\mu + \bar{\mu} - 4\bar{\gamma})\lambda + (3\alpha + \bar{\beta} - \bar{\tau})\bar{\tau},$$

NP 11 $$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - (3\alpha - \bar{\beta})\sigma + (\rho - \bar{\rho})\tau$$

NP 12 $$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + (\rho - \bar{\rho})\gamma - \Psi_2 + \frac{4\pi G\rho^*}{c^2},$$

NP 13 $$\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\bar{\tau} + (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda$$

NP 14 $$\Delta\mu - \delta\bar{\tau} = -\mu^2 - \lambda\bar{\lambda} + (\bar{\alpha} + 3\beta - \tau)\bar{\tau} - \frac{2\pi G\rho^*}{c^2}.$$

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NP 15 $\Delta\beta - \delta\gamma = (\bar{\alpha}+\beta-\tau)\gamma - \mu\tau + \sigma\bar{\tau} + (2\gamma-\mu)\beta - \alpha\bar{\lambda}$,

NP 16 $\Delta\sigma - \delta\tau = -(\mu - 4\bar{\gamma})\sigma - \bar{\lambda}\rho - (\tau+\beta-\bar{\alpha})\tau$,

NP 17 $\Delta\rho - \bar{\delta}\tau = -\bar{\mu}\rho - \sigma\lambda + (\bar{\beta}-\alpha-\bar{\tau})\tau - \Psi_2 - \frac{6\pi G\rho^*}{c^2}$,

NP 18 $\Delta\alpha - \bar{\delta}\gamma = \rho\bar{\tau} - (\tau+\beta)\lambda + (\bar{\gamma}-\bar{\mu})\alpha + (\bar{\beta}-\bar{\tau})\gamma$.

Bianchi Identities :

Using (2·30), (2·31), for Petrov type D in the Bianchi Identities of Chapter-2 Sec. 6 we get

BI 1 $\delta\rho^* = 2(\bar{\alpha}+\beta)\rho^*$,

BI 2 $\Psi_2 = \frac{2\pi G\rho^*}{3c^2}\,\frac{(\bar{\lambda}-\sigma)}{\sigma}$,

BI 3 $\Delta\rho^* + 3D\rho^* + \frac{c^2}{2\pi G}D\Psi_2 = \frac{3c^2}{2\pi G}\rho\Psi_2 - (\bar{\mu}-\rho)\rho^*$,

BI 4 $\delta\rho^* + \frac{c^2}{6\pi G}\delta\Psi_2 = \frac{c^2}{2\pi G}\tau\Psi_2$,

BI 5 $\bar{\delta}\rho^* + \frac{c^2}{6\pi G}\bar{\delta}\Psi_2 = 0$,

BI 6 $D\rho^* + 3\Delta\rho^* + \frac{c^2}{2\pi G}\Delta\Psi_2 = -\frac{c^2}{\pi G}\mu\Psi_2 - (\mu-\bar{\rho})\rho^*$,

BI 7 $\Psi_2 = \frac{2\pi G\rho^*}{3c^2}\,\frac{(\bar{\sigma}-\lambda)}{\lambda}$,

BI 8 $\bar{\delta}\rho^* = \frac{3c^2}{2\pi G}\bar{\tau}\Psi_2 - 2\rho^*(\bar{\beta}+\alpha)$,

BI 9 $5D\rho^* + \Delta\rho^* = (\rho+\bar{\rho}-\mu-\bar{\mu})\rho^*$,

BI 10 $\delta\rho^* = 0$,

BI 11 $D\rho^* + 5\Delta\rho^* = (\rho+\bar{\rho}-\mu-\bar{\mu})\rho^*$.