# CHAPTER - III

## RAY ANALYSIS OF FREE GRAVITATIONAL FIELD IN EMPTY SPACE

# CHAPTER - III

## 1. INTRODUCTION :

Szekeres (1964) has used the tensor

$$C_{ab} \overset{\text{def.}}{=} C_{acbd} u^c u^d$$

to study the interaction of the free gravitational field and the source of gravitation. The tensor field $C_{ab}$ is referred as the Gravitational tidal force in general relativity by Ellis (1971)

For the choice

$$u^a = (2)^{-1/2}(l^a + n^a) \qquad (3.1)$$

the ray equivalent of $C_{ab}$ (SINGH .E. 1983) is

$$C_{ab} = -(\Psi_2 + \bar{\Psi}_2)\,[v_a v_b - m_{(a}\bar{m}_{b)}] +$$
$$+ (\bar{\Psi}_1 - \Psi_3)\,[l_{(a}m_{b)} - m_{(a}n_{b)}] + c.c.$$
$$- \frac{1}{2}(\bar{\Psi}_0 + \Psi_4)\, m_a m_b - c.c. \qquad (3.2)$$

In Sec. 2 the necessary and sufficient conditions for the (Lie) invariance of the gravitational tidal force are obtained when the free gravitational field is of Petrov type I. It is found that Petrov type I is incompatible. Petrov type II, D, N and III are compatible and the corresponding necessary and sufficient condition of the compatibility are given in Sec. 3, Sec. 4, Sec. 5 and Sec. 6. respectively.

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## 2. RAY ANALYSIS OF FREE GRAVITATIONAL TIDAL FORCE IN EMPTY SPACE FOR PETROV TYPE I :

For Petrov type I

$$\Psi_0 = \Psi_4 = 0, \; \Psi_1 \Psi_3 \neq 0, \; 9\Psi_2^2 \neq 16\,\Psi_1\Psi_3. \tag{3.3}$$

Therefore, equation (3.2) gives

$$\underset{I}{C}_{ab} = -(\Psi_2+\bar{\Psi}_2)\left[V_a V_b - m_{(a}\bar{m}_{b)}\right] + \\ + (\bar{\Psi}_1 - \Psi_3)\left[l_{(a}m_{b)} - m_{(a}n_{b)}\right] + (\text{c.c.}) \tag{3.4}$$

From the definition of Lie derivative with respect to $l^a$

$$\underset{l}{\pounds}\, C_{ab} = C_{ab;c}\, l^c + C_{ac}\, l^c{}_{;b} + C_{cb}\, l^c{}_{;a}. \tag{3.5}$$

Substituting (3.4) in (3.5) and using Appendix-1 of this Chapter $\underset{l}{\pounds} C_{ab} = 0$ reduces to the following ten equations by equating coefficients $l_a l_b, \cdots l_a n_b, \cdots n_a n_b$ to zero separately.

Ia) $D(\mathrm{Re}\,\Psi_2) + 2\,\mathrm{Re}(\Psi_2)(\gamma+\bar{\gamma}-\epsilon-\bar{\epsilon}) + (\bar{\Psi}_1-\Psi_3)\bar{\pi} + $
$+ (\Psi_1-\bar{\Psi}_3)\pi = 0,$

Ib) $\mathrm{Re}\,\Psi_2\,(\bar{\kappa}+\nu+\pi-\alpha-\bar{\beta}+\bar{\tau}) + D(\bar{\Psi}_1-\Psi_3) + $
$+ (\bar{\Psi}_1-\Psi_3)(2\epsilon-\rho-\gamma-\bar{\gamma}) = 0,$

Ic) $D(\mathrm{Re}\,\Psi_2) + \mathrm{Re}\,\Psi_2\,(\epsilon+\bar{\epsilon}-\gamma-\bar{\gamma}) - (\bar{\Psi}_1-\Psi_3)(\tau+\bar{\pi}) - $
$-(\Psi_1-\bar{\Psi}_3)(\pi+\bar{\tau}) = 0,$

Id) $\mathrm{Re}\,\Psi_2\,(\bar{\kappa}+2\pi+\bar{\tau}-\alpha-\bar{\beta}) + D(\bar{\Psi}_1-\Psi_3) + (\bar{\Psi}_1-\Psi_3)\times$
$\times(\rho-\gamma-\bar{\gamma}) + (\Psi_1-\bar{\Psi}_3)\bar{\sigma} = 0,$

Ie) $(\bar{\Psi}_1-\Psi_3)(-\bar{\kappa}-\pi+2\alpha+2\bar{\beta})-2\,\mathrm{Re}\,\Psi_2\cdot\bar{\sigma}=0$

If) $D(\mathrm{Re}\,\Psi_2)-\mathrm{Re}\,\Psi_2(\rho+\bar{\rho})+(\bar{\Psi}_1-\Psi_3)(-\bar{\pi}-\bar{\alpha}+\beta)+$
$+(\Psi_1-\bar{\Psi}_3)(\alpha+\bar{\beta}-\bar{\kappa})=0$

Ig) $\mathrm{Re}\,\Psi_2(-\bar{\tau}-\pi+\alpha+\bar{\beta})-D(\bar{\Psi}_1-\Psi_3)+$
$+(\bar{\Psi}_1-\Psi_3)(\bar{\epsilon}-\epsilon+\rho)+(\Psi_1-\bar{\Psi}_3)\bar{\sigma}=0$

Ih) $D(\mathrm{Re}\,\Psi_2)+\mathrm{Re}\,\Psi_2(\epsilon+\bar{\epsilon}-\bar{\gamma}-\gamma)+(\bar{\Psi}_1-\Psi_3)(-\tau+\kappa)+$
$+(\Psi_1-\bar{\Psi}_3)(-\bar{\tau}+\bar{\kappa})=0$

Ii) $\mathrm{Re}\,\Psi_2(-\pi-\bar{\kappa}+\alpha+\bar{\beta})-D(\bar{\Psi}_1-\Psi_3)+$
$+(\bar{\Psi}_1-\Psi_3)(\rho-\epsilon-\bar{\epsilon})+(\Psi_1-\bar{\Psi}_3)\bar{\sigma}=0$

Ij) $-D(\mathrm{Re}\,\Psi_2)-\kappa(\bar{\Psi}_1-\Psi_3)-(\Psi_1-\bar{\Psi}_3)\bar{\kappa}=0$

By using freedom conditions along $l^a$ ($\kappa=\pi=\epsilon=0$)
we get

$$\underset{l}{\mathcal{L}}\ \underset{I}{C}_{ab}=0 \iff D(\mathrm{Re}\,\Psi_2)=0,\ D(\bar{\Psi}_1-\Psi_3)=0$$

($\mathrm{Re}\,\Psi_2$ and $\bar{\Psi}_1-\Psi_3$ conserved along $l^a$)

$\tau=0$, (null rotations about $l^a$),

$\sigma=0$, ($l^a$ is shear free),

$\nu=0$, (geodesic along $n^a$), (3.6)

$\rho=0$, (Expansion free, twist free),

$\alpha+\bar{\beta}=0$, ($l\bar{m}$-surface relative to $n^a$),

$\gamma+\bar{\gamma}=0$, ($ln$-surface relative to $n^a$)

Substituting these conditions (3.6) in NP equations of Ch. II P-10, equation (MG-2) gives

$$\Psi_1 = 0 \tag{3.7}$$

which concludes that the gravitational tidal force is incompatible for Petrov type I.

## 3. RAY ANALYSIS OF FREE GRAVITATIONAL TIDAL FORCE IN EMPTY SPACE FOR PETROV TYPE II :

For Petrov type II

$$\Psi_0 = \Psi_1 = \Psi_4 = 0, \quad \Psi_2 \Psi_3 \neq 0 \tag{3.8}$$

Using (3.8) in (3.2) we get

$$\underset{\mathrm{II}}{C_{ab}} = -(\Psi_2 + \bar{\Psi}_2)\left[V_a V_b - m_{(a}\bar{m}_{b)}\right] - \frac{1}{2}\Psi_4\, m_a m_b - (c.c.) \tag{3.9}$$

Substituting (3.9) in (3.5) and with AI-2, AI-3; $\underset{l}{\pounds} C_{ab} = 0$ reduces to nine equations.

II a) $\frac{Re\,\Psi_2}{2}(2\epsilon + 2\bar{\epsilon} - 2\gamma - 2\bar{\gamma}) - D(Re\,\Psi_2) = 0,$

II b) $\frac{Re\,\Psi_2}{2}(\gamma + \bar{\gamma} - \epsilon - \bar{\epsilon}) + D(Re\,\Psi_2) = 0,$

II c) $-\frac{Re\,\Psi_2}{2}(\bar{\pi} - 2\pi + \alpha + \bar{\beta} - \tau) - \frac{\Psi_4}{2}(\tau + \bar{\pi}) = 0,$

II d) $\frac{Re\,\Psi_2}{2}(\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma}) + D(Re\,\Psi_2) = 0,$

II e) $Re\,\Psi_2(\epsilon + \bar{\epsilon}) - D(Re\,\Psi_2) = 0,$

II f) $\frac{1}{2}Re\,\Psi_2(\bar{\kappa} + \pi + \alpha + \bar{\beta}) = 0,$

II g) $Re\,\Psi_2\,\bar{\pi} - \frac{1}{2}D\Psi_4 = 0,$

$$\text{II h)}\ -\tfrac{1}{2}\,\mathrm{Re}\,\Psi_2(\rho+\bar{\rho})+\tfrac{1}{2}\Psi_4\sigma+D(\mathrm{Re}\,\Psi_2)=0$$

$$\text{II i)}\ \mathrm{Re}\,\Psi_2(\bar{\kappa}+\pi-\alpha-\bar{\beta})=0$$

Using freedom conditions along $l^a$ $(\kappa=\pi=\epsilon=0)$

$$\underset{l}{\pounds}\ \underset{\mathrm{II}}{C_{ab}}=0 \iff D(\mathrm{Re}\,\Psi_2)=0,$$

($\mathrm{Re}\,\Psi_2$ conserved along $l^a$)

$$\gamma+\bar{\gamma}=0$$

$$\alpha+\bar{\beta}=0 \qquad (3.10)$$

Putting the conditions (3.10) in the Bianchi Identity (BI2) of Ch.I P-14 we get

$$\sigma=0\ ,\quad \text{for}\quad \Psi_2\neq 0$$

Hence for Petrov type II, $l^a$ is shear free.

## 4. RAY ANALYSIS OF FREE GRAVITATIONAL TIDAL FORCE IN EMPTY SPACE FOR PETROV TYPE D :

For Petrov type D

$$\Psi_0=\Psi_1=\Psi_3=\Psi_4=0,\ \Psi_2\neq 0 \qquad (3.12)$$

Therefore, equation (3.2) becomes

$$\underset{D}{C_{ab}}=-(\Psi_2+\bar{\Psi}_2)\left[v_a v_b-m_{(a}\bar{m}_{b)}\right] \qquad (3.13)$$

Putting (3.13) in (3.5) and using (AI-3) $\underset{l}{\pounds}\underset{D}{C_{ab}}=0$ reduces to nine equations.

D-a) $-D(Re\,\Psi_2)+2\epsilon+2\bar{\epsilon}-2\gamma-2\bar{\gamma}=0,$

D-b) $D\,(Re\,\Psi_2)+\gamma+\bar{\gamma}-\epsilon-\bar{\epsilon}=0,$

D-c) $-\bar{\kappa}-2\pi+\alpha+\bar{\beta}-\bar{\tau}=0,$

D-d) $D(Re\,\Psi_2)-\gamma-\bar{\gamma}-\epsilon-\bar{\epsilon}=0,$

D-e) $-D\,(Re\Psi_2)-2\epsilon-2\bar{\epsilon}=0,$

D-f) $\pi+\kappa+\alpha+\bar{\beta}=0,$

D-g) $\bar{\sigma}=0,$

D-h) $D\Psi_2+\rho+\bar{\rho}=0,$

D-i) $\pi+\bar{\kappa}-\alpha-\bar{\beta}=0.$

Using freedom conditions along $l^a$ ( $\kappa=\pi=\epsilon=0$ ), we get

$\underset{l}{£}\, C_{ab}^{\;D}=0 \Longleftrightarrow$ $D(Re\,\Psi_2)=0$,
($Re\Psi_2$ is conserved along $l^a$),
$\tau=0$, (null rotation about $l^a$),
$\sigma=0$, (shear free along $l^a$),
$\rho+\bar{\rho}=0$, (divergence free of $l^a$ congruence),
$\alpha+\bar{\beta}=0$, ($l\bar{m}$ - surface relative to $n$),
$\gamma+\bar{\gamma}=0$, ($ln$ - surface relative to $n$)

Substituting (3.14) in NP equations of Ch.I P-10 we get

$$(MG\text{-}1)\Rightarrow \quad D\gamma=\Psi_2,\ (MG\text{-}4)\Rightarrow D\nu=0,$$
$$(MG\text{-}5)\Rightarrow \quad \delta\rho=0\ ,\ (M4)\Rightarrow \bar{\lambda}\rho=0. \tag{3.15}$$

Which concludes that $\rho$ is conserved along $m^a$ and $\mu, \nu$ are conserved along $l^a$

Again by putting (3.14) in Bianchi Identities we get

$$(BI4) \Rightarrow \delta\Psi_2 = 0, \quad (BI5) \Rightarrow \bar{\delta}\Psi_2 = 0,$$
$$(BI7) \Rightarrow \lambda = 0, \ (\text{Since } \Psi_2 \neq 0),$$
$$(BI8) \Rightarrow \nu = 0, \ (\text{Since } \Psi_2 \neq 0). \tag{3.16}$$

which means $\Psi_2$ is conserved along $m^a$ and $\bar{m}^a$ and $\lambda = 0$ means shear free along $n^a$ and $\nu = 0$ geodesic along $n^a$.

## 5. RAY ANALYSIS OF FREE GRAVITATIONAL TIDAL FORCE IN EMPTY SPACE FOR PETROV TYPE N ... :

For Petrov type N, with propagation vector $l^a$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \ \Psi_4 \neq 0 \tag{3.17}$$

Hence (3.2) becomes

$$\underset{N}{C}_{ab} = -\frac{1}{2}\Psi_4 m_a m_b - (C.C.) \tag{3.18}$$

Substituting (3.18) in (3.5), (AI-1) with the freedom conditions along $l^a$ $(\kappa = \pi = \epsilon = 0)$ we get

$$\underset{l}{£}\ \underset{N}{C}_{ab} = 0 \Leftrightarrow D\Psi_4 = \rho$$

$\tau + \bar{\pi} = 0$ (hypersurface orthogonal along $m^a$)

$\sigma = 0$ (shear free ~~along~~ $l^a$)

(3.19)

Using (3.19) in NP equations (Ch.I,P-10) we get

$$(M1) \Rightarrow D\rho = 0, \qquad (M2) \Rightarrow D\alpha = 0,$$
$$(G2) \Rightarrow D\beta = 0, \qquad (MG\text{-}3) \Rightarrow D\gamma = 0, \tag{3.20}$$
$$(M3) \Rightarrow D\lambda = 0, \qquad (MG\text{-}1) \Rightarrow D\mu = 0,$$
$$(MG\text{-}4) \Rightarrow D\nu = 0.$$

Hence $\rho$, $\alpha$, $\beta$, $\gamma$, $\lambda$, $\mu$ and $\nu$ all conserved along $l^a$.

Substituting (3.19) in Bianchi Identities (Ch.I,P-14) we get.

$$(BI7) \Rightarrow D\Psi_4 = 0, \; (i.e.\ \rho = 0), \tag{3.21}$$
$$(BI8) \Rightarrow \delta\Psi_4 = 0.$$

i.e. $\Psi_4$ is conserved along $l^a$ as well as along $m^a$.

6. RAY ANALYSIS OF FREE GRAVITATIONAL TIDAL FORCE IN EMPTY SPACE FOR PETROV TYPE III. :

For Petrov type III

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0, \; \Psi_3 \neq 0. \tag{3.22}$$

Therefore (3.2·) becomes

$$\underset{III}{C_{ab}} = \Psi_3 [m_{(a} n_{b)} - l_{(a} m_{b)}] + (c.c). \tag{3.23}$$

Substitute (3.23) in (3.5), and (AI-2) with freedom conditions along $l^a$ we get

$$\underset{l}{\pounds}\ \underset{\text{III}}{C_{ab}} \Longleftrightarrow D(Re\,\Psi_3)=0$$

($Re\,\Psi_3$ is conserved along $l^a$),

$\tau = 0$, (null rotation about $l^a$),

$\sigma = 0$, (shear free along $l^a$),

$\rho = 0$, (Expansion free, twist free along $l^a$),

$\gamma+\bar{\gamma}=0$ (ln-surface relative to n). (3.24)

Substituting (3.24) in NP equations (Ch.I,P-10) we get

$$(M2) \Rightarrow D\alpha = 0, \qquad (G\text{-}2) \Rightarrow D\beta = 0,$$

$$(MG\text{-}3) \Rightarrow D\gamma = 0, \qquad (M3) \Rightarrow D\lambda = 0,$$

$$(MG\text{-}1) \Rightarrow D\mu = 0. \tag{3.25}$$

Which concludes that $\alpha$, $\beta$, $\gamma$, $\lambda$, $\mu$ all conserved along

Again by putting conditions (3.24) in Bianchi Identities, (Ch.I, P-14) we get

$$(BI5) \Rightarrow D\Psi_3 = 0.$$

Which implies that $\Psi_3$ also conserved along $l^a$.

## APPENDIX-I

NP equivalents of Lie Derivatives of $C_{ab}$ of all the Petrov types are given below

(AI-1)
$$
\begin{aligned}
\underset{l}{\pounds}\, \underset{N}{C}_{ab} = & -\tfrac{1}{2} D\Psi_4 m_a m_b - \tfrac{1}{2} D\bar{\Psi}_4 \bar{m}_a \bar{m}_b - \\
& - \Psi_4 \{ \bar{\pi}\, l_{(a} m_{b)} + (\epsilon - \bar{\epsilon})\, m_a m_b + \\
& \quad \tau\, l_{(a} m_{b)} - \rho\, m_a m_b - \sigma\, m_{(a} \bar{m}_{b)} \} \\
& - \bar{\Psi}_4 \{ \pi\, l_{(a} \bar{m}_{b)} + (\bar{\epsilon} - \epsilon)\, \bar{m}_a \bar{m}_b + \\
& \quad + \bar{\tau}\, l_{(a} \bar{m}_{b)} - \bar{\rho}\, \bar{m}_a \bar{m}_b - \bar{\sigma}\, \bar{m}_{(a} m_{b)} \}
\end{aligned}
$$

(AI-2)
$$
\begin{aligned}
\underset{l}{\pounds}\, \underset{III}{C}_{ab} = & \{ D\Psi_3 [ m_{(a} n_{b)} - l_{(a} m_{b)} ] + \tfrac{1}{2} \Psi_3 \times \\
& [ (-2\tau - \bar{\pi})\, l_a l_b + (\tau + \bar{\pi})\, l_a n_b + \\
& + (-\epsilon - \bar{\epsilon} + \rho + \gamma + \bar{\gamma})\, l_a m_b + (-\epsilon + \bar{\epsilon} + \sigma)\, l_a \bar{m}_b + \\
& + (\tau - \kappa)\, n_a l_b + \kappa\, n_a n_b + (\epsilon + \bar{\epsilon} - \rho)\, n_a m_b + \\
& + n_a \bar{m}_b (-\sigma) + (\gamma + \bar{\gamma} + \rho)\, m_a l_b + \\
& + (\epsilon - \bar{\epsilon} - \rho)\, m_a n_b + (\pi + \bar{\kappa} - 2\alpha - 2\bar{\beta})\, m_a m_b + \\
& + (\bar{\pi} + \kappa - \bar{\alpha} - \beta)\, m_a \bar{m}_b + (-\bar{\alpha} - \beta)\, \bar{m}_a m_b + \\
& + (-\sigma)\, \bar{m}_a n_b + \sigma\, \bar{m}_a l_b ] \} + (\text{c.c.})
\end{aligned}
$$



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$$(\mathrm{AI}\text{-}3)\quad \underset{l}{\pounds}\, \underset{D}{C}_{ab} = -\frac{D(\Psi_2+\overline{\Psi}_2)}{2}\left[l_a l_b - l_a n_b - n_a l_b + n_a n_b - \overline{m}_a m_b - m_a \overline{m}_b\right] -$$

$$-(\Psi_2+\overline{\Psi}_2)\Big\{(\epsilon+\overline{\epsilon})l_a l_b - \overline{\kappa} m_a l_b - \kappa \overline{m}_a l_b +$$

$$+(\epsilon+\overline{\epsilon})l_a l_b - \overline{\kappa} l_a m_b - \kappa l_a \overline{m}_b - \pi m_a l_b -$$

$$-\overline{\pi}\overline{m}_a l_b + (\epsilon+\overline{\epsilon}) n_a l_b - (\epsilon+\overline{\epsilon}) n_a l_b +$$

$$+\overline{\kappa} n_a m_b + \kappa n_a \overline{m}_b - (\epsilon+\overline{\epsilon}) l_a n_b +$$

$$+\overline{\kappa} m_a n_b - \kappa \overline{m}_a n_b - \pi l_a m_b - \overline{\pi} l_a \overline{m}_b$$

$$+(\epsilon+\overline{\epsilon}) l_a n_b + \pi m_a n_b + \overline{\pi}\overline{m}_a n_b -$$

$$-(\epsilon+\overline{\epsilon}) n_a n_b + \pi n_a m_b + \overline{\pi} n_a \overline{m}_b -$$

$$-(\epsilon+\overline{\epsilon}) n_a n_b - \overline{\pi} l_a \overline{m}_b - (\epsilon-\overline{\epsilon}) m_a \overline{m}_b$$

$$+\kappa n_a \overline{m}_b - \pi m_a l_b - (\overline{\epsilon}-\epsilon) m_a \overline{m}_b +$$

$$+\overline{\kappa} m_a n_b - \pi l_a m_b - (\overline{\epsilon}-\epsilon)\overline{m}_a m_b + \overline{\kappa} n_a m_b$$

$$-\overline{\pi}\overline{m}_a l_b - (\epsilon-\overline{\epsilon})\overline{m}_a m_b + \kappa \overline{m}_a n_b - (\gamma+\overline{\gamma}) l_a l_b +$$

$$+(\alpha+\overline{\beta}) l_a m_b + (\overline{\alpha}+\beta) l_a \overline{m}_b - (\epsilon+\overline{\epsilon}) l_a n_b - (\gamma+\overline{\gamma}) n_a l_b +$$

$$+(\alpha+\overline{\beta}) n_a m_b + (\overline{\alpha}+\beta) n_a \overline{m}_b - (\epsilon+\overline{\epsilon}) n_a n_b - \overline{\tau} m_a l_b +$$

$$+\overline{\sigma} m_a m_b + \overline{\rho} m_a \overline{m}_b - \overline{\kappa} m_a n_b - \tau \overline{m}_a l_b + \rho \overline{m}_a m_b +$$

$$+\sigma \overline{m}_a \overline{m}_b - \kappa \overline{m}_a n_b - (\gamma+\overline{\gamma}) l_a l_b + (\alpha+\overline{\beta}) m_a l_b +$$

$$+(\overline{\alpha}+\beta)\overline{m}_a l_b - (\epsilon+\overline{\epsilon}) n_a l_b + (\gamma+\overline{\gamma}) l_a n_b -$$

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$$-(\alpha+\bar{\beta})m_a n_b-(\bar{\alpha}+\beta)\bar{m}_a n_b +$$
$$+(\epsilon+\bar{\epsilon})n_a n_b-\tau l_a\bar{m}_b+\rho m_a\bar{m}_b+$$
$$+\sigma\bar{m}_a\bar{m}_b-\kappa n_a\bar{m}_b-\bar{\tau}l_a m_b+$$
$$+\bar{\sigma}m_a m_b+\bar{\rho}\bar{m}_a m_b-\kappa n_a m_b]$$

Using AI-2 and AI-3, $\underset{I}{\pounds}C_{ab}$ can be obtained by replacing $-\Psi_3$ by $(\bar{\Psi}_1-\Psi_3)$ in AI-2 and using AI-3. Also $\underset{II}{\pounds}C_{ab}$ can be obtained by adding AI-2 and AI-3.

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## APPENDIX-II

## COMPUTATIONAL AIDS

I Ray concomitants for covariant derivatives of NP-tetrad

$$l_{a;b} = (\gamma+\bar{\gamma}) l_a l_b - (\alpha+\bar{\beta}) l_a m_b - (\bar{\alpha}+\beta) l_a \bar{m}_b + (\epsilon+\bar{\epsilon}) l_a n_b - \bar{\tau} m_a l_b + \bar{\sigma} m_a m_b + \bar{\rho} m_a \bar{m}_b - \bar{\kappa} m_a n_b - \tau \bar{m}_a l_b + \rho \bar{m}_a m_b + \sigma \bar{m}_a \bar{m}_b - \kappa \bar{m}_a n_b. \quad \text{(A2.1)}$$

$$n_{a;b} = \nu m_a l_b - \lambda m_a m_b - \mu m_a \bar{m}_b + \pi m_a n_b + \bar{\nu} \bar{m}_a l_b - \bar{\mu} \bar{m}_a m_b - \bar{\lambda} \bar{m}_a \bar{m}_b + \bar{\pi} \bar{m}_a n_b - (\gamma+\bar{\gamma}) n_a l_b + (\alpha+\bar{\beta}) n_a m_b + (\bar{\alpha}+\beta) n_a \bar{m}_b - (\epsilon+\bar{\epsilon}) n_a n_b. \quad \text{(A2.2)}$$

$$m_{a;b} = \bar{\nu} l_a l_b - \bar{\mu} l_a m_b - \bar{\lambda} l_a \bar{m}_b + \bar{\pi} l_a n_b + (\gamma-\bar{\gamma}) m_a l_b + (\bar{\beta}-\alpha) m_a m_b + (\bar{\alpha}-\beta) m_a \bar{m}_b + (\epsilon-\bar{\epsilon}) m_a n_b - n_a l_b + \rho n_a m_b + \sigma \bar{m}_b n_a - \kappa n_a n_b. \quad \text{(A2.3)}$$

$$\bar{m}_{a;b} = \nu l_a l_b - \lambda l_a m_b - \mu l_a \bar{m}_b + \pi l_a n_b + (\bar{\gamma}-\gamma) \bar{m}_a l_b + (\alpha-\bar{\beta}) \bar{m}_a m_b + (\beta-\bar{\alpha}) \bar{m}_a \bar{m}_b + (\bar{\epsilon}-\epsilon) \bar{m}_a n_b - \bar{\tau} n_a l_b + \bar{\sigma} n_a m_b + \bar{\rho} n_a \bar{m}_b - \bar{\kappa} n_a n_b. \quad \text{(A2.4)}$$

The intrinsic derivatives of the tetrad vectors :

$$\begin{aligned}
l_{a;b} l^b &= (\epsilon+\bar{\epsilon}) l_a - \bar{\kappa} m_a - \kappa \bar{m}_a, \\
l_{a;b} m^b &= (\bar{\alpha}+\beta) l_a - \bar{\rho} m_a - \sigma \bar{m}_a, \\
l_{a;b} \bar{m}^b &= (\alpha+\bar{\beta}) l_a - \rho \bar{m}_a - \bar{\sigma} m_a. \\
l_{a;b} n^b &= (\gamma+\bar{\gamma}) l_a - \bar{\tau} m_a - \tau \bar{m}_a.
\end{aligned} \quad \text{(A2.5)}$$

$$
\begin{aligned}
n_{a;b}l^b &= \pi m_a + \bar{\pi}\bar{m}_a - (\epsilon+\bar{\epsilon})n_a, \\
n_{a;b}m^b &= \mu m_a + \bar{\lambda}\bar{m}_a - (\bar{\alpha}+\beta)n_a, \\
n_{a;b}\bar{m}^b &= \lambda m_a + \bar{\mu}\bar{m}_a - (\alpha+\bar{\beta})n_a, \\
n_{a;b}n^b &= \nu m_a + \bar{\nu}\bar{m}_a - (\gamma+\bar{\gamma})n_a.
\end{aligned}
\tag{A2.6}
$$

$$
\begin{aligned}
m_{a;b}l^b &= \bar{\pi} l_a + (\epsilon-\bar{\epsilon})m_a - \kappa n_a, \\
m_{a;b}m^b &= \bar{\lambda} l_a - (\bar{\alpha}-\beta)m_a - \sigma n_a, \\
m_{a;b}\bar{m}^b &= \bar{\mu} l_a - (\bar{\beta}-\alpha)m_a - \rho n_a, \\
m_{a;b}n^b &= \bar{\nu} l_a + (\gamma-\bar{\gamma})m_a - \tau n_a.
\end{aligned}
\tag{A2.7}
$$

$$
\begin{aligned}
\bar{m}_{a;b}l^b &= \pi l_a + (\bar{\epsilon}-\epsilon)\bar{m}_a - \bar{\kappa} n_a, \\
\bar{m}_{a;b}m^b &= \mu l_a + (\bar{\alpha}-\beta)\bar{m}_a - \bar{\rho} n_a, \\
\bar{m}_{a;b}\bar{m}^b &= \lambda l_a - (\alpha-\bar{\beta})\bar{m}_a - \bar{\sigma} n_a, \\
\bar{m}_{a;b}n^b &= \nu l_a + (\bar{\gamma}-\gamma)\bar{m}_a - \bar{\tau} n_a.
\end{aligned}
\tag{A2.8}
$$

The projections of the tetrad vectors can be obtained from (A2.1) to (A2.4) in the following forms :

$$
\begin{aligned}
l^a l_{a;b} &= 0 \\
m^a l_{a;b} &= \tau l_b - \rho m_b - \sigma \bar{m}_b + \kappa n_b, \\
\bar{m}^a l_{a;b} &= \bar{\tau} l_b - \bar{\sigma} m_b - \bar{\rho}\bar{m}_b + \kappa n_b \\
n^a l_{a;b} &= (\gamma+\bar{\gamma}) l_b - (\alpha+\bar{\beta})m_b - (\bar{\alpha}+\beta)\bar{m}_b + (\epsilon+\bar{\epsilon})n_b
\end{aligned}
\tag{A2.9}
$$

$$
\begin{aligned}
l^a n_{a;b} &= -(\gamma+\bar{\gamma})\, l_b + (\alpha+\bar{\beta})\, m_b + (\bar{\alpha}+\beta)\, \bar{m}_b - (\epsilon+\bar{\epsilon})\, n_b, \\
m^a n_{a;b} &= -\bar{\nu}\, l_b + \bar{\mu}\, m_b + \bar{\lambda}\, \bar{m}_b - \bar{\pi}\, n_b, \\
\bar{m}^a n_{a;b} &= -\nu\, l_b + \lambda\, m_b + \mu\, \bar{m}_b - \pi\, n_b, \\
n^a n_{a;b} &= 0.
\end{aligned}
\tag{A3.0}
$$

$$
\begin{aligned}
l^a m_{a;b} &= -\tau\, l_b + \rho\, m_b + \sigma\, \bar{m}_b - \kappa\, n_b, \\
m^a m_{a;b} &= 0, \\
\bar{m}^a m_{a;b} &= -(\gamma-\bar{\gamma})\, l_b - (\bar{\beta}-\alpha)\, m_b - (\bar{\alpha}-\beta)\, \bar{m}_b - (\epsilon-\bar{\epsilon})\, n_b, \\
n^a m_{a;b} &= \bar{\nu}\, l_b - \bar{\mu}\, m_b - \bar{\lambda}\, \bar{m}_b + \bar{\pi}\, n_b.
\end{aligned}
\tag{A3.1}
$$

$$
\begin{aligned}
l^a \bar{m}_{a;b} &= -\bar{\tau}\, l_b + \bar{\rho}\, \bar{m}_b + \bar{\sigma}\, m_b - \bar{\kappa}\, n_b, \\
m^a \bar{m}_{a;b} &= -(\bar{\gamma}-\gamma)\, l_b - (\beta-\bar{\alpha})\, \bar{m}_b - (\alpha-\bar{\beta})\, m_b - (\bar{\epsilon}-\epsilon)\, n_b, \\
\bar{m}^a \bar{m}_{a;b} &= 0, \\
n^a \bar{m}_{a;b} &= \nu\, l_b - \mu\, \bar{m}_b - \lambda\, m_b + \pi\, n_b.
\end{aligned}
\tag{A3.2}
$$

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