CHAPTER-I

INTRODUCTION

A function f analytic in the whole of the complex plane $\mathbb{C}$ is called an entire function. Liouville proved that a bounded entire function must be a constant. Consequently if $M(r,f) = \max_{|z| \leq r} |f(z)|$ then $M(r,f) \to \infty$ as $r \to \infty$. To study the growth of entire functions we compare $M(r,f)$ with e^{r^t} where $t > 0$.

An entire function $f(z)$ is said to be a function of finite order if there exists a positive constant t such that the inecuality

$$M(r,f) < e^{r^t}$$

is valid for sufficiently large values of r.

The greatest lower bound of such numbers t is called the order of f and is denoted by $\rho(f)$ or simply by ρ. It can be easily be verified that

$$\rho(f) = \limsup_{r \to \infty} \frac{\log \log M(r,f)}{\log r}$$

A function which is analytic in $\mathbb{C}$ except possibly for poles is called a meromorphic function. So obviously every entire function is meromorphic function (as it does not have any poles). In this dissertation we shall mostly be interested with meromorphic functions. A fundamental role in the Nevanlinna's theory of meromorphic functions is played by the theorem known as "Poisson Jensen formula". We shall briefly develop this theory which we shall require in our dissertation.

<u>Theorem 1.1 (Poisson Jensen):</u> If $f(z)$ is meromorphic in $|z| \leq R$ $(0 < R < \infty)$ and a_μ $(\mu = 1, 2, \ldots, M)$ are the zeros and b_ν $(\nu = 1, 2, \ldots, N)$ are the poles of $f(z)$ in $|z| \leq R$ and if $z = re^{i\theta}$ $(0 < r < R)$ and if $f(z) \neq 0$, then

$$\log |f(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi$$

$$+ \sum_{\mu=1}^{M} \log \left| \frac{R(z - a_\mu)}{R^2 - \bar{a}_\mu z} \right| - \sum_{\nu=1}^{N} \log \left| \frac{R(z - b_\nu)}{R^2 - \bar{b}_\nu z} \right|. \quad (1.1)$$

The case when $z = 0$ is called Jensen's formula. We define

$$\log^+ x = \log x \quad \text{if } x \geq 1$$
$$= 0 \quad \text{if } 0 \leq x < 1$$

then since $\log x = \log^+ x - \log^+ \frac{1}{x}$ for all $x > 0$, it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{iO})| dO = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(Re^{i\phi})| d\phi$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(Re^{i\phi})| d\phi.$$

Let $|a_\mu| = r_\mu$, $\mu = 1, 2, \ldots, M$

then

$$\sum_{\mu=1}^{M} \log \left| \frac{R}{a_\mu} \right| = \int_0^R \log \frac{R}{t} dn(t)$$

where $n(t)$ is the number of zeros of $f(z)$ in $|z| \leq t$.

Integrating by parts, we get

$$\sum_{\mu=1}^{M} \log \frac{R}{r_\mu} = \int_0^R \frac{n(t)}{t} dt \qquad (1.2)$$

We denote the R.H.S. of (1.2) by $N(R,0)$ and also by $N(R,\frac{1}{f})$.

Similarly $$\sum_{\nu=1}^{N} \log \left| \frac{R}{b_\nu} \right| = \int_0^R \frac{n(t,\infty)}{t} dt$$

$$= N(R,f) = N(R,\infty)$$ where $n(t,\infty)$ denotes the number of poles of $f(z)$ in $|z| \leq t$.

Set

$$m(R,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(Re^{i\phi})| d\phi$$

then (1.1) with $z=0$ gives

$$\log |f(0)| = m(R,f) - m(R,\frac{1}{f}) + N(R,f) - N(R,\frac{1}{f})$$

or

$$m(R,f) + N(R,f) = m(R,\frac{1}{f}) + N(R,\frac{1}{f}) + \log |f(0)|. \qquad (1.3)$$

We set

$$m(R,f) + N(R,f) = T(R,f),$$ and (1.3) becomes

$$T(R,f) = T(R,\frac{1}{f}) + \log |f(0)|. \qquad (1.4)$$

The term $T(R,f)$, is called the Nevanlinna characteristic function, plays a cardinal role in the whole theory of meromorphic functions.

The equation (1.4) is a particular case of a more general theorem, known as Nevanlinna's first fundamental theorem.

Theorem 1.2 : If a is any complex number then

$$T(r,\frac{1}{f-a}) = T(r,f) - \log|f(0)-a| + \epsilon(a,r)$$

where

$$|\epsilon(a,r)| \leq \log^{+}|a| + \log 2 .$$

The above theorem gives directly some of the elementary following properties of $T(r,f)$

(i) If $k>0$ is a constant then

$$T(r,kf) = T(r,f)+O(1)$$

(ii) $T(r,fg) \leq T(r,f)+T(r,g)$

(iii) $T(r,f+g) \leq T(r,f)+T(r,g)+O(1)$

(iv) $T(r,\frac{af+b}{cf+d}) = T(r,f)+O(1)$ where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$

For an entire function f, $T(r,f)$ has many properties similar to $\log M(r,f)$. For instance, like $\log M(r,f)$, $T(r,f)$ is an increasing function of r and a convex function of $\log r$. Hence it is natural to define the order of a meromorphic function $f(z)$ by

$$\rho = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

For an entire function, it can be proved that

$$\rho = \limsup_{r\to\infty} \frac{\log \log M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} .$$

This follows from the fact that if f(z) is analytic in $|z| \leq R$, then for $0 \leq r < R$

$$T(r,f) \leq \log^{+}M(r,f) \leq \frac{R+r}{R-r}\, T(R,f). \tag{1.5}$$

For functions of given order, a better measure of growth is obtained by the term type of an entire (or meromorphic) function.

If f(z) is an entire function of finite order $\rho\,(>0)$ then the type τ of f is defined by

$$\tau = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} . \tag{1.6}$$

τ is said to be minimal type, maximal type or mean type according as $\tau = 0$, $\tau = \infty$ or $0 < \tau < \infty$ respectively.

For a meromorphic function of finite order $\rho\,(>0)$ the type σ of f is defined by

$$\sigma = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho}} , \tag{1.7}$$

Thus for entire functions type can be defined by (1.6) and as well as by (1.7) and it follows on using (1.5) that

$\tau = \infty$ if and only if $\sigma = \infty$

$\tau = 0$ if and only if $\sigma = 0$

$0 < \tau < \infty$ if and only if $0 < \sigma < \infty$.

Let us emphasize here that the third case does not imply that $\tau = \sigma$, see for instance [4 ,19].

The other important theorem of Nevanlinna theory viz. the second fundamental theorem of Nevanlinna states:

Theorem 1.3: Let f(z) be a nonconstant meromorphic function in $|z| \leq r$. Let $a_1, \ldots, a_q$, where $q > 2$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\nu| \geq \delta$ for $1 \leq \mu < \nu \leq q$ then

$$m(r,\infty) + \sum_{\nu=1}^{q} m(r,a_\nu) \leq 2T(r,f) - N_1(r) + S(r,f) \tag{1.8}$$

where

$N_1(r)$ is positive and is given by

$$N_1(r) = N(r,\frac{1}{f'}) + 2N(r,f) - N(r,f')$$

and

$$s(r,f) = m(r,\frac{f'}{f}) + m(r,\sum_{\nu=1}^{q} \frac{f'}{f-a_\nu}) + q \log^+ \frac{3q}{\delta}$$

$$+ \log 2 + \log \frac{1}{|f'(0)|}$$

with modifications if $f(0)=0$ or ∞ or $f'(0)=0$.

The term s(r,f) in general plays an unimportant role. Infact if f is of finite order then $s(r,f)=O(T(r,f))$ as $r \to \infty$. And if f is of infinite order then $s(r,f)=O(T(r,f))$ as $r \to \infty$ except possibly for a set of r of finite linear measure.

Adding $N(r,f)+\sum_{\nu=1}^{q} N(r,\frac{1}{f-a_\nu})$ on both the sides of (1.8) and using the first fundamental theorem of Nevanlinna it is easy to see that (1.8) yields

$$(q-2)T(r,f) \leq \sum_{\nu=1}^{q} N(r,\frac{1}{f-a_\nu})-N_1(r)+s(r,f)$$

$$\leq \sum_{\nu=1}^{q} N(r,\frac{1}{f-a_\nu})+s(r,f).$$

Thus we observe that in

$$T(r,f) = m(r,a)+N(r,a)+O(1),$$

the term m(r,a) is small as compared to T(r,f) and consequently N(r,a) comes near to T(r,f). This leads to defining the deficiency relation for which we start with the following.

Let n(t,a)=n(t,a,f) denote the number of roots of the equation f(z)=a in $|z|\leq t$, multiple roots being counted with their multiplicity and let $\bar{n}(t,a)$ be the number of distinct roots of f(z)=a in $|z|\leq t$.
We set

$$N(r,a) = N(r,a,f) = \int_0^r \frac{n(t,a)-n(o,a)}{t}\,dt + n(o,a)\log r$$

$$\bar{N}(r,a) = \bar{N}(r,a,f) = \int_0^r \frac{\bar{n}(t,a)-\bar{n}(o,a)}{t}\,dt + \bar{n}(0,a)\log r .$$

As usual we set

$$N(r,\infty,f) = N(r,f),$$

$$T(r,\infty,f) = T(r,f).$$

Let

$$\delta(a) = \delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)}$$

$$\Theta(a) = \Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a)}{T(r,f)}$$

$$\lambda(a) = \lambda(a,f) = 1 - \liminf_{r\to\infty} \frac{N(r,a)}{T(r,f)}$$

$$\theta(a) = \theta(a,f) = \liminf_{r\to\infty} \frac{N(r,a)-\overline{N}(r,a)}{T(r,f)} .$$

The quantity $\delta(a)$ is called the deficiency of the value a and $\Theta(a)$ is called the index of multiplicity. Clearly for any $a\in\overline{\mathbb{C}}$, the extended complex plane, $0 \leq \delta(a) \leq 1$. Infact more is true! Nevanlinna proved that if f(z) is a meromorphic function then the set of values a for which $\Theta(a) > 0$ is countable and

$$\sum_a \{\delta(a) + \theta(a)\} \leq \sum_a \Theta(a) \leq 2.$$

The quantity $\delta(a)$ is called the deficiency of the value a. Finally the term s(r,f) will denote any quantity satisfying $s(r,f) = o(T(r,f))$ as $r\to\infty$ through all values of r if f is of finite order and as $r\to\infty$ possibly outside a set of finite linear measure if f is of infinite order.

The study of exceptional values plays a fundamental role in theory of meromorphic function. If for a meromorphic function f, $\delta(a,f) > 0$, then a is called an exceptional value in the sense of Nevarlinna (e.v.N.). If $n(r,a)=n(r,a,f)=O(1)$ where

$n(r,a)$ denotes the number of zeros of $f-a$ then a is called exceptional value Picard (e.v.P.).

We denote

$$\limsup_{r\to\infty} \frac{\log^+ n(r,a)}{\log r}$$

by $\rho_1(a)$ and call it the exponent of convergence of the a-points of $f(z)$. It is well known that $\rho_1(a) \leq \rho$. Also if the order of f is not an integer then $\rho_1(a) = \rho$. And in the general case $\rho_1(a) = \rho$ except possibly for one or two values of a depending on whether f is entire or meromorphic. This exceptional value is called exceptional value in the sense of Borel (e.v.B.). Further a is called e.v.E. if $\liminf_{r\to\infty} \frac{T(r,f)}{n(r,a)\phi(r)} > 0$ where $\phi(r)$ is any positive non-decreasing function such that $\int_A^\infty \frac{dx}{x\phi(x)} < \infty$. There are relations involving these exceptional values. For it is known see [12] that for an entire function e.v.P. $\Rightarrow$ e.v.B. $\Rightarrow$ e.v.E. $\Rightarrow$ e.v.N. For a meromorphic function e.v.P. $\Rightarrow$ e.v.B. $\Rightarrow$ e.v.N.. However in this case e.v.B. $\Rightarrow$ e.v.N. is not true. Infact Valiron [14] has shown by an example that if α is e.v.B. for a meromorphic function then α may not be an e.v.N..

Let $f(z)$ be an entire function and Γ be a curve starting from $z=0$ and proceeding towards infinity. If $f(z)\to a$ (a finite) as $z\to\infty$ along Γ we say that a is an asymptotic value for $f(z)$, and Γ is called an asymptotic path.

For an entire function every e.v.P. is an asymptotic value, and every e.v.B. is also an asymptotic value in case if the function is of finite order. Nevanlinna put the question whether every e.v.N. is also an asymptotic value. This was disproved by Arakelion, a Russian mathematician, who constructed an entire function of finite order having infinity of e.v.N which obviously cannot all be asymptotic values because by Ahlfor's theorem, an entire function can have at most 2ρ asymptotic values where ρ is the order of f. See Arakelian,"Doklady AKademy Nayuk, U.S.S.R.,1966". S.M.Shah in 1952 proved that if f(z) is an entire function of finite order ρ having a as e.v.E. then the number of asymptotic values of f(z) is precisely ρ and each asymptotic value is a. Nevanlinna conjectured that if α is e.v.N. for an entire function or meromorphic function then α must be an asymptotic value. But this was proved to be false in 1941 by Madame Laurent Schwartz. She constructed a meromorphic function f(z) for which $\delta(0)=\delta(\infty)>0$, and thus 0 and ∞ are e.v.N., but they were not asymptotic values, See [6]. For an entire function of infinite oredr it was proved to be false by W.K.Hayman and for finite order it was proved to be false by A.A.Goldberg, see [3]. But with some additional hypothesis the conjecture of Nevanlinna is true. Edrei and Fuchs have proved that if f(z) is an entire function of finite order and if $\sum_{i}\delta(a_i)=2$, that is, the total deficiency is attained, then each deficient value of f(z) is also an asymptotic value, See [2]. Later on by replacing some other smoother condition in place of $\sum_i \delta(a_i)=2$, Edrei and Fuchs proved that the restriction that f(z) must be of finite order can be removed and each deficient value will be asymptotic value, See A. Edrei [1].

The deficient values corresponding to zeros and poles being counted only once have also been studied extensively. Nevanlinna's theorem on deficient values states that if f(z) is meromorphic function then the set of values of a, for which $\delta(a)>0$ or $\Theta(a)>0$ is countable and $\sum_a \Theta(a) \leq 2$. This clearly implies that $\sum_a \delta(a) \leq 2$. If $\sum \delta(a)=2$ then we say that the total deficiency is attained , S.K.Singh and H.S. Gopalkrishna [13] have shown by an example that a meromorphic function may be such that $\sum_{a\in\overline{\mathbb{C}}} \delta(a)=1$ where as $\sum_{a\in\overline{\mathbb{C}}} \Theta(a)=2$.

As mentioned earlier

$$\delta(a,f) = 1-\limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)} .$$

Consequently one can speak of the term

$$1 - \limsup_{r\to\infty} \frac{N(r,\frac{1}{f'-a})}{T(r,f')} \tag{1.9}$$

which can be denoted by $\delta(a,f')$. Milloux [7] introduced the concept of relative defects where he defined the term

$$1 - \limsup_{r\to\infty} \frac{N(r,\frac{1}{f'-a})}{T(r,f)} \tag{1.10}$$

and in contrast the usual defect given by (1.9) was called the absolute defect. This definition was later extended by Xiong-Qing Lai [15] where he defined the terms

$$\delta_r^{(k)}(\alpha, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f^{(k)} - \alpha})}{T(r, f)} \tag{1.11}$$

and

$$\delta(\alpha, f^{(k)}) = \delta_a^{(k)}(\alpha, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f^{(k)} - \alpha})}{T(r, f^{(k)})}. \tag{1.12}$$

The suffixes "r" and "a" in the left hand side of (1.11) and (1.12) are just to distinguish between the term "relative" and "absolute". Xiong-Qing Lai found various relations involving the relative defects and the usual or the absolute defects.

Later A.P.Singh [11] defined the relative defects corresponding to the distinct zeros and distinct poles of a meromorphic function. He introduced the term

$$\Theta_r^{(k)}(\alpha, f) = 1 - \limsup_{t\to\infty} \frac{\overline{N}(t, \frac{1}{f^{(k)} - \alpha})}{T(t, f)}$$

and found various bounds for $\Theta_r^{(k)}(\alpha, f)$ in terms of the usual defects and the relative defects. Also he studied the behaviour of the relative defects when f satisfied certain conditions with regards to the deficient values.

The concept of relative defects was further carried over to two meromorphic functions in A.P.Singh's subsequent paper [11]. For this we shall first need some notations.

Let $f_1(z)$, $f_2(z)$ be two non-constant meromorphic functions and let a be any complex number. Let $n_0(r,a)$ denote the number of common roots in the disk $|z| \leq r$ of the two equations $f_1(z)=a$ and $f_2(z)=a$, and let $\bar{n}_0(r,a)$ denote the number of common roots in the disk $|z| \leq r$ of the two equations $f_1(z)=a$ and $f_2(z)=a$, where the multiplicity is disregarded (i.e. each root being counted only once). Set

$$\bar{N}_0(r,a) = \int_0^r \frac{\bar{n}_0(t,a)-\bar{n}_0(o,a)}{t}\,dt + \bar{n}_0(0,a)\log r$$

$$\bar{N}_{1,2}(r,a) = \bar{N}\left(r,\frac{1}{f_1-a}\right) + \bar{N}\left(r,\frac{1}{f_2-a}\right) - 2\bar{N}_0(r,a).$$

Let $\bar{n}_0^{(k)}(r,a)$, $N_{1,2}^{(k)}(r,a)$ etc. denote the corresponding quantities with respect to $f_1^{(k)}$ and $f_2^{(k)}$. Set

$$\Theta_{1,2}(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_{1,2}(r,a)}{T(r,f_1)+T(r,f_2)}$$

$$\Theta_{1,2}^{(k)}(a) = 1 - \limsup_{r\to\infty} \frac{N_{1,2}^{(k)}(r,a)}{T(r,f_1)+T(r,f_2)}$$

$$\delta_{1,2}(a) = 1 - \limsup_{r\to\infty} \frac{N_{1,2}(r,a)}{T(r,f_1)+T(r,f_2)} \quad ,$$

$\Theta_0(a)$, $\Theta_0^{(k)}(a)$ being similarly defined. A.P.Singh in the above mentioned paper [11] has proved several relations dealing with these relative defects. For instance he has shown that if $f_1(z)$ and $f_2(z)$ are two meromorphic functions such that $N(r,\frac{1}{f_1})=s(r,f_1)$ and $N(r,\frac{1}{f_2})=s(r,f_2)$ then for any $a\neq 0,\infty$

$$\Theta_{1,2}^{(k)}(a) + 2\,\Theta_0^{(k)}(a) \leq 5 - (\Theta_{1,2}(\infty) + 2\,\Theta_0(\infty)),$$

and for any finite non-zero distinct α and β

$$\Theta_{1,2}^{(k)}(\alpha) + \Theta_{1,2}^{(k)}(\beta) \leq 5-2(\Theta_0^{(k)}(\alpha) + \Theta_0^{(k)}(\beta)).$$

Our second chapter deals with this concept of relative defects where we have found several bounds for relative defects in terms of the Nevanlinna deficient values and bounds for relative defects corresponding to two meromorphic functions in terms of absolute defects corresponding to the two functions. Thus, for instance, we have shown in Theorem 2.1 that if f is meromorphic and a and b are distinct finite complex numbers and further if $b\neq 0$ then for all positive integers k

$$\Theta_r^{(k)}(b,f) \leq 2-(\Theta(\infty, f)+\delta(a,f)),$$

and in theorem 2.7 we show that if f is meromorphic function and a and $b(\neq 0)$ are distinct finite complex numbers then for every non-negative integer k,

$$\Theta_r^{(k)}(a,f) + \Theta_r^{(k)}(b,f) \leq 4 - \{2\,\Theta(\infty,f) + \delta(0,f)\} .$$

As regards to the relative defects corresponding to two meromorphic functions we have proved in Theorem 2.2 that for f_1 and f_2 meromorphic and a, b distinct finite non-zero complex numbers

$$\Theta_{1,2}^{(k)}(b) + 2\,\Theta_o^{(k)}(b) \leq 8 - [\Theta_{1,2}(\infty) + \delta_{1,2}(a) + 2\,\Theta_o(\infty) + 2\,\delta_o(a)]$$

for every positive integer k. And in Theorem 2.6 we have shown

$$\delta_{1,2}^{(k)}(\infty) + 2\delta_o^{(k)}(\infty) \leq \frac{11}{2} - \frac{1}{2}\{\delta_{1,2}(b) + 2\,\delta_o(b) + \delta_{1,2}(a) + 2\,\delta_o(a)\} .$$

Several other theorems of similar nature have been proved in this chapter. In the proofs of these results a fundamental role is played by a theorem of Milloux which we now state.

Theorem 1.4 (Milloux) : Let p be a positive integer and

$$\psi(z) = \sum_{\nu=0}^{p} a_\nu(z) f^{(\nu)}(z).$$

Then

$$m(r, \frac{\psi(z)}{f(z)}) = s(r,f)$$

and

$$T(r,\psi) \leq (p+1)T(r,f) + s(r,f).$$

The proof of this theorem can be found in [4 ,55]. Another important theorem which we shall need frequently is :

Theorem 1.5 : Let f and g be two meromorphic functions with $g(0)\neq 0$. Then

$$N(r,\frac{f}{g}) - N(r,\frac{g}{f}) = N(r,f)+N(r,\frac{1}{g}) - N(r,g) - N(r,\frac{1}{f}).$$

For the proof of this theorem, one can refer [8 ,73].

A function L(r) is said to be slowly increasing function if $L(ct)\sim L(t)$ as $t\to\infty$ for every fixed positive C. In chapter III we have used the comparison function $r^{\rho}L(r)$ where ρ is the order of f to obtain bounds for $\overline{n}(r,\frac{1}{f^{(k)}-a})$ and $\overline{N}(r,\frac{1}{f^{(k)}-a})$. Thus for instance we have shown in Theorem 3.2 that if

$$\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho}L(r)} = a \quad \text{and} \quad \limsup_{r\to\infty} \frac{N(r,\frac{1}{f-a})}{r^{\rho}L(r)} = b \qquad (b\neq 0,a)$$

then for every positive integer k and $c\neq 0,a,b$,

$$\limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f^{(k)}-b})}{r^{\rho}L(r)} + \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f^{(k)}-c})}{r^{\rho}L(r)} \geqslant a-b .$$

And in theorem 3.6 we have shown that if f is an entire function of order ρ $(0<\rho<\infty)$, with

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}L(r)} = \alpha \quad \text{and} \quad \limsup_{r\to\infty} \frac{N(r,\frac{1}{f})}{r^{\rho}L(r)} = \beta \quad \text{, then}$$

for any distinct $a_1,\ldots,a_p$ and for every positive integer k,

$$\sum_{i=1}^{p} \limsup \frac{\overline{N}(r, \frac{1}{f^{(k)}-a_i})}{r^{\rho}L(r)} \geqslant p\left(\frac{\alpha}{h(\rho)} - \beta \right)$$

and

$$\sum_{i=1}^{p} \limsup \frac{\overline{n}(r, \frac{1}{f^{(k)}-a_i})}{r^{\rho}L(r)} \geqslant \rho\, p\left(\frac{\alpha}{h(\rho)} - \beta \right)$$

where $h(\rho) = \{\rho + (1+\rho^2)^{\frac{1}{2}}\} \{ \frac{1+(1+\rho^2)^{\frac{1}{2}}}{\rho} \}^{\rho}$.

Several other theorems of similar nature have been proved in this chapter. In the proofs of these theorems we require the following theorem, the proof of which can be found in [5].

<u>Theorem 1.6</u> [Lemma 5,5]: If $\phi(kt) \sim \phi(t)$ when $t \to \infty$ for any fixed positive k. Then for every positive δ

$$\int_1^t u^{\delta-1}\phi(u)\,du \sim \frac{t^{\delta}}{\delta}\,\phi(t).$$