CHAPTER - II

C H A P T E R - II

INTRODUCTION :

Here we are interested to study the practical application of the boundary layer theory which are non-linear parabolic differential equations. These equations are very difficult and lengthy to be solved. Therefore, it becomes essential to device approximate methods which may be much quicker to apply, although less accurate but may yield acceptable results.

J.L. Bansal [12] studied the asymptotic suction temperature profiles in a laminar boundary layer over a porous flat plate. In this note it has been shown that in the case of laminar boundary layer over a porous flat plate. In this note it has been shown that in the case of laminar boundary layer over a flat plate with homogeneous suction as we have the 'asymptotic suction velocity profiles' for various values of the Prandtl number (Pr). The recovery factor (Tr) in such a case is found to be independent of Pr and has a constant value 1.

Holt, M. and Modarress, D. [13] studied the 'Application of the method of integral relations to laminar boundary layers in three-dimensions'. It has been traditional along fluid dynamicists to employ some numerical means (such as the finite difference techniques) to solve two dimensional nonlinear compressible boundary layer equations. But as an alternative to this numerical procedure, the boundary layer equations have been more successfully solved in an integral form for example, with the classical Ka'rman-Pohlhausen momentum integral method. The main principle of the method of integral equation is based on the idea of representing the streamwise velocity gradient (normal to the wall) as a simple algebraic function of the streamwise velocity itself. These authors extended the method of integral relations to the problem of three-dimensional compressible boundary layer flows with and without separation. By reducing the equations of motion to a quasi-incompressible form they solved the resulting hyperbolic partial differential equations.

B.P.Acharya & S.Pandhy [6] studied the free convective viscous flow past hot vertical porous plate with periodic temperature. In this problem he obtained an analysis of a free convective flow of viscous liquid past a hot vertical porous wall was presented under the assumption that the suction velocity is constant and normal to the wall, and the wall temperature is spanwise consinusoidal approximate solution of equation of motion and energy equation have been obtained by the method of regular perturbation.

Krishna Lal [7] investigated "free convection laminar boundary layer in the unsteady flow". In this paper he was studied the effect of unsteady flow in the magnitude of surface temperature on the free convective laminar velocity and thermal boundary layer on a flat plate was studied. In section one, the general equation of motion and the temperature distribution are given. In section two, the solution are obtained when the fluctuations in the velocity components and temperature distribution are in the form

$$(u,\ v,\ G) = (\ v_o,\ V,\ G_o) + E\ (u_1,\ V_1,\ G)\ X\ \exp\ (wt)$$

and lastly solution is given when the fluctuations is an exponentially decreasing function of time.

R. Sharma [8] explained a two parameter method for calculating the two dimensional boundary layer with suction or injection. Detailed calculation of the boundary layer parameters made by this method indicates that the error are within 5% of the exact value.

M.G.Palekar and D.P.Sharma [9] studied "Approximate solution of boundary layer equation with suction blowing according to him the problem under consideration is that of the boundary layer flow along a flat plate with suction or blowing.

G.N.Sharma and D.P.Singh [10] investigated "The effect of viscosity temperature law in unsteady boundary layer on a flat plate". They studied the effect of viscosity temperature law, when the wall is in arbitrary motion with steady stream velocity. Prandtl number being unity.

D. Surma Devi and G. Nath [19] investigated 'similarity solution of the unsteady boundary layer equation for a moving wall'. In this problem we obtained the similarity solution of the unsteady laminar for two dimensional incompressible and of axisymmetric boundary layer equation for the case of surface which moves with the velocity which varies inversely as a linear function of time. The governing equation has been solved numerically.

R.P.Agrawal [11] studied 'Non-linear two point boundary value problem'. In this problem he obtained existance and uniqueness of the solution of third order non-linear differential equation with boundary conditions prescribed at two points. R. Sharma [8] obtained the exact solution of the incompressible laminar boundary layer equations with zero pressure gradient and variable suction. In this paper a numerical solution of the boundary layer equations with zero pressure gradient and with the general distribution of sunction is obtained.

It is discovered by Holstein and Bohlen [1] that the momentum integral equation may be solved easily if δ_2, instead of δ, is regarded as the unknown function. In order to avoid the numerical integration of the differential equation we turn about a simple method suggested by Waltz and Thwaites [3]. Seeing the nature of the curve $L(\lambda)$ plotted against λ, which is almost a straight line, it was suggested by Walz that the function L (λ) can be approximated by

$$L(\lambda) = a - b\lambda .$$

The solution of the equations for velocity field are obtained. The solution of the equation for the temperature field, which was obtained by E. Pohlhausen [2] . The solution of the thermal boundary layer equation

$$v_r \frac{\partial \theta}{\partial r} + v_z \frac{\partial \theta}{\partial z} = \frac{\vartheta}{Pr} \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial \theta}{\partial r} \right)$$

was obtained by Yih [5] . The solution of the integral equations was obtained by Squire [4] . The recovery factor obtained by Pohlhausen was computed by E. Eckert and R.M.Drakes [14] for a large range of Prandtl numbers namely Pr = 0.4 or Pr = 1000.

1. Approximate solution of Pohlhausen's problem of free convection on a heated vertical plate :

Notations

$a = K/eC_p$: thermal diffusivity,

K : Coefficient of thermal conductivity,

C_p : Specific heat at constant pressure,

Pr : Prandtl number = $\frac{\mu C_p}{K}$,

$$Nu(x) = \frac{-\left(\frac{\partial T}{\partial y}\right)_{y=0} \cdot x}{T_w - T_\infty} \text{ ; Nusselt Number}$$

ν : Kinematic viscosity

$\eta = y/\delta$

u and v - velocity components in x and y directions.

U_∞ - free-stream velocity in x-direction

x, y : Co-ordinates along and normal to the wall respectively.

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \text{ , dimensionless temperature}$$

δ : displacement thickness

$u_1(x)$: Any arbitrary function

C_1, C_2: Any two constants

$$Gr = \frac{g(T_w - T_\infty)}{\nu^2 T_\infty} \cdot x^3, \text{ Grasshof's Number}$$

The equations governing the motion of the fluid in the neighbourhood of heated vertical plate are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g\alpha\,\theta \qquad \ldots (2)$$

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = a \frac{\partial^2 \theta}{\partial y^2} \qquad \ldots (3)$$

where $\theta = \dfrac{T - T_\infty}{T_w - T_\infty}$ and $\alpha = \dfrac{T_w - T_\infty}{T_\infty}$

with the boundary conditions

$$\left.\begin{array}{l} y = 0,\ u = 0,\quad v = 0,\quad \theta = 1 \\ y = \delta,\ u = 0,\quad \theta = 0 \end{array}\right\} \qquad \ldots (4)$$

Integrating equations (2) and (3) with respect to y between the limits y = 0 to y = δ we get from equation (2)

$$\int_0^\delta u \frac{\partial u}{\partial x}\, dy + \int_0^\delta v \frac{\partial u}{\partial y}\, dy = \nu \int_0^\delta \frac{\partial^2 u}{\partial y^2}\, dy + g\alpha \int_0^\delta \theta dy$$

$$\ldots (5)$$

Consider $\displaystyle\int_0^\delta v \frac{\partial u}{\partial y}\, dy = vu \Big|_{y=0}^{y=\delta} - \int_0^\delta u . \frac{\partial v}{\partial y}\, dy$

$$= v.u \Big|_{y=\delta} - v.u \Big|_{y=0} - \int_0^\delta u . \frac{\partial v}{\partial y} \, dy$$

$$\int_0^\delta v \frac{\partial u}{\partial y} \, dy = + \int_0^\delta u \frac{\partial u}{\partial x} \, dy \qquad \text{by (1)}$$

Consider the first part of R.H.S. of equation (5)

$$\nu \int_0^\delta \frac{\partial^2 u}{\partial y^2} \, dy = \nu \left(\frac{\partial u}{\partial y} \right)_{y=0}^{y=\delta}$$

$$= - \nu \left(\frac{\partial u}{\partial y} \right)_{y=0} + \nu \left(\frac{\partial u}{\partial y} \right)_{y=\delta}$$

$$= - \nu \left(\frac{\partial u}{\partial y} \right)_{y=0}$$

Then equation (5) will take the following form

$$\int_0^\delta u \frac{\partial u}{\partial x} \, dy + \int_0^\delta u \frac{\partial u}{\partial x} \, dy = - \nu \left(\frac{\partial u}{\partial y} \right)_{y=0}$$

$$\text{i.e.} \quad \frac{d}{dx} \int_0^\delta u^2 \, dy = - \nu \left(\frac{\partial u}{\partial y} \right)_{y=0} \qquad \dots (6)$$

From equation (3) we have

$$\int_0^\delta u \frac{\partial \theta}{\partial x} \, dy + \int_0^\delta v \frac{\partial \theta}{\partial y} \, dy = a \int_0^\delta \frac{\partial^2 \theta}{\partial y^2} \, dy \qquad \dots (7)$$

$$\text{Consider} \quad \int_0^\delta v \frac{\partial \theta}{\partial y} \, dy = v \, \theta \Big|_{y=0}^{y=\delta} - \int_0^\delta \theta \frac{\partial v}{\partial y} \, dy$$

$$= v\theta \Big|_{y=\delta} - v\theta \Big|_{y=0} - \int_0^{\delta} \theta \frac{\partial v}{\partial y}\, dy$$

$$= + \int_0^{\delta} \theta \frac{\partial u}{\partial x}\, dy \qquad \text{by (1)}$$

and

$$a \int_0^{\delta} \frac{\partial^2 \theta}{\partial y^2}\, dy = a \frac{\partial \theta}{\partial y} \Big|_{u=0}^{y=\delta}$$

$$= a \frac{\partial \theta}{\partial y}\Big|_{y=\delta} - a \frac{\partial \theta}{\partial y}\Big|_{y=0}$$

$$= - a \left(\frac{\partial \theta}{\partial y} \right)_{y=0}$$

Then equation (7) will take the following form

$$\int_0^{\delta} u \frac{\partial \theta}{\partial x}\, dy + \int_0^{\delta} \theta \frac{\partial u}{\partial x}\, dy = -a \left(\frac{\partial \theta}{\partial y} \right)_{y=0}$$

i.e. $$\frac{d}{dx} \int_0^{\delta} (u\,\theta)\, dy = - a \left(\frac{\partial \theta}{\partial y} \right)_{y=0} \qquad \ldots (8)$$

Solving the above integral equations by taking the following polynomials in $\eta = y/\delta$ for the distributions of u and θ, satisfying the respective boundary conditions

$$u = u_1(x)\eta\,(1-\eta)^4$$

and $$\theta = (1-\eta)^4 \qquad \ldots (9)$$

where $u_1(x)$ is an arbitrary function has the dimension of velocity to be determined.

$$(0.6669)\ \frac{d}{dx}\ (u_1^2\ \delta) = \frac{1}{5}\ g\alpha\delta - \nu\ \frac{u_1}{\delta} \qquad \ldots (10)$$

and equation (8) will reduce to

$$(1.0131)\ \frac{d}{dx}\ (u_1\delta) = \frac{4a}{\delta} \qquad \ldots (11)$$

Let us try to find the solutions of the above equations in the forms of

$$u_1 = C_1\ x^m \text{ and}$$

$$\delta = C_2\ x^n \qquad \ldots (12)$$

Then from equation (12), equations (10) and (11) will take the form

$$(2m + n)\ (0.6669)\ C_1^2\ C_2\ x^{2m+n-1}$$

$$= \frac{1}{5}\ g\alpha\ C_2\ x^n - \frac{C_1}{C_2}\ \nu\ \ x^{m-n} \qquad \ldots (13)$$

$$(m+n)\ (1.0131)\ C_1C_2\ x^{m+n-1} = \frac{4a}{C_2}\ x^{-n} \qquad \ldots (14)$$

must be identically satisfied.

This gives

$$2m + n - 1 = n = m - n\ ;\ m + n - 1 = -n$$

$$m = 1/2 \text{ and } n = 1/4$$

Then we have

$$C_1 = (1.0261)\ \nu\ (Pr + 4.3886)^{-\frac{1}{2}}\ \left(\frac{g\alpha}{\nu^2} \right)^{\frac{1}{2}} \qquad \ldots (15)$$

$$C_2 = (2.2651)\ Pr^{-\frac{1}{2}}\ (Pr + 4.3886)^{\frac{1}{4}}\ \left(\frac{g\,\alpha}{\nu^2}\right)^{-\frac{1}{4}}$$

From equations (12) and (15), we have

$$\frac{\delta}{x} = (2.261)\ Pr^{-1/2}\ (Pr + 4.38886)^{1/4}\ \left(\frac{g\,\alpha}{\nu}\right)^{-1/4} X\ x^{-1/4}$$

$$\frac{\delta}{x} = (2.261)\ Pr^{-1/2}\ (Pr + 4.3886)^{1/4}\ (Gr)^{-1/4}$$

where

$$Gr = \frac{g\ (T_w - T_\infty)}{\nu^2\ T_\infty} = \frac{g\ \alpha\ x^3}{\nu^2}$$

The temperature gradient at the wall is given by

$$\left(\frac{\partial\ \theta}{\partial\ \eta}\right)_{\eta=0} = -\ 2$$

The local Nusselt number for the heat transfer in the present case is given by

$$Nu(x) = \frac{-\left(\frac{\partial\ T}{\partial\ Y}\right)_{y=0} \cdot x}{(T_w - T_\infty)}$$

$$Nu(x) = -\left(\frac{\partial\ \theta}{\partial\ \eta}\right)_{\eta=0} \cdot \frac{x}{\delta}$$

$$= 2.0000 \text{ X } 0.4415 \text{ X } (0.733)^{1/2}\ (0.6647)$$

$$Nu(x) = (0.502)\ (Gr)^{1/4}\ ,\quad Pr = 0.733\ .$$

2. Approximate solution of Karman's Pohlhausen's Method :

Karman's Momentum Integral Equation :

The boundary conditions for a steady two dimensional compressible flow are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \qquad \ldots (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots (2)$$

The boundary conditions are

$y = 0 : u = v = 0 : y = \delta\ (x) : u = U\ (x)$

where $\delta(x)$ is the boundary layer thickness.

The modification of boundary condition in (3) from the usual condition $y = \infty : u = U(x)$ should be noted. In fact there is no edge to the boundary layer and

$u \text{ --- } U$ and $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial y^2}$ etc. tends to zero asymptotically.

However the integral methods, it is often assumed that the boundary layer has a finite thickness and the outer boundary conditions are modified so that some point

$y = \delta(x) : u = U(x)$ and

$\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial y^2}$ etc. are zero.

Integrating the momentum equation (1) with respect to y between y = 0 to y = δ(x) we get

$$\int_0^\delta u \frac{\partial u}{\partial x} dy + \int_0^\delta v \frac{\partial u}{\partial y} dy = \int_0^\delta U \frac{dU}{dx} dy + \nu \int_0^\delta \left(\frac{\partial^2 u}{\partial y^2} \right) dy$$

$$= \int_0^\delta U \frac{du}{dx} dy - \nu \left(\frac{\partial u}{\partial y} \right)_{y=0} \qquad \ldots (4)$$

Integrating by parts the second integral in the L.H.S. of equation (4) we find

$$\int_0^\delta v \frac{\partial u}{\partial y} dy = vu \Big|_{y=0}^{y=\delta} - \int_0^\delta u \frac{\partial v}{\partial y} dy$$

$$= (v)_{y=\delta} \cdot U + \int_0^\delta u \frac{\partial u}{\partial x} dy \qquad \ldots (5)$$

where the boundary condition (3) and the equation of continuity has been used. It may be noted that at the outer edge of the boundary layer the velocity component v is not equal to zero and in order to substitute its value in terms of u, we integrate the equation of continuity also with respect to y between y = 0 and y = δ(x) which gives

$$(v)_{y=\delta} = - \int_0^{\delta} \frac{\partial u}{\partial x} dy \qquad \ldots (6)$$

From equations (5) and (6)

$$\int_0^{\delta} v \frac{\partial u}{\partial y} dy = - U \int_0^{\delta} \frac{\partial u}{\partial x} dy + \int_0^{\delta} u \frac{\partial u}{\partial x} dy \ldots (7)$$

putting (7) in (4), we obtain

$$\int_0^{\delta} \frac{\partial}{\partial x} (u^2 dy) - U \int_0^{\delta} \frac{\partial u}{\partial x} dy = \int_0^{\delta} U \frac{dU}{dx} dy - \nu \left(\frac{\partial u}{\partial y}\right)_{y=0} \ldots (8)$$

But

$$\int_0^{\delta} \frac{\partial}{\partial x} (u^2) dy = \frac{d}{dx} \int_0^{\delta} u^2 dy - U^2 \frac{d\delta}{dx} \qquad \ldots (9)$$

and

$$U \int_0^{\delta} \frac{\partial u}{\partial x} dy = U \frac{d}{dx} \int_0^{\delta} u \, dy - U^2 \frac{d\delta}{dx} \qquad \ldots (10)$$

Hence equation (8) reduces to

$$\frac{d}{dx} \int_0^{\delta} u^2 dy - U \frac{d}{dx} \int_0^{\delta} u dy = \int_0^{\delta} U \frac{dU}{dx} dy - \nu \left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad \ldots (11)$$

$$\text{i.e.} \quad \frac{d}{dx} \int_0^\delta u(U - u)\,dy + \frac{dU}{dx} \int_0^\delta (U-u)\,dy = \nu \left(\frac{\partial u}{\partial y} \right)_{y=0} \qquad \dots (12)$$

Introducing the following quantities,

(i) $$\delta_1 = \int_0^\delta \left(1 - \frac{u}{U} \right) dy \quad \text{(displacement thickness)}$$

$$\text{or} \quad U\delta_1 = \int_0^\delta (U - u)\,dy \qquad \dots (13)$$

(ii) $$\delta_2 = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy \quad \text{(momentum thickness)}$$

$$\text{or} \quad U^2\delta_2 = \int_0^\delta u\,(U - u)\,dy \qquad \dots (14)$$

(iii) and $$\tau_0 = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} \quad \text{(shearing stress on the wall)} \qquad \dots (15)$$

Equation (12) can be written as

$$\frac{d}{dx} \left(U^2\delta_2 \right) + U \frac{dU}{dx} \delta_1 = \frac{\tau_0}{\rho}$$

$$\text{or} \quad U^2 \frac{d\delta_2}{dx} + (2\delta_2 + \delta_1)\, U \frac{dU}{dx} = \frac{\tau_0}{\rho} \qquad \dots (16)$$

which is known as the Ka'rma'ns momentum integral equation for the two-dimensional steady incompressible boundary layers.

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A.

It may be noted that the Ka'rm'an momentum integral equation (16) is an ordinary differential equation for the boundary layer thickness $\delta(x)$, provided a suitable form of the velocity profile is assumed, K. Pohlhausen, on the suggestion of Ka'rm'an assumed a polynomial of fourth degree for the velocity profile and worked out the solution.

In this problem we studied a polynomial of sixth degree for the velocity profile and worked out the solution.

Karm'an-Pohlhausen Method :

The "similar solutions" of the boundary layer thickness, reveal that the velocity distribution in the boundary layer is some function of the radio y/δ which we approximated by a polynomial of sixth degree velocity profile in y/δ as follows :

$$\frac{u}{U} = f(\eta) = \sum_{i=0}^{6} a_i \eta^i \quad : \quad 0 \leq \eta \leq 1$$

$$\frac{u}{U} = 1 \quad \text{for } \eta > 1 \qquad \dots (1)$$

where $\eta = y/\delta$ ... (2)

In order to determine the coefficients a_0, a_1, a_2, a_3, a_4, a_5, a_6 the following boundary and compatibility conditions are used.

$$y = 0 : u = 0; \quad \nu \left(\frac{\partial^2 u}{\partial y^2} \right)_{y=0} = - U \frac{dU}{dx}, \quad \left(\frac{\partial^3 u}{\partial y^3} \right)_{y=0} = 0 \,..$$

... (3)

$$y = \delta : u = U, \quad \left(\frac{\partial u}{\partial y} \right)_{y=0} = \left(\frac{\partial^2 u}{\partial y^2} \right)_{y=0} = \left(\frac{\partial^3 u}{\partial y^3} \right) = 0$$

The first condition at $y = 0$ is the usual no slip boundary condition whereas the second, known as the compatibility condition, at the surface, follows from the boundary layer equation. It may be pointed out that compatibility condition is always satisfied by all exact solutions.

The conditions at $y = \delta$ follows from the consideration that the outer edge of the boundary layer the velocity u in the boundary layer passes smoothly to the potential velocity U. The coefficients a_0, a_1, a_2, a_3, a_4, a_5, a_6 are obtained by the above conditions (3), They are given by

$$a_0 = 0, \quad a_1 = \frac{2\Lambda + 20}{10}, \quad a_2 = -\frac{\Lambda}{2} \qquad \ldots (4)$$

$$a_3 = 0, \quad a_4 = \Lambda - 5, \quad a_6 = \frac{3\Lambda - 20}{10}$$

$$\text{where } \Lambda = \frac{\delta^2}{\nu} \frac{dU}{dx} \quad \text{(Shape factor)} \qquad \ldots (5)$$

Hence the sixth degree velocity profile which satisfies the boundary condition (3) are

$$\frac{u}{U} = f(\eta) = F(\eta) + \Lambda\, G(\eta) \qquad \ldots (6)$$

where

$$F(\eta) = 2\eta - 5\eta^4 + 6\eta^5 - 2\eta^6 \qquad \ldots (7)$$

$$G(\eta) = \frac{1}{5}\,\eta - \frac{\eta^2}{2} + \eta^4 - \frac{\eta^5}{5} + \frac{3}{10}\,\eta^6$$

The velocity profile (6) is a one parameter family of curves with parameter Λ , known as the shape factor because the shape of velocity profile, plotted against η depend on the value of ? .

Range of Λ :

The range of Λ for which the above distribution is used, will be determined by the following consideration.

The value of $\Lambda = 0$ i.e. when $\frac{dU}{dx} = 0$ corresponds to the profile in the boundary layer on a flat plate. The profile at separation point with $\left(\frac{\partial u}{\partial y}\right)_0 = 0$, with $a_1 = 0$ occurs at $\Lambda = -12$. It will be seen later that the profile at the stagnation point corresponds to $\Lambda = 11.76$. The value of greater than 12 must be excluded, since for such value of u would exceed U within the boundary layer, which is physically unreasonable. For values of Λ less than -12, which corresponds to the conditions behind the point of separation, the calculations based, as it is on the boundary layer concept loses significance. Therefore the shape factor ? is limitted to the range

$$-12 \leq \Lambda \leq 12 \quad \ldots (8)$$

We are now, in a position to calculate the value of $\delta(x)$ from the Ka'rm'an momentum integral equation with the sixth degree velocity profile.

For this we calculate δ_1, δ_2 and τ_o

(i) $$\delta_1 = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

$$= \delta \int_0^{1} \left(1 - \frac{u}{U}\right) dy$$

$$\delta_1 = \delta \left\{ \left(\frac{2}{7}\right) - \left(\frac{156}{4260}\right) \Lambda \right\} \quad \ldots (9)$$

(ii) $$\delta_2 = \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

$$= \delta \int_0^{1} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

$$\delta_2 = \delta \left\{ \left(\frac{10}{49}\right) - \left(\frac{468}{29820}\right) \Lambda - \left(\frac{24336}{18147600}\right) \Lambda^2 \right\} \quad \ldots (10)$$

and

(iii) $$\tau_o = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U}{\delta} \left(\frac{\partial f}{\partial \eta}\right)_{\eta=0}$$

$$\tau_o = \frac{\mu U}{\delta} \left(\frac{2\Lambda + 20}{10}\right) \quad \ldots (11)$$

It was discovered by Holstein and Bohlen that the momentum integral equation may be solved easily if δ_2 instead of δ_1 is regarded as the unknown function.

For this we wrote the equation as

$$\frac{U\delta_2}{\nu}\frac{d\delta_2}{dx} + \left(2 + \frac{\delta_1}{\delta_2}\right)\frac{\delta_2^2}{\nu}\frac{dU}{dx} = \frac{\tau_o\,\delta_2}{\mu\,U} \quad \ldots (12)$$

Equation (12) may be simplified as, if we introduce the following parameters

$$\lambda = \frac{\delta_2^2}{\nu} \cdot \frac{dU}{dx} = \frac{\delta_2^2}{\delta^2} \cdot \frac{\delta^2}{\nu} \cdot \frac{dU}{dx}$$

$$\lambda = \left[\left(\frac{10}{49}\right) - \left(\frac{468}{29820}\right)\Lambda - \left(\frac{24336}{18147600}\right)\Lambda^2\right]^2 \quad \ldots (13)$$

$$H(\lambda) = \delta_1 / \delta_2$$

$$= \frac{\left[\left(\frac{2}{7}\right) - \left(\frac{156}{4260}\right)\Lambda\right]}{\left[\left(\frac{10}{49}\right) - \left(\frac{468}{29820}\right)\Lambda - \left(\frac{24336}{18147600}\right)\Lambda^2\right]} \quad \ldots (14)$$

and $\quad I(\lambda) = \dfrac{\tau_o\delta_2}{\mu\,U}$

$$= \left(\frac{2\Lambda + 20}{10}\right)\left[\left(\frac{10}{49}\right) - \left(\frac{468}{29820}\right)\Lambda - \left(\frac{24336}{18147600}\right)\Lambda^2\right] \ldots (15)$$

Thus equation (12) reduces to

$$\frac{d}{dx}\left(\frac{\delta_2^2}{\nu}\right) = \frac{L(\lambda)}{U}, \qquad \lambda = \frac{\delta_2^2 U}{\nu} \qquad \ldots (16)$$

where

$$L(\lambda) = 2\left[I - \lambda(H + 2)\right] \qquad \ldots (17)$$

is a Universal function.

Equation (16) is ordinary nonlinear differential equation of first order and may be integrated numerically.

Since $\frac{d}{dx}\left(\frac{\delta_2^2}{\nu}\right)$ can not be infinite at x = 0, L(?) must be equal to zero at the stagnation point.

This gives

$$I - \lambda(H + 2) = 0 \qquad \ldots (18)$$

Substituting the values of λ, H and I from (13), (14) and (15) respectively in equation (18) and on simplification we find

$$\left[0.0000016\,\Lambda^4 + 0.000039\,\Lambda^3 - 0.0004\,\Lambda^2 - 0.062\,\Lambda\right] + 0.0417\left[0.0011\,\Lambda^2 - 2.1837\,\Lambda + 11.918\right] = 0. \qquad \ldots (19)$$

gives the roots as

= 343.72, 288.51, 42.44, 11.76, two roots are imaginary.

But due to limitations on the range of Λ , i.e. -12 ? 12 only. One root 11.76 is permissible. The initial value of Λ is

$$\Lambda = 11.7600 \qquad \dots (20)$$

Therefore the corresponding initial equation (10) values of from equation (13) is given by

$$\lambda = 0.1026 \qquad \dots (21)$$

Hence

$$\left(\frac{\delta_2^2}{\nu} \right)_{x=0} = \frac{0.1026}{\left(\frac{dU}{dx} \right)^{o}} \qquad \dots (22)$$

and a simple process by taking limit

$$\frac{d}{dx} \left(\frac{\delta_2^2}{\nu} \right)_{x=0} = -0.685 \left(\frac{U^{11}}{U^{12}} \right)_0 \qquad \dots (23)$$

where the prime denotes the differentiation with respect to x.

Application of Ka'rm'an Pohlhausen Method :

(a) **Boundary layer over a flat plate :**

In this case U(x) = constant therefore $\frac{dU}{dx} = 0$

Hence, $\Lambda = 0$, and $\lambda = 0$ and the equation (16) will take the following form

$$\frac{d}{dx}\left(\frac{\delta_2^2}{\nu}\right) = \frac{L(0)}{U} = \frac{0.8082}{U} \qquad \ldots (24)$$

The solution of equation (24) with the initial value $\delta_2 = 0$ at $x = 0$ gives the momentum thickness

$$\delta_2 = 0.9035 \sqrt{\frac{\nu x}{U}} \qquad \ldots (25)$$

and therefore by using equation (14) and equation (25), the displacement thickness δ_1 is given by

$$\delta_1 = 1.3308 \sqrt{\frac{\nu x}{U}} \qquad \ldots (26)$$

and the shearing stress τ_o, by using equation (15) and equation (25) is calculated as

$$\tau_o = \mu \sqrt{\frac{U}{\nu x}} \; (0.562) \qquad \ldots (27)$$

(1) Two-dimensional stagnation point flow :

In the case of two-dimensional stagnation point flow the potential flow velocity is given by

$$U(x) = a.x$$

therefore, $U'(x) = a$ and $U''(x) = 0$

Hence, from the equation (16) and equation (22) we have

$$\frac{\delta_2^2}{\nu} = \frac{0.1026}{a}$$

It may be noted that it is not only true at $x = 0$ but is true for all values of x, since the equation (22) gives a zero increment in its initial value. Also the value of remains the same as 11.76.

Thus the momentum thickness δ_2 is obtained as

$$\delta_2 = 0.320 \sqrt{\nu/a} \qquad \ldots (28)$$

and the displacement thickness δ_1 is obtained as from equations (28) and (14)

$$\delta_1 = 0.471 \sqrt{a/\nu} \qquad \ldots (29)$$

and the shearing stress τ_o on the wall from equations (15) and (28) is obtained as

$$\tau_o = 1.3089 \, \mu U \sqrt{a/\nu} \qquad \ldots (30)$$

3. Suction velocity at the point of separation for a boundary layer flow over a flat porous plate :

In the general case of arbitary body and the arbitrary law of suction we shall give the approximate methods for the momentum integral equation. The displacement thickness,

$$\delta^* = \int_0^\infty \frac{1}{U} (U - u)\, dy$$

The momentum thickness,

$$\theta = \int_0^\infty \frac{u}{U^2} (U - u)\, dy$$

and the shearing stress,

$$\frac{\tau_o}{\rho} = \frac{d}{dx} (U^2 \theta) + \delta^* U \frac{dU}{dx}$$

Then the equation of normal component of velocity at a distance y = h is given by

$$v_h = v_0 - \int_0^h \frac{\partial u}{\partial x}\, ds \qquad \ldots (1)$$

The momentum integral equation is given by

$$U^2 \frac{d\theta}{dx} + (2\theta + \delta^*) U \frac{dU}{dx} - v_o U = \frac{\tau_o}{\rho} \qquad \ldots (2)$$

where the addition term $(- v_o U)$ denotes the change in momentum with the suction at the wall. This equation was used by L. Prandtl, for the simple estimate of suction velocity which makes sufficiently to prevent separation. It was assumed that $\Lambda = -12$ by L. Pohlhausen.

Now

$$\frac{u}{U} = F(\eta) + \Lambda G(\eta)$$

where $F(\eta) = 2\eta - 5\eta^4 + 6\eta^5 - 2\eta^6$

$$G(\eta) = \frac{1}{5}\eta - \frac{\eta^2}{2} + \eta^4 - \frac{\eta^5}{5} + \frac{3}{10}\eta^6$$

with $\Lambda = -12$

then

$$\frac{u}{U} = F(\eta) + \Lambda G(\eta)$$

$$= \left(-\frac{2}{5}\eta - 17\eta^4 + \frac{42}{5}\eta^5 + \frac{8}{3}\eta^6\right)$$

$$\therefore \quad u = U\left[\left(\frac{-2}{5}\right)\left(\frac{y}{\delta}\right) - (17)\left(\frac{y}{\delta}\right)^4 + \left(\frac{42}{5}\right)\left(\frac{y}{\delta}\right)^5 + \right.$$

$$\left. + \left(\frac{8}{3}\right)\left(\frac{y}{\delta}\right)^6\right]$$

Now we take from previous problem,

$$\frac{\delta^*}{\delta} = \left(\frac{2}{7} - \frac{156}{4260}\Lambda\right)$$

and

$$\frac{\theta}{\delta} = \left(\frac{10}{49} - \frac{468}{29820}\Lambda - \frac{24336}{18147600}\Lambda^2\right)$$

Now for $\Lambda = -12$

$$\delta^* = \delta(0.7249)$$

and $\theta = \delta\,(0.2281)$

$$\delta^* + 2\,\theta = (0.7249)\delta + 2(0.2281)\,\delta$$

$$= \delta\,(0.7249 + 0.4562)$$

$$\delta^* + 2\theta = (1.1801)\,\delta$$

substituting this value in equation (2) and taking $\frac{d\theta}{dx} = 0$, because of the assumption of constant boundary layer thickness.

$$U^2 \frac{d\theta}{dx} + (2\theta + \delta^*)\,U\,\frac{dU}{dx} - v_o\,U = \frac{\tau_o}{\rho}$$

and $\frac{\tau_o}{\rho}$ = constant

On simplification it gives,

$$v_o = (1.1801)\,\delta\,\frac{dU}{dx} \qquad \ldots (3)$$

Now we shall try to calculate the value of δ, the momentum equation is necessary to satisfy the equation of motion at the wall is required by E. Pohlhausen

$$v_o\left(\frac{\partial u}{\partial y}\right)_{y=0} = U\,\frac{dU}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$$

under the consideration

$$\left(\frac{\partial u}{\partial y} \right)_{y=0} = 0, \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \frac{12U}{\delta^2}$$

Then above equation take the following form

$$\delta = \sqrt{\frac{12 \nu}{\left(- \frac{dU}{dx} \right)}} \qquad \ldots (4)$$

Substituting equation (4) in equation (3) we obtain

$$v_o = (1.1801) \left(\sqrt{\frac{12 \nu}{\left(- \frac{dU}{dx} \right)}} \right) \cdot \frac{dU}{dx}$$

$$\therefore \quad v_o = - 3.71 \sqrt{- \nu \left(\frac{dU}{dx} \right)} \qquad \ldots (5)$$

which is the required suction velocity.

4. Approximate solution of Pohlhausen's problem of forced convection on a heated vertical plate.

Notations :

$a = \frac{k}{eC_p}$: thermal diffusivity

K : Coefficient of thermal conductivity

C_p : Specific heat at constant pressure

a_i : Coefficient of η^i in velocity profile

C_i : Coefficient of η_t^i in velocity profile

$\gamma = \frac{T_a - T_\infty}{U_\infty^2 / 2C_p}$, recovery factor

$Pr = \frac{\mu C_p}{K}$, Prandtl Number

$Re = \frac{U_\infty x}{\nu}$, Reynold Number

T : Temperature

T_a : a diabatic wall temperature

$$Nu(x) = \frac{-\left(\frac{\partial T}{\partial y}\right)_{y=0} \cdot x}{(T_w - T_\infty)}, \quad \text{Nusselt Number}$$

u, v : velocity components in x and y directions

U_∞ : Free stream velocity in x direction.

x, y : co-ordinates along and normal to the wall

Greek Symbols :

δ : velocity boundary layer thickness

δ_t : thermal boundary layer thickness

$\Delta = \frac{\delta_t}{\Delta}$, ratio of the thickness of the temperature and velocity boundary layer.

δ_1 : displacement thickness

δ_2 : momentum thickness

$_o = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$, shearing stress at the wall

μ : dynamic viscosity of the fluid

ρ : density of the fluid

$\nu = \frac{\mu}{\rho}$, kinematic viscosity of the fluid

$\eta = \frac{y}{\delta}$, $\eta_t = \frac{y}{\delta_t}$

$\theta_1 = \frac{T - T_\infty}{T_w - T_\infty}$, dimensionless temperature in the case of cooling problem

$\theta_2 = \frac{T - T_\infty}{T_w - T_\infty}$, dimensionless temperature in the case of adiabatic wall.

W : conditions at the wall

∞ : conditions at the outer edge of the boundary layer.

Thermal Energy Integral Equation :

The method of obtaining the thermal energy integral equation from the thermal boundary layer equation is similar to the momentum integral equation from the velocity boundary layer equation.

In the present case

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \left(\frac{\partial u}{\partial y} \right)^2 \quad \ldots (1)$$

with boundary conditions

$$y = \delta t, \quad \frac{\partial T}{\partial y} = 0$$

$$y = 0 , \quad T = T_\infty \quad \ldots (2)$$

Integrating equation (1) with respect to y between the limits at $y = 0$ to $y = \delta_t$ we get

$$\int_0^{\delta t} u \frac{\partial T}{\partial x} dy + \int_0^{\delta t} v \frac{\partial T}{\partial y} dy = \int_0^{\delta t} a \frac{\partial^2 T}{\partial y^2} dy + \frac{\mu}{e c_p} \int_0^{\delta t} \left(\frac{\partial u}{\partial y}\right)^2 dy, \quad \ldots (3)$$

Consider

$$\int_0^{\delta_t} a \frac{\partial^2 T}{\partial y^2} dy = a \frac{\partial T}{\partial y} \Big|_{y=0}^{y=\delta_t}$$

$$= a \frac{\partial T}{\partial y}_{y=\delta_t} - a \frac{\partial T}{\partial y}_{y=0}$$

$$= - a \left(\frac{\partial T}{\partial y} \right)_{y=0}$$

and consider

$$\int_0^{\delta_t} v \frac{\partial T}{\partial y} dy = v \int_0^{\delta_t} \frac{\partial T}{\partial y} dy - \int_0^{\delta_t} T \frac{\partial v}{\partial y} dy$$

$$= vT \Big|_{y=0}^{y=\delta_t} - \int_0^{\delta_t} T \frac{\partial v}{\partial y} dy$$

$$= - vT_{y=0} + \int_0^{\delta_t} T \frac{\partial u}{\partial x} dy$$ by equation of continuity

$$= - vT_\infty + \int_0^{\delta_t} T \frac{\partial u}{\partial x} dy$$

$$= - T_\infty \int_0^{\delta_t} \frac{\partial u}{\partial x} dy + T \int_0^{\delta_t} \frac{\partial u}{\partial x} dy$$

? $$\int_0^{\delta_t} v \frac{\partial T}{\partial y} dy = (T - T_\infty) \int_0^{\delta_t} \frac{\partial u}{\partial x} dy$$

Combining all the results in equation (3) we obtain

$$\int_0^{\delta_t} u \frac{\partial T}{\partial x} dy + \int_0^{\delta t} (T - T_\infty) \frac{\partial u}{\partial x} dy = - a \left(\frac{\partial T}{\partial y} \right)_{y=0} + \frac{\mu}{\rho C_p} \int_0^{\delta_t} \left(\frac{\partial u}{\partial y} \right)^2 dy$$

i.e.

$$\frac{d}{dx} \int_0^{\delta_t} u \, (T-T_\infty) dy = -a \left(\frac{\partial T}{\partial y} \right)_{y=0} + \frac{\mu}{\rho C_p} \int_0^{\delta_t} \left(\frac{\partial u}{\partial y} \right)^2 dy \qquad \ldots (4)$$

This is the required thermal energy integral equation taking frictional heat in to account. It is also known as heat flux equation. An approximate solution of the thermal boundary layer based on thermal energy integral equation, has been studied by a number of authors including Krojuline [15] Dinemann [16] Bansal [17] . From among the numerous procedures which are available for the solution of thermal energy integral equation.

We propose to study in detail a method based on tenth degree velocity profile because it is an extension of Pohlhausen's method and gives result more near to known exact solution. Taking Pohlhausen's tenth degree velocity profile, for boundary layer flow over a flat plate,

$$\frac{u}{U_\infty} = f(\eta) = \sum_{i=0}^{10} a_i \, \eta^i \qquad 0 \leq \eta \leq 1 \qquad \ldots (1)$$

$$u = U_\infty , \qquad \eta > 1$$

where $\eta = y/\delta$

with the following boundary and compatibility conditions.

$$y = 0,\ u = 0,\ \frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 u}{\partial y^3} = 0 = \frac{\partial^4 u}{\partial y^4} = \frac{\partial^5 u}{\partial y^5} \quad \ldots (2)$$

$$y = \delta,\ u = U_\infty,\ \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 u}{\partial y^3} = \frac{\partial^4 u}{\partial y^4} = \frac{\partial^5 u}{\partial y^5} = 0$$

Then using these boundary and compatibility conditions we obtain eleven constants as,

$$a_0 = 0, \quad a_1 = 1.0412, \quad a_2 = 0$$

$$a_3 = 0, \quad a_4 = 0, \quad a_5 = 0 \quad \ldots (3)$$

$$a_6 = -\ 1.2624 \quad a_7 = -1.2290 \quad a_8 = 1.8146$$

$$a_9 = 1.0376 \quad a_{10} = -\ 0.002$$

Now we calculate $F(\eta)$ by using above constants

$$\frac{u}{U_\infty} = F(\eta) = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 + a_5\eta^5 + a_6\eta^6 + $$
$$+ a_7\eta^7 + a_8\eta^8 + a_9\eta^9 + a_{10}\eta^{10}$$

$$F(\eta) = 0 + (1.0412)\eta + 0 + 0 + 0 + 0 - (1.2624)\eta^6 -$$
$$- (1.2290)\eta^7 + (1.8146)\eta^8 + (1.0376)\eta^9 - (0.002)\eta^{10}.$$

Now we determine the displacement thickness, momentum thickness and shearing stress at the wall .

Now we determine the displacement thickness, momentum thickness and shearing stress at the wall

(i) Displacement thickness (δ_1) :

$$\delta_1 = \delta \int_0^1 \left(1 - \frac{u}{U_\infty} \right) d\eta$$

$$= \delta \int_0^1 \left(1 - (1.0412)\eta - (1.2624)\eta^6 - (1.2290)\eta^7 + (1.8146)\eta^8 + (1.0376)\eta^9 - (0.002)\eta^{10} \right) \times d\eta$$

$$= \delta \left(1 - 0.5206 + 0.1803 + 0.1536 - 0.2016 - 0.1037 + 0.0001 \right)$$

$$\delta_1 = \delta\,(0.3278) \qquad \ldots (4)$$

(ii) Momentum thickness (δ_2) :

$$\delta_2 = \delta \int_0^1 \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty} \right) d\eta$$

$$\delta_2 = \delta \int_0^1 \left((1.0412)\eta - (1.2624)\eta^6 - (1.2290)\eta^7 + (1.8146)\eta^8 + (1.0376)\eta^9 - (0.002)\eta^{10} \right) \times$$

$$\left(1 - (1.0412)\eta - (1.2624)\eta^6 - (1.2290)\eta^7 + (1.8146)\eta^8 + (1.0376)\eta^9 - (0.002)\eta^{10} \right) d\eta$$

$$\delta_2 = \delta \int_0^1 \Big((1.0412)\eta - (1.2624)\eta^6 - (1.2290)\eta^7 + (1.8146)\eta^8 + (1.0376)\eta^9 - (0.002)\eta^{10} \Big) \times (0.3278)\, d\eta$$

$$\delta_2 = 1.505$$

(iii) Shearing stress at the wall (τ_o) :

$$\tau_o = \frac{\mu U_\infty}{\delta} \left(\frac{\partial f}{\partial \eta} \right)_{\eta=0}$$

$$\tau_o = \frac{\mu U_\infty}{\delta} \left(\frac{\partial}{\partial \eta} \Big((1.0412)\eta - (1.2624)\eta^6 - (1.2290)\eta^7 + (1.8146)\eta^8 + (1.0376)\eta^9 - (0.002)\eta^{10} \Big) \right)_{\eta=0}$$

$$= \frac{\mu U_\infty}{\delta} \Big((1.0412) - 6(1.2624)\eta^5 - 7(1.2290)\eta^6 + 8(1.8146)\eta^7 + 9(1.0376)\eta^8 - 10(0.002)\eta^9 \Big)_{\eta=0}$$

$$\tau_o = \frac{\mu U_\infty}{\delta} (1.0412) \qquad \ldots (6)$$

(a) <u>Solution of the cooling problem</u> :

Introducing the dimensionless temperature θ_1 as

$$\theta_1 = \frac{T - T_\infty}{T_w - T_\infty} \qquad \ldots (7)$$

the heat flux equation in the present case is given by

$$\frac{d}{dx} \int_0^{\delta_t} (\theta_1 . \frac{u}{U_\infty}) dy = \frac{-a}{U_\infty} . (\frac{\partial \theta_1}{\partial y})_{y=0} \quad \ldots (8)$$

For the temperature distribution we consider the following polynomial in $\eta_t (= y/\delta_t)$

$$\theta_1 = L(\eta_t) = 1 - \sum_{i=0}^{10} a_i \eta_t^i \quad 0 \leq \eta_t \leq 1 \quad \ldots (9)$$

$$\theta_1 = \Theta, \quad \eta_t > 1$$

Satisfying the following boundary and compatibility conditions

$$\eta_t = 0, \ \theta_1 = 1, \ (a \frac{\partial^2 \theta_1}{\partial \eta_t^2}) = (a \frac{\partial^3 \theta_1}{\partial \eta_t^3}) = (a \frac{\partial^4 \theta_1}{\partial \eta_t^4}) =$$

$$= (a \frac{\partial^5 \theta_1}{\partial \eta_t^5}) = 0 \quad \ldots (10)$$

$$\eta_t = 1, \ \theta_1 = 0, \quad \frac{\partial \theta_1}{\partial \eta_t} = \frac{\partial^2 \theta_1}{\partial \eta_t^2} = \frac{\partial^3 \theta_1}{\partial \eta_t^3} = \frac{\partial^4 \theta_1}{\partial \eta_t^4} =$$

$$= \frac{\partial^5 \theta_1}{\partial \eta_t^5} = 0$$

The compatibility conditions are obtained from the thermal boundary layer equation after neglecting the term due to dissipation. On a similar manner the boundary and compatibility conditions of previous results from the velocity boundary layer equation. Moreover in the form of temperature distribution is so selected as to ensure identical velocity and temperature distribution in the case of Pr = 1 (Prandtl number) for the existance of Crocco's first integral.

We have

$$\frac{d}{dx} \left(\delta_t H(\Delta) \right) = \frac{2a}{U_\infty \delta_t} \qquad \ldots (11)$$

where

$$\Delta = \frac{\delta_t (x)}{\delta (x)}$$

$$H(\Delta) = \int_0^1 f(\eta) \, L(\eta_t) \, d\eta_t$$

It may be noted that

$$\eta = \frac{y}{\delta} = \frac{y}{\delta_t} \cdot \frac{\delta_t}{\delta} = \eta_t \cdot \Delta$$

Performing the indicated integration we get

$$\Delta \leq 1 : \; H(\Delta) = {}_0^1 \; f(\eta_t \Delta) \, L(\eta_t) \, d\eta_t$$

$$\Delta \leq 1 : \; H(\Delta) = \int_0^1 \left(\sum_{i=0}^{10} a_i (\eta_t \Delta)^i \right) \left(1 - \sum_{i=0}^{10} a_i \eta_t{}^i\right) d\eta_t$$

$$= \Delta \quad 0.5206 - 0.3613 + 0.1643 + 0.1421 - 0.1889 -$$

$$- 0.0984 - 0.00017$$

$$+ \Delta^6 \quad - 0.1803 + 0.1643 - 0.1225 - 0.1108 + 0.1527 +$$

$$+ 0.018 - 0.00013$$

$$+ \Delta^7 \quad - 0.1537 + 0.1421 + 0.1108 - 0.1006 + 0.1393 +$$

$$+ 0.0750 - 0.00014$$

$+ \Delta^8 \quad 0.2016 - 0.1889 + 0.1527 + 0.1393 - 0.1936 -$

$- 0.1046 + 0.00019$

$+ \Delta^9 \quad 0.1037 - 0.0982 + 0.0818 + 0.0750 - 0.1046 -$

$- 0.0566 + 0.00010$

$+ \Delta^{10} \quad - 0.00018 - 0.00017 - 0.00014 + 0.00019 -$

$- 0.00013 + 0.00010 - 0.00019$

$$H(\Delta) = (0.5563)\,\Delta - (0.0149)\,\Delta^6 + (0.2129)\,\Delta^7 - (0.1829)\Delta^8 + (0.0012)\,\Delta^9 - (0.0007)\,\Delta^{10} \qquad \ldots (12)$$

$$\Delta \geqslant 1 : \quad H(\Delta) = \frac{1}{\Delta}\, f(\eta).\, L\,(\eta/\Delta)\, d\eta$$

$$= \frac{1}{\Delta} \int_0^1 f(\eta).\, L(\eta/\Delta)\, d\eta + \frac{1}{\Delta} \int_1^\Delta L(\eta/\Delta)\, d\eta$$

$\Delta \geqslant 1$:

$$\begin{aligned} H(\Delta) = \; & 0.5081 \\ & + \frac{1}{\Delta}\,(-1.0000 + 0.4959) \\ & + \frac{1}{\Delta^2}\,(0.5206 - 0.3279) \\ & + \frac{1}{\Delta^7}\,(-0.1803 + 0.1664) \\ & + \frac{1}{\Delta^8}\,(0.1536 - 0.1449) \end{aligned}$$

$$+ \frac{1}{\Delta^9} \left(- 0.2016 + 0.0904\right)$$

$$+ \frac{1}{\Delta^{10}} \left(0.1037 - 0.1025\right)$$

$$+ \frac{1}{\Delta^{11}} \left(- 0.0010 + 0.0007\right)$$

$$H(\Delta) = (0.5081) - (0.5041) \frac{1}{\Delta} + (0.1967) \frac{1}{\Delta^2}$$

$$- (0.0138) \frac{1}{\Delta^7} + (0.0087) \frac{1}{\Delta^8} - (0.1712) \frac{1}{\Delta^9} +$$

$$+ (0.0012) \frac{1}{\Delta^{10}} - (0.0003) \frac{1}{\Delta^{11}} . \qquad \ldots (13)$$

Now Ka'rm'ans momentum integral equation for reduced form is

$$\frac{d}{dx} \left(\frac{\delta_2^2}{\vartheta} \right) = \frac{L(\lambda)}{U_\infty} \qquad \ldots (14)$$

$$L(\lambda) = 2 \left(I(\lambda) - \lambda (2 + H(\lambda))\right)$$

$$= \frac{\delta_2^2}{\nu} \cdot \frac{dU}{dx} , \quad H(\lambda) = \frac{\delta_1}{\delta_2}$$

$$I(\lambda) = \frac{\tau_0 \, \delta_2}{\mu \, U_\infty}$$

In the case of the boundary layer over a flat plate, we have

$$U(x) = \text{constant or } \frac{dU}{dx} = 0$$

and hence $\lambda = 0$

Then the reduced form of Ka'rm'ans momentum integral equation take form

$$\frac{d}{dx}\left(\frac{\delta_2^2}{\nu}\right) = \frac{2\,I(\lambda)}{U_\infty} = \frac{2\,\tau_o\,\delta_2}{U_\infty^2\,\mu}$$

$$= \frac{2(1.0142)\,(1.505)}{U_\infty} \qquad \ldots (15)$$

On integrating with respect to x we get

$$\frac{\delta_2^2}{\nu} = \frac{2\,(1.0142)\,(1.505\,)}{U_\infty}\,.\,x$$

$$\delta^2 = \frac{2(1.0142)}{1.505}\;\frac{\nu\,x}{U_\infty} \qquad \ldots (16)$$

Now from equation (1)()

$$\frac{d}{dx}\left(\delta_t\,H(\Delta)\right) = \frac{1.0142}{U_\infty\delta_t}\,.a$$

$$\Delta = \frac{\delta_t\,(x)}{\delta(x)}$$

$$\delta_t^2\,H^2\,(\Delta) = \frac{2(1.0412\,)}{U_\infty}\int_0^x H(\Delta)\,dx$$

$$\Delta^2\;H^2(\Delta) = 1.505\;\frac{1}{x}\;\frac{1}{Pr}\int_0^x H(\Delta)\,dx$$

$$\Delta^2\,H(\Delta) = 1.505\,\frac{1}{Pr} \qquad \ldots (17)$$

(i) For very small Prandtl number (i.e. for very large value of Δ)

$$H(\Delta) = 0.312 \text{ (approximately)}$$

$$\Delta = 1.7903 \, Pr^{-1/2} \quad \text{as } Pr \longrightarrow 0 \qquad \ldots (18)$$

(ii) For very large Prandtl number (i.e. for very small value of Δ)

$$H(\Delta) = 0.161 \, \Delta$$

$$\Delta = 1.6823 \, Pr^{-1/3} \quad \text{as } Pr \longrightarrow \infty \qquad \ldots (19)$$

(iii) For moderate values of Prandtl number

$$\Delta = Pr^{-1/3} \qquad \ldots (20)$$

Constitute a good approximation.

(iv) The temperature gradient at the wall

$$\left(\frac{\partial \theta_1}{\partial \eta_t} \right)_{\eta_t = 0} = -1.0142$$

Therefore the local Nusselt Number

$$Nu(x) = \frac{-\left(\frac{\partial T}{\partial y} \right)_{y=0} \cdot x}{(T_w - T_\infty)}$$

$$= -\left(\frac{\partial \theta_1}{\partial \eta_t} \right)_{\eta_t=0} \cdot \frac{x}{\delta t}$$

$$= 1.0142 \, \frac{1}{\Delta} \left(\frac{1.5050}{2.0284} \right)^{1/2} \sqrt{\frac{U_\infty x}{\nu}}$$

$$Nu(x) = \frac{0.873}{\Delta} R_{ex}^{1/2} \quad \ldots (21)$$

Then results (i), (ii), (iii) take the following forms,

$$Nu(x) = 0.873\, Pr^{1/3} R_{ex}^{1/2} \text{ as} \quad \ldots (22)$$

moderate values of Prandtl number ... (23)

$$Nu(x) = 0.487\, Pr^{1/2} R_{ex}^{1/2} \text{ as } Pr \rightarrow 0 \quad \ldots (24)$$

$$Nu(x) = 0.518\, Pr^{1/3} R_{ex}^{1/2} \text{ as } Pr \rightarrow \infty \quad \ldots$$

Now it can be easily checked that $\Delta = 1$, $Pr = 1$ is a solution of the above equation. Therefore if $Pr = 1$, $\delta_t = \delta$ and $\eta_t = \eta$, then we have

$$\theta_1 = 1 - \sum_{i=0}^{10} a_i \eta^i = 1 - f(\eta),\ Pr = 1 \quad \ldots (25)$$

which is known as Crocco's [18] first integral.

(b) Adiabatic Wall :

$$\text{Introducing, } \theta_2 = \frac{T - T_\infty}{U_\infty^2/2\, C_p} \quad \ldots (26)$$

and keeping in view that for the adiabatic wall $\left(\frac{\partial T}{\partial y}\right)_{y=0} = 0$

the thermal energy integral equation takes the form

$$\frac{d}{dx} \int_0^{\delta t} \left(\theta_2 \frac{u}{U_\infty} \right) dy = \frac{2\nu}{U_\infty} \int_0^{\delta t} \left(\frac{\partial}{\partial t} \left(\frac{u}{U_\infty} \right) \right)^2 dy$$

... (27)

$$\text{Let} \quad \theta_2 = r - \left(\sum_{i=0}^{10} c_i \eta_t^i \right)^2 \quad \ldots (28)$$

where the coefficients C_0 to C_{10} are to be obtained by using the following boundary and compatibility conditions.

$$\eta_t = 0 : \quad \frac{\partial \theta_2}{\partial \eta_t} = 0, \quad \frac{\partial^2 \theta_2}{\partial \eta_t^2} = -8 \, Pr \, \Delta^2, \quad \frac{\partial^3 \theta_2}{\partial \eta_t^3} = \frac{\partial^4 \theta_2}{\partial \eta_t^4} = \frac{\partial^5 \theta_2}{\partial \eta_t^5} = 0 \qquad \ldots (29)$$

$$\eta_t = 1; \quad \theta_2 = 0, \quad \frac{\partial \theta_2}{\partial \eta_t} = \frac{\partial^2 \theta_2}{\partial \eta_t^2} = \frac{\partial^3 \theta_2}{\partial \eta_t^3} = \frac{\partial^4 \theta_2}{\partial \eta_t^4} = \frac{\partial^5 \theta_2}{\partial \eta_t^5} = 0$$

$$\text{and} \quad r = \frac{T_r - T_\infty}{U_\infty^2 / 2\, C_p} \qquad \text{(recovery factor)}$$

The compatibility conditions are obtained from the thermal boundary layer equation for the adiabatic wall, in the usual manner.

The form of θ_2 is so selected as to ensure the Crocco's second integral when Pr = 1,

We find

$$C_0 = 0 = C_2 = C_3 = C_4 = 0$$

$$C_1 = 2 \sqrt{Pr}\, \Delta \qquad C_5 = 182 \sqrt{r} - 557\, C_1 \qquad \ldots (30)$$

$C_6 = -\ 390\ \sqrt{r} + 2010\ C_1 \qquad C_7 = 365\sqrt{r} - 2590\ C_1$

$C_8 = -\ 175\sqrt{r} + 1280\ C_1 \qquad C_9 = 5\sqrt{r} - 45\ C_1$

$C_{10} = 13\sqrt{r} - 98\ C_1$

Putting equations (1) and (28) in (27) we obtain

$$\frac{d}{dx}(\delta_t\, G) = \frac{2\ \nu\ \Delta}{U_\infty \delta_t}\, J \qquad \ldots (31)$$

where $$G = \int_0^\Delta \left(\theta_2 \frac{u}{U_\infty} \right) d\eta_t \qquad \ldots (32)$$

and $$J = \int_0^\Delta \left(\frac{\partial}{\partial\ \eta} \left(\frac{u}{U_\infty} \right)^2 \right) d\eta \qquad \ldots (33)$$

Performing the indicated integration in above equations we find .

$$G = \sum_{i=1}^{10} \quad \sum_{j=1}^{10} \quad \sum_{k=1}^{10} \quad a_i\ \Delta^i \left(\frac{r}{i+1} - \frac{C_j\ C_k}{i+j+k+1} \right)$$

$$= (1.0412)\ \Delta \quad 62.5057\ r + 0.0814/r\ C_1 - 0.8971\ C_1^2$$

$$-\ (1.2624)\ \Delta \quad 20.5152\ r - 347.9584/r\ C_1 - 0.008\ C_1{}^2$$

$$-\ (1.2290)\Delta^7 \quad 11.7550r + 0.07600/rC_1 - 0.0012\ C_1^2$$

$$+\ (1.8146)\ \Delta^8 \quad 8.7584r - 12.8454/r\ C_1 + 10.4933\ G^2$$

$$+\ (1.0376)\ \Delta^9 \quad 7.5547\ r + 0.038/rC_1 + 0.3442\ C_1^2$$

$$- (0.002)\Delta^{10} \quad 7.4044r - 9.5444/rC_1 + 8.5070\, C_1^2 \qquad \ldots (34)$$

and

$$J = \sum_{i=1}^{10} \sum_{j=1}^{10} \sum_{k=1}^{10} \frac{ij}{i + j - 1} \Delta^{i+j-1} a_i a_j$$

$$= 5.7544\, \Delta - 26.15\, \Delta^6 + 34.5075\, \Delta^7 - 9.7455\, \Delta^8 -$$

$$- 0.3453\, \Delta^9 + 87.5544\, \Delta^{10} - 315.5179\, \Delta^{11} +$$

$$+ 330.5053\, \Delta^{12} - 187.5543\, \Delta^{13} + 155.446\, \Delta^{14} -$$

$$- 95.9490\, \Delta^{15} + 25.4757\, \Delta^{16} - 0.9096\, \Delta^{17} +$$

$$+ 0.0097\, \Delta^{18}. \qquad \ldots (35)$$

and

$$J = \frac{544}{317} \qquad \ldots (36)$$

Integrating equation (31) and taking the value of δ from equation (16) we have

$$G\Delta = 4.2035\, J \qquad \ldots (37)$$

Now, it can be easily checked that when $Pr = 1$ and $\Delta = 1$ then $r = 1$ is a solution of the above equation. Therefore, in such a case $C_i = a_i$ and $\eta_t = \eta$. Thus from (28) we have

$$\theta_2 = 1 - \left(\sum_{i=0}^{10} a_i\, \eta^i \right)^2 \qquad \ldots (38)$$

$$\theta_2 = 1 - F^2\,(\eta)$$

which is the Crocco's second integral. Equation (37) indicates that r is a function of Δ and Pr. But as we know from equation (17), Δ is a function of Pr. Therefore, the recovery factor r will be a function of Pr only. Hence, equation (37) which is algebraic equation, can be solved in r for a given value of the Prandtl number Pr, taking the corresponding value of Δ. From (17) it is found that

(i) for moderate values of the Prandtl number the expression

$$r = Pr^{-1/2}$$

Constitute a good approximation to the solution of equation (37).

(ii) for very large Prandtl number (i.e. for very small value of Δ).

$$G\Delta = \Delta\ (1.0412) \quad 62.5057\ r + 0.0814/rC_1 - 0.8971\ C_1^2$$

and $J = 4.0250\ \Delta$ (approximately)

Putting these values of G and J in equation (37) and using the relation (19) we find

$$r = 0.0831\ Pr^{1/3} \text{ for } Pr \longrightarrow \infty$$

which is about 4% higher than the value obtained in Exact Soluti

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