CHAPTER - III

FIXED POINT THEOREMS IN NORMED spaces AND BANACH SPACES

Introduction :

This chapter consists two sections. Section-I deals with the study of some fixed point theorems in normed spaces, and Section-II consists the study of some fixed point theorems in Banach spaces.

Many authors have defined contractive type mappings on a complete metric spaces, which are generalizations of the well-known Banach's contraction principle; and which have the property that each such mapping has a unique fixed point. The fixed point can always be obtained by using Picard iteration scheme, beginning with some initial choice of $x_0 \in X$.

Rhoades [13] has shown that, if a mapping T satisfying certain contraction conditions and a sequence of Mann iterates converges, then it converges to a fixed point of T. He also compared various definations of contractive mappings studied by different authors (Rhoades [14]).

SECTION - I

In this section we have made a generalization of Banach contraction principle and used technique of Youel A.K., Sharma P.L. [15] to prove two fixed point theorems in normed spaces. Theorem first is associated with Mann iteration Scheme and theorem second is extension of a pair of mappings in normed spaces.

Rhoades in 1974 has proved some fixed point theorems, for a mapping satisfying certain contractive condition associated with Mann iteration process. H.K. Pathak [11] in 1988 used this technique and he has proved some fixed point theorems. Following the same procedure for our contraction mapping, we prove two theorems.

Theorem (3.1) :

Let X be a closed, convex subset of a normed space N and T be a continuous mapping on X into itself such that,

$$\|Tx-Ty\| \leq q \max \left\{ \|x-y\|, \frac{\|y-Ty\|[1-\|x-Tx\|]}{1+\|x-y\|} \right.$$

$$\left. \frac{1}{2} \frac{\|x-Ty\|[1-\|x-Tx\|-\|y-Tx\|]}{1+\|x-y\|} \right\} \qquad \ldots (3.1.1)$$

for all $x, y \in X$, $x \neq y$, $0 < q < 1$,

and let for any $x_0 \in X$, sequence $\{x_n\}$ be defined as follows.

$$x_{n+1} = (1-C_n)x_n + C_n Tx_n, \quad (n \geq 0) \qquad \ldots (3.1.2)$$

where C_n satisfyies the conditions,

$$\left.\begin{array}{ll} \text{(i)} & C_0 = 1 \\ \text{(ii)} & 0 < C_n < 1, \quad n > 0 \\ \text{(iii)} & \lim_{n \to \infty} C_n = h > 0 \quad \text{and} \quad h < 1 \end{array}\right] \qquad \ldots (3.1.3)$$

If $\{x_n\}$ converges in X, then it converges to a fixed point of T.

Proof :

Let $z \in X$ such that $\lim_{n \to \infty} x_n = z$.

Now we shall show that z is the fixed point of T. consider,

$$\|z-Tz\| \leq \|z-x_{n+1}\|+\|x_{n+1}-Tz\|$$

using (3.1.2) on r.h.s. we get

$$\|z-Tz\| \leq \|z-x_{n+1}\| + \|(1-C_n)x_n+C_nTx_n-Tz\|$$

i.e. $$\|z-Tz\| \leq \|z-x_{n+1}\|+\|(1-C_n)x_n-(1-C_n)Tz+C_nTx_n-C_nTz\|$$

$$\leq \|z-x_{n+1}\|+(1-C_n)\|x_n-Tz\|+C_n\|Tx_n-Tz\|$$

using (3.1.1) on r.h.s. we get,

$$\|z-Tz\| \leq \|z-x_{n+1}\|+(1-C_n)\|x_n-Tz\|+C_nq \max \{\|x_n-z\|,$$

$$\frac{\|z-Tz\|[1-\|x_n-Tx_n\|]}{1+\|x_n-z\|}, \tfrac{1}{2} \frac{\|x_n-Tz\|[1-\|x_n-Tx_n\|-\|z-Tx_n\|]}{1+\|x_n-z\|}\}$$

Since, from (3.1.2), $\|x_n-Tx_n\| = \|x_n-x_{n+1}\|1/C_n$

Thus the above inequality becomes as,

$$\|z-Tz\| \leq \|z-x_{n+1}\|+(1-C_n)\|x_n-Tz\|+C_n\, q \max \{ \|x_n-z\|,$$

$$\frac{\|z-Tz\|[1-\|x_n-x_{n+1}\|1/C_n]}{1+\|x_n-z\|},$$

$$\frac{1}{2}\ \frac{\|x_n - Tz\|[1-\|x_n - x_{n+1}\| 1/C_n - \|z - Tx_n\|]}{1+\|x_n - z\|}\ \}$$

Now taking the limit as $n \to \infty$ and using (3.1.3) and continuity of T, we have

$$\|z-Tz\| \leq (1-h)\|z-Tz+\| + hq \max\{0,\ \|z-Tz\|,\ \tfrac{1}{2}\|z-Tz\|[1-\|z-Tz\|]\}$$

If $\max\{0, \|z-Tz\|, \tfrac{1}{2}\|z-Tz\|[1-\|z-Tz\|]\} = \|z-Tz\|$

then

$$\begin{aligned} \|z-Tz\| &\leq (1-h)\|z-Tz\| + hq\|z-Tz\| \\ &\leq (1-h+hq)\|z-Tz\| \\ &\leq [1-h(1-q)]\|z-Tz\| \end{aligned}$$

Clearly it is contradiction when $z \neq Tz$

hence $Tz = z$

i.e. z is fixed point of T.

Hence the proof.

Now we extend theorem (3.1) for pair of mappings.

Theorem (3.2) :

Let X be closed, convex subset of a normed space N and let T_1, T_2 be contractive pair of mappings of X, T_1 and T_2 are continuous on X such that,

$$\|T_1x - T_2y\| \leq q \max\{\|x-y\|,\ \frac{\|y-T_2y\|[1-\|x-T_1x\|]}{1+\|x-y\|},$$

$$\tfrac{1}{2}\, \frac{\|x-T_2y\|[1-\|x-T_1x\|-\|y-T_1x\|]}{1+\|x-y\|}\} \qquad \ldots (3.2.1)$$

for all $x, y \in X$, $x \neq y$, $0 < q < 1$,

and the sequence $\{x_n\}$ of Mann iterates associated with T_1 and T_2 as given below,

For $x_0 \in X$, Set $x_{2n+1} = (1-C_n)x_{2n}+C_nT_1x_{2n}$

and $x_{2n+2}=(1-C_n)x_{2n+1}+C_nT_2x_{2n+1}$, for $n=0,1,2..$ ... (3.2.2)

Where $\{C_n\}$ satisfy (i), (ii), (iii) of (3.1.3).

If x_n converges to z in X and is a fixed point of either T_1 or T_2 then z is the common fixed point of T_1 and T_2.

Proof :

Let $z \in X$ such that $\lim_{n\to\infty} x_n=z$, let $T_1z=z$.

Now we shall show that z is common fixed point of T_1 and T_2.

Now we have

$$\|z-T_2z\| \leq \|z-x_{2n+1}\|+\|(1-C_n)x_{2n}+C_nT_1x_{2n}-T_2z\|$$

$$\leq \|z-x_{2n+1}\|+\|(1-C_n)x_{2n}-(1-C_n)T_2z+C_nT_1x_{2n}-C_nT_2z\|$$

$$\leq \|z-x_{2n+1}\|+(1-C_n)\|x_{2n}-T_1z\|+C_n\, q \max\{\|x_{2n}-z\|,$$

$$\frac{\|z-T_2z\|[1-\|x_{2n}-T_1x_{2n}\|]}{1+\|x_{2n}-z\|}, \tfrac{1}{2}\, \frac{\|x_{2n}-T_2z\|[1-\|x_{2n}-T_1x_{2n}\|=\|z-T_1x_{2n}\|]}{1+\|x_{2n}-z\|}\}$$

by ... (3.2.1)

Since from (3.2.2), $\|x_{2n}-T_1x_{2n}\|=\|x_{2n}-x_{2n+1}\|1/C_n$.

Thus the above inequality reduces to

$$\|z-T_2z\| \leq \|z-x_{2n+1}\|+(1-C_n)\|x_{2n}-T_2z\|+C_n\ q\ \max\ \{\|x_{2n}-z\|,$$

$$\frac{\|z-T_2z\|[1-\|x_{2n}-x_{2n+1}\|1/C_n]}{1+\|x_{2n}-z\|},$$

$$\frac{1}{2}\ \frac{\|x_{2n}-T_2z\|[1-\|x_{2n}-x_{2n+1}\|1/C_n-\|z-T_1x_{2n}\|}{1+\|x_{2n}-z\|}\}$$

Now taking the limit as $n \to \infty$ and using (iii) of (3.1.3) and continuity of T_2 we obtain for $T_1z=z$.

$$\|z-T_2z\| \leq (1-h)\|z-T_2z\|$$

$$+\ hq\ \max\{0,\ \|z-T_2z\|,\ \tfrac{1}{2}\ \|z-T_2z\|\ [1-\|z-z\|]\}$$

$$\leq\ (1-h)\|z-T_2z\|+hq\ \max\{\ 0,\|z-T_2z\|,\ \tfrac{1}{2}\|z-T_2z\|\}$$

if $\max\ \{\ 0,\ \|z-T_2z\|,\ \tfrac{1}{2}\ \|z-T_2z\|\}\ =\ \|z-T_2z\|$

then $q\ \max\{0,\ \|z-T_2z\|,\ \tfrac{1}{2}\ \|z-T_2z\|\}\ =\ q\ \|z-T_2z\|$

Thus the above inequality becomes,

$$\|z-T_2z\| \leq\ (1-h)\|z-T_2z\|+hq\ \{\ \|z-T_2z\|\ \}$$

$$\leq\ (1-h+hq)\|z-T_2z\|$$

$$\leq\ [1-h(1-q)]\|z-T_2z\|$$

if suppose $z \neq T_2z$ then it is contradiction.

Thus it follows $T_2z = z$.

i.e. z is the fixed point of T_2.

Similarly we can prove that if $T_2 z = z$, there $T_1 z = z$.

i.e. z is the common fixed point of T_1 and T_2.

This completes the proof.

Queries :

1. If T satisfies contractive condition (3.1.1) and the continuity of T is removed, does T have a fixed point?

2. Does the same conclusion hold if thee Mann iteration procedure is replaced by that of Ishikawa iteration procedure?

SECTION - II

Fixed point theorem in Banach spaces :

The Banach contraaction principle has been generalized in different directions. Many authors have been given some variants, such as Chatterjea [1], Edelstein [4], Hardy and Rogers [5], Kannan [9], Reich [12] etc. In 1980 D.S. Jaggi and Bal Kishan Dass [8] have generalized this principle through rational expression to prove the following theorem.

Theorem A :

Let f be a self map defined on a metric space

(X,d) satisfying the following -

(i) for some $\alpha, \beta \in [0, 1)$ with $(\alpha + \beta) < 1$,

$$d(fx,fy) \leq \frac{\alpha\, d(x,fx)\; d(y,fy)}{d(x,fy)+d(y,fx)-d(x,y)} + \beta\, d(x,y),$$

$x,\ y \in X, \quad x \neq y,$

(ii) there exists $x_0 \in X : \{ f^n x_0\} \supset \{ f^{n_k} x_0\}$ with

$\lim_{k \to \infty} f^{n_k} x_0 \in X,$

Then f has a unique fixed point $u = \lim_{k \to \infty} f^{n_k} x_0$

Now we state and prove our main theorems on fixed points of self-mappings of a Banach space.

Theorem (3.3) :

Let T be a self map of a Banach space X with norm $\|.\|$ satisfying the condition,

$$\|Tx-Ty\| \leq q \max \left\{ \|x-y\|, \frac{\|y-Ty\| \; [1+\|x-Tx\|]}{1+\|x-y\|}, \right.$$

$$\left. \frac{1}{2} \frac{\|x-Ty\| \; [1+\|x-Tx\|+\|y-Tx\|]}{1+\|x-y\|} \right\} \quad \ldots (3.3.1)$$

for all $x,\ y \in X$, $x \neq y$ and $0 < q < 1$.

Then T has unique fixed point in X.

Proof :

For some arbitrary $x_0 \in X$, we define the sequence $\{x_n\}$ as,

$$x_1 = Tx_0,\ x_{n+1} = Tx_n, \quad n = 1, 2, \ldots$$

Then by (3.3.1) we have,

$$\|x_n - x_{n+1}\| = \|Tx_{n-1} - Tx_n\|$$

$$\leq q \max \Big\{ \|x_{n-1} - x_n\|, \ \frac{\|x_n - Tx_n\| \, [1 + \|x_{n-1} - Tx_{n-1}\|]}{1 + \|x_{n-1} - x_n\|},$$

$$\tfrac{1}{2} \ \frac{\|x_{n-1} - Tx_n\| [1 + \|x_{n-1} - Tx_{n-1}\| + \|x_n - Tx_{n-1}\|]}{1 + \|x_{n-1} - x_n\|} \Big\}.$$

$$\leq \quad q \max \Big\{ \|x_{n-1} - x_n\|, \ \frac{\|x_n - x_{n+1}\| [1 + \|x_{n-1} - x_n\|]}{[1 + \|x_{n-1} - x_n\|]},$$

$$\tfrac{1}{2} \ \frac{\|x_{n-1} - x_{n+1}\| [1 + \|x_{n-1} - x_n\| + \|x_n - x_n\|]}{[1 + \|x_{n-1} - x_n\|]} \Big\}$$

$$\leq \quad q \max \{ \|x_{n-1} - x_n\|, \quad \|x_n - x_{n+1}\|, \ \tfrac{1}{2}\|x_{n-1} - x_{n+1}\| \}$$

Since $\quad \|x_n - x_{n+1}\| \leq q \ \|x_n - x_{n+1}\| \quad$ is impossible (as $q < 1$),

Now one has

$$\|x_n - x_{n+1}\| \leq q \ \max\{\|x_{n-1} - x_n\|, \ \tfrac{1}{2}\|x_{n-1} - x_{n+1}\|\}$$

If max $\quad \{\|x_{n-1} - x_n\|, \ \tfrac{1}{2}\|x_{n-1} - x_{n+1}\|\} = \|x_{n-1} - x_n\|$ ... (3.3.2)

Then $\quad \|x_n - x_{n+1}\| \leq q \ \|x_{n-1} - x_n\|$ ... (3.3.3)

If maximum of two memebrs in (3.3.2) is

$\frac{1}{2}\ \|x_{n-1}-x_{n+1}\|$ then,

$$\|x_n-x_{n+1}\| \le \tfrac{1}{2}\, q\ \|x_{n-1}-x_{n+1}\| \le \tfrac{1}{2}\, q\{\|x_n-x_n\|+\|x_n-x_{n+1}\|\}$$

Thus $$\|x_n-x_{n+1}\| \le \frac{q}{2-q}\ \|x_{n-1}-x_n\| \le q\ \|x_{n-1}-x_n\| \quad \ldots(3.3.4)$$

from (3.3.3) and (3.3.4) we have

$$\|x_n-x_{n+1}\| \le q\ \|x_{n+1}-x_n\|$$

proceeding in this manner we obtain.

$$\|x_n-x_{n+1}\| \le q\ \|x_{n-1}-x_n\|$$

$$\le q^2\,\|x_{n-2}-x_{n-1}\|$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$\le q^n\,\|x_0-x_1\|$$

and hence for $m > n$

$$\|x_n-x_m\| \le q^n[1+q+q^2+\ldots\ldots\ldots-q^{m-n+1}]\,\|x_0-x_1\| \quad \ldots (3.3.5)$$

$$\le \frac{q^n}{1-q}\ \|x_0-x_1\|$$

Since $q < 1$, $q^n \to 0$ as $n \to \infty$, so from (3.3.5) it follows that the sequence $\{x_n\}$ is Cauchy sequence.

Since X is complete then there exists a some point $u \in X$ such that

$$u = \lim_{n\to\infty} x_n$$

Now we shall prove that u is a fixed point of T. By (3.3.1) and triangle inequality, we have,

$$\|u-Tu\| \leq \|u-Tx_n\|+\|Tx_n-Tu\|$$

$$\leq \|u-Tx_n\|+q \max \{ \|x_n-u\|, \frac{\|u-Tu\|[1+\|x_n-Tx_n\|]}{1+\|x_n-u\|},$$

$$\frac{1}{2} \frac{\|x_n-Tu\| [1+\|x_n-Tx_n\|+\|u-Tx_n\|]}{1+\|x_n-u\|}\}$$

$$\|u-Tu\| \leq \|u-Tx_n\|+q \max \{ \|x_n-u\|, \frac{\|u-Tu\|[1+\|x_n-x_{n+1}\|]}{1+\|x_n-u\|},$$

$$\frac{1}{2} \frac{\|x_n-Tu\| [1+\|x_n-x_{n+1}\|+\|u-Tx_n\|]}{1-\|x_n-u\|} \}$$

Since $x_n \to u$, $x_{n+1} \to u$, then above inequality holds that

$$\|u-Tu\| \leq \|u-u\|+q \max \{ \|u-u\|, \frac{\|u-Tu\| [1+\|u-u\|]}{1+\|u-u\|},$$

$$\frac{1}{2} \frac{\|u-Tu\| [1+\|u-u\|+\|u-Tu\|]}{1+\|u-u\|}\}$$

$$\leq q \max \{ \|u-Tu\|, \|u-Tu\|\}$$

i.e. $\|u-Tu\| \leq q \|u-Tu\|$

i.e. $\|u-Tu\|<\|u-Tu\|$ (as $q < 1$)

Which is contradition

Thus it implies that

$$\|u-Tu\| \to 0 \quad \text{as} \quad n \to \infty$$

i.e $\quad u = Tu$

i.e. $\quad$ u is fixed point T.

For uniqueness, let $v \neq u$ be another fixed point of T,

Consider, $\|u-v\| = \|Tu-Tv\|$

$$\leq q \max \left\{ \|u-v\|, \frac{\|v-Tv\|[1+\|u-Tu\|]}{1+\|u-v\|}, \frac{1}{2} \frac{\|u-Tv\| \; [1+\|u-Tu\|+\|v-Tu\|]}{1+\|u-v\|} \right\}$$

i.e. $\quad \|u-v\| \leq q \max\{\|u-v\|, \frac{1}{2} \|u-v\|\}$

Clearly

$$\|u-v\| < q \|u-v\|$$

Which is impossible (as $q < 1$)

Thus $u = v$

i.e. u is a unique common fixed point of T.

This completes the proof.

Now we extend this theorem (3.3) for a pair of mappings in Banach space.

Theorem (3.4) :

Let T_1 and T_2 be two continuous self mappings of a Banach space X such that,

$$\|T_1x-T_2y\| \leq q \max\left\{\|x-y\|, \frac{\|y-T_2y\| \; [1+\|x+T_1x\|]}{1+\|x-y\|},\right.$$

$$\left. \frac{1}{2}\,\frac{\|x-T_2y\|[1+\|x-T_1x\| + \|y-T_1x\|]}{1+\|x-y\|}\right\} \quad \ldots (3.4.1)$$

for all $x, y \in X$, $x \neq y$ and $0 < q < 1$,

Then T_1 and T_2 have unique common fixed point.

Proof :

For any arbitrary $x_0 \in X$, let us define the sequence $\{x_n\}$ as follows.

$$x_1 = T_1x_0, \quad x_2 = T_2x_1,$$

$$x_{2n+1} = T_1x_{2n}, \quad x_{2n+2} = T_2x_{2n+1}, \quad n = 0,1,2, \ldots$$

By (3.4.1) we have

$$\|x_{2n+1}-x_{2n+2}\| = \|T_1x_{2n}-T_2x_{2n+1}\|$$

$$\leq q \max\left\{\|x_{2n}-x_{2n+1}\|,\right.$$

$$\frac{\|x_{2n+1}-T_2x_{2n+1}\| \; [1+\|x_{2n}-T_1x_{2n}\|]}{1+\|x_{2n}-x_{2n+1}\|},$$

$$\tfrac{1}{2}\ \frac{\|x_{2n}-T_2x_{2n+1}\|\ [1+\|x_{2n}-T_1x_{2n}\|+\|x_{2n+1}-T_1x_{2n}\|]}{1+\|x_{2n}-x_{2n+1}\|}\ \}$$

$$\leq \quad q \max \{\ \|x_{2n}-x_{2n+1}\|,$$

$$\frac{\|x_{2n+1}-x_{2n+2}\|\ [1+\|x_{2n}-x_{2n+1}\|]}{1+\|x_{2n}-x_{2n+1}\|}$$

$$\frac{\|x_{2n}-x_{2n+2}\|\ [1+\|x_{2n}-x_{2n+1}\|+\|x_{2n+1}-x_{2n+1}\|]}{1+\|x_{2n}-x_{2n+1}\|}\ \}$$

$$\leq \quad q \max \{\ \|x_{2n}-x_{2n+1}\|,\ \|x_{2n+1}-x_{2n+2}\|,\ \tfrac{1}{2}\ \|x_{2n}-x_{2n+2}\|\}$$

Since $\|x_{2n+1}-x_{2n+2}\| \leq q\ \|x_{2n+1}-x_{2n+2}\|$ is impossible,

$(q < 1)$

Then one has $\|x_{2n+1}-x_{2n+2}\| \leq q \max\{\|x_{2n}-x_{2n+1}\|, \tfrac{1}{2}\|x_{2n}-x_{2n+2}\|\}$

If max $\{\|x_{2n}-x_{2n+1}\|, \tfrac{1}{2}\|x_{2n}-x_{2n+1}\|\} = \|x_{2n}-x_{2n+1}\|$ ...(3.4.2)

Then $\|x_{2n+1}-x_{2n+2}\| \leq q\ \|x_{2n}-x_{2n+1}\|$ ...(3.4.3)

If maximum of two members in (3.4.2) is

$$\tfrac{1}{2}\ \|x_{2n}-x_{2n+2}\|$$

then $\|x_{2n+1}-x_{2n+2}\| \leq \tfrac{1}{2}\ q\ \|x_{2n}-x_{2n+2}\|$

$$\leq \tfrac{1}{2}\ q\{\|x_{2n}-x_{2n+1}\|+\|x_{2n+1}-x_{2n+2}\|\}$$

Thus $\|x_{2n+1}-x_{2n+2}\| \leq \frac{q}{2-q} \|x_{2n}-x_{2n+1}\|$

$$\leq q \|x_{2n-1}-x_{2n}\| \qquad \ldots (3.4.4)$$

then from (3.4.3) and (3.3.4) we have

$$\|x_{2n+1}-x_{2n+2}\| \leq q \|x_{2n-1}-x_{2n}\|$$

proceding in this manner we obtain

$$\|x_{2n+1}-x_{2n+2}\| \leq q \|x_{2n-1}-x_{2n}\|$$

$$\leq q^2 \|x_{2n-2}-x_{2n-1}\| \leq \cdots\cdots$$

$$\leq q^n \|x_0-x_1\|$$

For any $m > n$ we can show that

$$\|x_n-x_m\| \leq \frac{q^n}{1-q} \|x_0-x_1\| \qquad \ldots (3.4.5)$$

Which is cauchy sequence (Since $q \to 1$, $q^n \to 0$ as $n \to \infty$)

from (3.4.5)

Again since X is complete, then there exists any point $u \in X$ such that,

$$\lim_{n \to \infty} x_n = u$$

i.e. $x_n \to u$, consequently $x_{2n} \to u$.

Now we shall prove that u is a fixed point of T_1 and T_2.

Let $T_1u = u$

Consider

$$\|u-T_2u\| \leq \|u-x_{2n+1}\|+\|x_{2n+1}-T_2u\|$$

$$\leq \|u-x_{2n+1}\|+\|T_1x_{2n}-T_2u\|$$

$$\leq \|u-x_{2n+1}\|+q \max \{ \|x_{2n}-u\|,$$

$$\frac{\|u-T_2u\| \ [1+\|x_{2n}-T_1x_{2n}\|]}{1+\|x_{2n}-u\|},$$

$$\tfrac{1}{2} \ \frac{\|x_n-T_2u\| \ [1+\|x_{2n}-T_1x_{2n}\|+\|u-T_1x_{2n}\|]}{1+\|x_{2n}-u\|}\}$$

$$\leq q \ \max \{\|u-T_2u\|, \tfrac{1}{2} \ \|u=T_2u\|\}$$

Clearly $\|u-T_2u\| = 0$

Which follows that u is fixed find T_2.

Similary we can show that $T_1u = u$,

Now we shall show that u is unique common fixed point of T_1 and T_2.

Suppose $v \neq u$ is another fixed point of T_1 and T_2.

then,

$$\|u-v\| = \|T_1u-T_2v\|$$

$$\leq q \max \{\|u-v\|,$$

$$\frac{\|v-T_2v\| \ [1+\|u-Tu\|]}{1+\|u-v\|},$$

$$\tfrac{1}{2} \ \frac{\|u-T_2v\| \ [1+\|u-T_1u\|+\|u-T_1u\|]}{1+\| \ u \ -v\|} \}$$

$$\|u-v\| \leq q \ \max \{\|u-v\|, \tfrac{1}{2}\|u-v\|\}$$

Clearly,

$$\|u-v\| = 0$$

which implies $u = v$.

i.e. u is common fixed point of T_1 and T_2.

This completes the proof.

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