CHAPTER - IV

FIXED POINT THEOREM IN HILBERT SPACE

FIXED POINT THEOREM FOR NON-EXPANSIVE MAPPING IN HILBERT SPACE

Introduction :

This chapter is tatally devoted to study, a fixed point theorem in Hilbert space.

Many problems of applied mathematics like, non linear integral equations, boundary value problems for non linear ordinary or partial differential equations can be reduced to find the solutions, formulating in terms of finding the fixed points of a given non-linear mapping in function space. A mapping satisfying compactness conditions has fixed points. Browder F.E. and Petryshyn [1] discussed the existence of fixed points for non linear mappings of a compact subset C of a Hilbert space H into itself. They have also introduced four classes of mappings. On closed, convex subset C of a Hilbert space H; which admititerative methods for construction of their fixed points. This work is very useful to our investigation.

Since non-expansive mapping is most useful tool by which one can obtain fixed point. Browder and Petryshyn had shown that, "Every non-expansive self mapping of closed, bounded convex subset C of Banach space X must have a fixed point". One can choose any of the iteration methods like, Mean value iteration method, Mann iteration method, Ishikawa iteration method, etc. Gardon G. Johnson [4] has shown that a technique of W.R.Mann [5] is fruitful in finding a fixed of a function eventhough a Picard iteration may fail. Troy L. Hicks and John D Kubicek [2] have proved some fixed point theorems for different types of contractive mappings in Hilbert space and using Mann iteration procedure.

Recently Naimpally and Singh [6] have proved very interesting results related to Ishikawa scheme.

They have proved the following theorem,

Theorem A :

Let H be a Hilbert space and C be a convex subset of H. Suppose $T : C \to C$ is mapping satisfies Tricomi condition. (i.e. for all $x \in C$ and $y \in F(T)$, $\|Tx-y\| \leq \|x-y\|$) Suppose F(T) ((F(T) = the set of fixed points) is non empty. Suppose $\sum_{n=1}^{\infty} \alpha_n \beta_n$ diverges and $\beta_n \to \beta < 1$. Then $\underline{\lim} \| x_n - Tx_n \| = 0$ for each $x_0 \in C$, where x_{n+1} is defined as in the Ishikawa scheme.

This result is applicable to obtain fixed point of different mappings. Shiro Ishikawa [3] has shown that, if T is a Lipschitzian pseudo-contractive map of compact convex subset C of Hilbert space into itself and x_1 is any point in C, then certain mean value sequance defined, converges strongly to a fixed point of T.

Considering these two results we prove our theorem.

Theorem (4.1) :

Let C be a convex, compact subset of a Hilbert space H, T is a non-expansive map from C into itself (i.e. for x, y $\in$ C, $\|Tx-Ty\| \leq \|x-y\|$). Suppose set of fixed points F(T) is non empty, and x_1 is any point in C then the sequence $\{x_n\}_{n=1}^{\infty}$. Converges strongly to a fixed point of T, where $\{x_n\}$ is defined for each positive integer n by

$$x_0 \in C \qquad \dots (4.1.1)$$

$$y_n = \beta_n Tx_n + (1-\beta_n)x_n, \quad n \geq 0 \qquad \dots (4.1.2)$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \quad n \geq 0 \qquad \dots (4.1.3)$$

in the Ishikawa scheme,

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences of positive real numbers that satisfy the following conditions :

$$0 < \alpha_n < \beta_n < 1 \quad \text{for all positive integer } n, \qquad \dots (4.1.4)$$

$$\lim_{n \to \infty} \beta_n = 0, \qquad \dots (4.1.5)$$

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty \qquad \dots (4.1.6)$$

As particular case, we choose for instance $\alpha_n = \beta_n = n^{-\frac{1}{2}}$.

Proof :

For any x, y, z in Hilbert space and a real number λ , we have,

$$\|\lambda x+(1-\lambda)y-z\|^2 = \|\lambda(x-y)+y-z\|^2$$

$$= \lambda^2\|x-y\|^2+\|y-z\|^2+2\lambda\ Re(x-y,y-z)$$

$$= \lambda^2\|x-y\|^2+\|y-z\|^2+\lambda Re[(\|x\|^2-2(x,z)+\|z\|^2)$$

$$-(\|x\|^2-2(x,y)+\|y\|^2)-(\|z\|^2-2(z,y)+\|y\|^2)]$$

$$= \lambda^2\|x-y\|^2+\|y-z\|^2+\lambda(\|x-z\|^2-\|x-y\|^2-\|y-z\|^2)$$

$$= \lambda\|x-z\|^2+(1-\lambda)\|y-z\|^2-\lambda(1-\lambda)\|x-y\|^2) \quad ..(4.1.7)$$

Since F(T) is nonempty, and C is convex compact set and T is continuous. Let p denote any point of F(T).

From (4.1.7) in which λ stands for α_n or β_n, and from (4.1.2),(4.1.3) we have

$$\|x_{n+1}-p\|^2 = \|\alpha_n-Ty_n+(1-\alpha_n)x_n-p\|^2$$

$$= (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|Ty_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-Tx_n\|^2$$

$$= (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|Ty_n-Tp\|^2-\alpha_n(1-\alpha_n)\|x_n-Tx_n\|^2$$

$$\leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|y_n-p\|^2-\alpha_n(1-\alpha_n)\|Ty_n-x_n\|^2$$

(Since T is non-expansive)

$$\leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|y_n-p\|^2 \quad ... (4.1.8)$$

But $\|y_n-p\|^2 = \|\beta_n Tx_n+(1-\beta_n)x_n-p\|^2$

$$= \beta_n\|Tx_n-p\|^2+(1-\beta_n)\|x_n-p\|^2-\beta_n(1-\beta_n)\|Tx_n-x_n\|^2$$

$$\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|x_n-p\|^2-\beta_n(1-\beta_n)\|Tx_n-x_n\|^2$$

$$\leq \|x_n-p\|^2-\beta_n(1-\beta_n)\|Tx_n-x_n\|^2 \quad \ldots (4.1.9)$$

from (4.1.8) and (4.1.9) we have,

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n[\|x_n-p\|^2-\beta_n(1-\beta_n)\|Tx_n-x_n\|^2]$$

$$\leq \|x_n-p\|^2-\alpha_n\beta_n(1-\beta_n)\|Tx_n-x_n\|^2$$

Summing this inequality by putting m, m+1, ... n for n we get

$$\|x_{n+1}-p\|^2 \leq \|x_m-p\|^2-\sum_{k=m}^{n}\alpha_k\beta_k(1-\beta_k)\|Tx_k-x_k\|^2 \quad \ldots (4.1.10)$$

which implies that

$$\sum_{k=m}^{n} \alpha_k\beta_k(1-\beta_k)\|Tx_k-x_k\|^2 \leq \|x_m-p\|^2-\|x_{n+1}-p\|^2 \quad \ldots (4.1.11)$$

Since $\beta_n \to 0$ as $n \to \infty$ then there exists an integer N such that $\beta_k < \frac{1}{2}$ for all $k \geq N$.

Thus if m is large than N, (4.1.11) becomes.

$$\tfrac{1}{2}\sum_{k=m}^{n} \alpha_k\beta_k\|Tx_k-x_k\|^2 \leq \|x_m-p\|^2-\|x_{n+1}-p\|^2$$

Since C is bounded then R.H.S. of the above inequality and therefore series on the L.H.S.

is also bounded. But as $\Sigma\ \alpha_n \beta_n$ diverges, then this implies that

$$\liminf_{n \to \infty} \|Tx_n - x_n\| = 0 \qquad \ldots (4.1.12)$$

As C is compact then there exist a sub-sequence $\{x_{n_i}\}_{i=1}^{\infty}$ which converges to some fixed point q of F(T).

Since q of fixed point of T, then from (4.1.6) we see that if $n \geq N$.

$$\|x_{n+1} - q\| \leq \|x_n - q\| \qquad \ldots (4.1.13)$$

Now for any $\varepsilon > 0$, there is a positive number n_{i_o} such that

$$\|x_{n_{i_o}} - q\| \leq \varepsilon \quad \text{for all } n_{i_o} \geq N.$$

Thus from (4.1.13) we have,

$$\|x_{n+1} - q\| \leq \varepsilon \quad \text{for } n \geq n_{i_o}$$

which shows that $\{x_n\}$ converges strongly to a fixed point of T.

This completes the proof.

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