CHAPTER - I

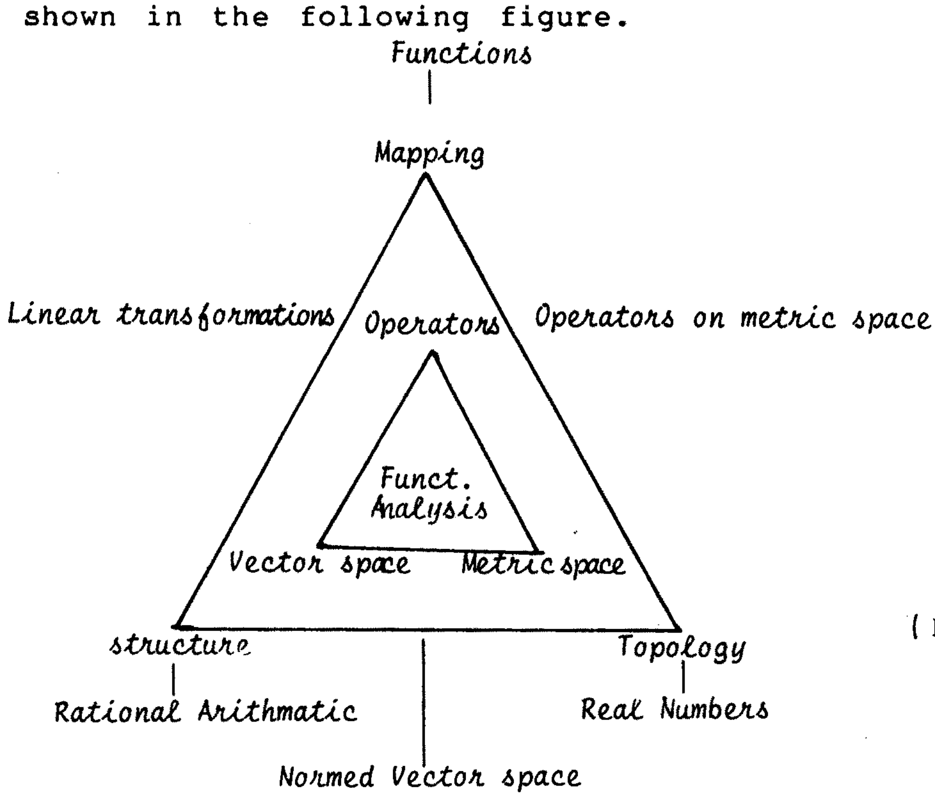
I N T R O D U C T I O N

INTRODUCTION :

This chapter deals with some basic concepts, methods, fixed point theorems and their applications as preliminaries which are needed for our investigations.

Functional analysis has been developed in the last 40 years. It is one of the modern branches of mathematics; that efficiently treats functions as points in a suitably defined space. The functional analysis is combination of several areas of mathematics. Functional anlytic ideas help us in the characterization of the problem and its solution. Then one can determine the line of attack and obtain a solution of a problem. This language of functional analysis is now penetrating into the technical literature of science and engineering. [30].

The building blocks of the functional analysis are shown in the following figure.



(Fig. 1.)

§ 1.1 Basic Concepts -

Definition 1.1.1 : Metric Space : [8]

Let X be any nonempty set. A metric on X is a mapping $d : X \times X \rightarrow R$ which satisfies the following axioms : for all $x,y,z \in X$,

(i) $d(x,y) \geq 0$,

$d(x,y) = 0 \Leftrightarrow x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,z) \leq d(x,y)+d(y,z)$,

(iv) $d(x,x) = 0$

Then set X together with a metric d on it is called a metric space and is denoted by (X,d).

The existance of a metric on a set allows the convergence of a sequence of elements and the associated concept of limit to be defined.

Definition 1.1.2 : Limit of a Convergent Sequence : [8]

A sequence $\{x_n\}$ of elements of (X,d) converges to an element $x \in X$ if

$$\lim_{n \to \infty} d(x_n,x) = 0$$

then limit of $\{x_n\}$ is x which is denoted by $x_n \rightarrow x$ as $n \to \infty$

Defination 1.1.3 : Cauchy Sequence [8]

Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in it. Then the sequence $\{x_n\}$ is said to be cauchy sequence if for every $\varepsilon>0$, there exists a positive integer n_0 such that

$$d(x_n, x_m) < \varepsilon \text{ for all } n,m \geq n_0$$

i.e. if $d(x_n, x_m) \to 0$ as $m,n \to \infty$.

Note : Every convergent sequence in a metric space is a Cauchy Sequence but the converse need not be true.

Definition 1.1.4 : Complete Metric Space : [8]

A metric space (X,d) is said to be complete if every Cauchy Sequence in (X,d) converges to x in X.

Definition 1.1.5 : Totally bounded Set : [30]

A subset 'A' of a metric space (X,d) is said to be totally bounded if, for every $\varepsilon > 0$, A ~~continuous~~ covered with finite set $A\varepsilon$ called an ε-net, such that for each $x \in (X,d)$ there is a $y \in A\varepsilon$ such that $d(X,y) < \varepsilon$.

Definition 1.1.6 : Compact Space : [30]

A metric space (X,d) is said to be compact (or sequentially compact) if it is complete and totally bounded.

Definition 1.1.7 : Closed Set : [33]

A subset 'A' of metric space (X,d) is said to be closed if the complement of A is open.

Definition 1.1.8 : Open Set : [33]

Let (X,d) be a metric space. A subset 'A' of X is said to be open if and only if to each $x \in A$, there exists $r > 0$, such that $S(x,r) \subset A$. where S is open sphere of radius r.

Definition 1.1.9 : Topological Space : [8]

Let X be a nonempty set. A topology on X is a collection of subsets of X which satisfies the following axioms :

(i) $\emptyset, X \in T$,

(ii) any union of members of T is a member of T,

(iii) the intersection of finite number of members of T is a member of T.

Then the set X together with T is called topological space and it is denoted by (X,T).

Definition 1.1.10 : Hausdorff Space : [8]

A topological space (X,T) is called a Hausdorff space if, for any pair of distinct points x and y in X, there exist two disjoint open sets, one containing x and other containing y.

Definition 1.1.11 : Normed Linear Space : [8]

Let X be a vector space. A norm on X is a real valued function $\|.\|$: $X \to R$ defined on X such that for any $x,y \in X$ and all $\lambda \in K$, (the set of complex numbers or the set of real numbers).

(i) $\| x \| > 0$,

(ii) $\| x + y \| \leq \| x \| + \| y \|$,

(iii) $\| \lambda x\| = |\lambda| . \| x \|$

(iv) $\| x \| = 0 \iff x = 0$.

Then vector space X together with $\|.\|$ is called normed vector space and denoted by $(X,\|.\|)$.

Definition 1.1.12 : Banach Space : [30]

A normed space X is a complete if every Cauchy sequence of vectors $\{x_n\}$ converges to x, in X,

i.e. if $\| x_n - x\| \to 0$ as $n \to \infty$

Then complete normed vector space X is called Banach space.

Definition 1.1.13 : Inner Product Space : [8]

Let X be vector space over the complex field C. An inner product on X is a scalar valued function $<,> : X \times X \to C$ which satisfies the following conditions :

(i) $< x,y > = \overline{< y,x >}$ for all $x,y \in X$.

(ii) $< \lambda x+\mu y,z > = \lambda < x,z > +\mu<y,z>$

for $\lambda , \mu \in C$, $x,y,z \in X$

(iii) $< x,x >> 0,$

$$< x,x > = 0 \Longleftrightarrow x = 0.$$

<u>Hilbert Space</u> :

A German mathematician of 20th century David Hilbert [16] is a founder of these spaces.

An axiomatic bases [8] for Hilbert Space are provided by the famous mathematician J.Von.Neuman in 1926. [26]

<u>Definition 1.1.14</u>

An inner on vector space H is a function $<,> : H \times H \to K$ satisfaying the conditions (i) to (iii) and defined a norm on H given by

$$\| x \| = < x,x >^{\frac{1}{2}}$$

with metric d on H such that

$$d(x,y) = \|x-y\| = < x-y,\ x-y >^{\frac{1}{2}}.$$

If H is complete with respect to $\|x-y\|$.
(i.e. $\|x_n - x_m\| \to 0$ as $m,n \to \infty$ $\lim x_n = x$)
then complete inner product space H is called Hilbert space.

The concept of Hilbert Space [12] is of great importance in many branches of mathematics and theoritical physics. It is remarkable fact that every Hilbert space is Banach Space.

Now we make brief survey of some major developments of fixed point theorems in metric spaces, in Banach spaces and in Hilbert spaces along with their applications.

§ 1.2 Development of fixed point theorems and their applications -

Fixed points have long been used in analysis to solve various kinds of differential equations and integral equations.

Many operator equations can be solved by writting it into the form $x = Tx$. Then the solution of the equations is represented by a fixed point. [30]

Definition 1.2.1 : Fixed Point : [8]

Let X be a set and T be a map from X to X. A fixed point of T is a point $x \in X$ such that

$$Tx = x$$

i.e. fixed point of T is a solution of the functional equation $Tx = x, \quad x \in X$.

Examples :

(i) A translation has no fixed points.

(ii) A rotation of the plane has a single fixed point.

(iii) The mapping $T:R \to R$ defined by $Tx=x^3$ has three fixed points $\{0, -1, 1\}$.

(iv) A mapping $Tx=4x-9$ defined on R has $x=3$ is a fixed point.

(v) A mapping $Tx=e^x+1$ has no fixed point in R.

There are various kinds of fixed points; which play an important role in thermometry as they must be part of a discussion of temperature [10]

¶ Fixed points in general topology : [10]

The information regarding those points that remain fixed under certain transformations is very useful. It can be put to work in many different fields. (eg. differential equations and algebric topology.)

¶ The odd behaviour of a surface [10]

"Transformed into itself"

To the topologist a crumpled sheet of a paper and a disc with various points on its surface radiating outwards are both undergoing the same kind of change. They are in mathematicians words -

"being transformed into themselves in a continuous fashion".

From such transformations, mathematicians have derived the "Fixed point theorem."--

When a surface is transformed into itself in this way one point on the surface will remain where it was.

¶ A fixed point on sheet : [10]

Crumpling a sheet of paper illustrates the fixed point theorem -

First a numbered paper sheet is placed over on exact duplicate so that all points on both the sheets are aligned. Then the top sheet is crumpled above the bottom sheet. One point on the crumpled sheet must still be over its starting point.

¶ A fixed point on disc : [10]

Under the fixed point theorem, if all the points arround the black dot on the disc at the right radiate outwards in regular flowing pattern - towards, but not beyond. In the boundary of the disc - one point (dot) must remain fixed. This holds true for any bounded unknown area.

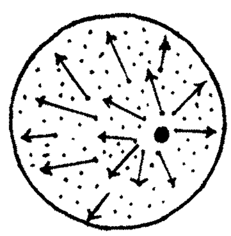


Fig 2.

¶ A topological red head : [10]

Most human heads have a fixed point, in the form of a whorl, from which all the hair radiates.

Topologically it would be impossible to cover a sphere with hair - or with radiating lines - without at least one such a fixed point.

For the same reason the wind can not blow everywhere over the earth's surface at once, there must be a point of calm.

The work of Cauchy[8] on differential equations has been fundamental to the concept of existance theorems in mathematics.

<u>Definition 1.1.2</u> : <u>Contraction mapping</u> : [8]

Let (X,d) be a complete metric space and $T:X \longrightarrow X$ be any mapping. The mapping T is called contraction mapping if it satisfies a Lipschitz condition with constant where $0<\alpha<1$ such that

$$d(Tx,Ty) \leq \alpha\, d(x,y)$$

for all $x, y \in X$.

The result that guarantees the existance and uniquness of fixed point is known as contraction principle. Which is introduced by a famous mathematician Banach [8].

<u>Theorem 1.2.1</u> : <u>Contraction principle</u> : (<u>Banach Fixed Point Theorem</u>) : Let T be a contraction mapping on a complete metric space X. Then T has a unique fixed point.

This well known Banach contraction principle in applied to establish existance and uniqueness theorems for

(i) Linear equations

(ii) differential equations (famous Picard's existance and uniqueness theorem for ordinary differential equations)

(iii) Integral equations (Fredholom and Voltera integral equations.)

The most important result and a classical example in fixed point theory is the famous theorem of Brouwer [8]. A Dutch topologist and logician Brouwer (1881-1966) is also known as founder of modern mathematics. [10]

<u>Theorem 1.2.2</u> : <u>Brouwer's fixed point theorem</u> :

In various versions -

(A) Every continuous map of the closed unit ball $S = \{x : \|x\| < 1\}$ in R^n to itself has a fixed point. [8]

(B) Let C be a circular disc consisting of a circle and region within the circle. Then for any continuous transformation which transforms each point of C which the transformation leaves fixed." [10]

This theorem is also true for closed n-cells.

(C) Brouwer fixed point theorem for polyhedra :-

Let X be a simplex and $f:X \to X$ a continuous mapping. Then f has a fixed point in X. (Math.Ann. 68, (1910) 71, (1912) [11]. Brouwer's theorem does not give any computational scheme [19] for obtaining fixed point. Scarf [31] considered additional conditions and developed a computational scheme in 1967 for computing fixed point of a mapping. Parron's theorem [8] is an application of Brouwer's fixed point theorem to the theory of matrices. Which play an important role in many applied fields. Poincare [14] proved a slightly different version of it much earlier in 1886. which was rediscorvered by Bhole in 1904.

Birkhoff and Kellog [1] were first prove fixed theorems in infinite-dimensional spaces. J.P. Schauder [32] generalized these results.

Theorem 1.2.3 : Schauder's fixed point theorem :

Let C be a nonempty convex compact subset of a normed linear space X. Then every continuous self map of C has a fixed point.

Many author's have generalized Brouwer's and Shauder's fixed point theorems in different spaces. Tikhonov [12] generalized Brouwer's theorem. (Maths.Ann. 111 (1935.) Which can be applied in m-dimensional space

F^m into a k-dimensional euclidean space E^k to show that existance of solutions of certain differential equations. Tychonoff [38] considered a general locally convex topological space instead of normed linear space and extended Schauder's theorem Kakutani [11] prove of fixed theorem for bounded closed convex subset in finite dimensional Euclean space.

(Duke Math. J.8 [1941])

Krein [10] a Russian functional analyst and applied mathematician (1907) with Milman proved theorem on compact convex subset of topological space. These prominent mathematican's have worked in the area of fixed point theory. Who put their remarks for further work.

From last three decades numbes of author's have worked in this field and put a step in medical sciences, (eg. Infectious disease model. Leggett [24]), Monetary economics model (J.Von Neumann [26]).

§ 1.3 Motivation of the Work -

In recent years many new notions have been introduced into mathematics. Among these there is the notion of fixed point. The fixed point theorems are used as a handy tools in proving the existance and uniqueness of the solution of certain problems in differential equations and integral equations. It is now widely known that the Schauder's fixed point theorem is a powerful method for proving existance theorems. If one wishes to use it to prove that a given problem has a solution, he proceeds by associating with the problem of a convex compact set in some Banach space and a continuous self map. Then there is a fixed point which is a solution. Mathematical literature since about 1935 abounds with illustrations of this technique.

A number of author's have defined contractive type mappings on a complete metric space X. Which are generalizations of the well known Banach contraction and which have the property that each of has unique fixed point. The fixed point can always be found by using Picard iteration beginning with some $x_0 \in X$. (Rhoades - [29] .)

Browder (1965) [3,4,5,6] initiated the study of fixed point theory of non-expansive mappings in Hilbert space. Petryshyn [27] studied an iteration method for the actual construction of fixed points of non linear contraction map of closed ball in Hilbert space. Browder and Petryshyn [27] introduced the four classes of mappings -

If T is a self map of a closed bounded convex subset C of a Hilbert space H then we define the following

<u>Definition 1.3.1</u> : <u>Strictly contractive</u> :

T is said to be strictly contractive if there exists a contstant k with $0 < k < 1$ such that

$$\|Tx-Ty\| < k\|x-y\| \quad \text{for all } x,y \in C.$$

<u>Definition 1.3.2</u> : <u>Contractive</u> (<u>or non-expansive</u>) :

T is said to be contractive or non-expansive if for all x,y,$\in$ C,

$$\|Tx-Ty\| \leq \|x-y\|.$$

<u>Definition 1.3.3</u> : <u>Strictly pseudo contractive</u> :

T is said to be strictly pseudo contractive if there exists a constant k with $0 < k < 1$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2,$$

for all x,y, $\in$ C.

<u>Definition 1.3.4</u> : (<u>pseudo contractive</u>) :

T is said to be pseudo contractive if for all x, y $\in$ C

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x-(I-T)y\|^2.$$

Which admit iterative methods for the construction of their fixed points. They established the following basic existance result.

Theorem 1.3.1 [7]

Let T be a self map of closed bounded covex subset C of a Hilbert space H such that

$$\|Tx-Ty\| \leq \|x-y\| \quad \text{for all } x,y \in C.$$

Then T has atleast one fixed point in C.

Many author s have proved their results regarding this basic theorem.

Definition 1.3.5 : Strongly Convergence : [23]

A sequence $\{x_n\}$ in a normed space X is said to be strongly convergent if there is $x \in X$ such that

$$\lim_{n \to \infty} \|x_n - x\| = 0 \Rightarrow \lim_{n \to \infty} x_n = x$$

i.e. $x_n \to x$. as $n \to \infty$.

Definition 1.3.6 : Weak Convergence : [22]

A sequence $\{x_n\}$ in a normed space X is said to be weakly convergent if there is an $x \in X$ such that for every $f \in X^*$ (Dual space of X)

$$\lim_{n \to \infty} f(x_n) = f(x)$$

i.e. $x_n \to x$. as $n \to \infty$

Ishikawa [17] introduced a new iteration scheme for the construction of fixed points of contractive mappings and obtained the following result.

<u>Theorem 1.3.2</u> : [17]

If T is a Lipschitizian pseudocontractive self-map of C, where C is a convex compact subset of a Hilbert space H and x is any point in C, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point x of T, where x_n is defined iteratively for each positive integer n by

$$x_{n+1} = \alpha_n Ty_n + (1-\alpha_n)x_n \qquad \text{... (1.3.3)}$$

$$y_n = \beta_n Tx_n + (1-\beta_n)x_n \qquad \text{... (1.3.4)}$$

Where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences of positive numbers that satisfy the following three conditions :

$$\left.\begin{array}{l} \text{(i)} \quad 0 \leq \beta_n \leq {}_n < 1, \\ \text{(ii)} \quad \lim_{n \to \infty} \beta_n = 0 \\ \text{(iii)} \quad \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty \end{array}\right] \qquad \text{... (1.3.5)}$$

Ishikawa derived the following technique and used it to prove above theorem.

For any x, y, z in Hilbert space H and a real number λ ,

$$\|\lambda x+(1-\lambda)y-z\|^2 = \lambda\|x-z\|^2+(1-\lambda)\|y-z\|^2-\lambda(1-\lambda)\|x-y\|^2 \quad \text{...(1.3.6)}$$

Mann [25] gave the following iteration process. For a self map T of a compact interval of the real line having a unique fixed point. The iteration process.

$$x_{n+1} = (1-\alpha_n)x_n+\alpha_n Tx_n \quad \ldots (1.3.7)$$

where $\alpha_n = \frac{1}{n+1}$, converges to the fixed point of T.

Hicks and Kubicek [15] studied Mann iteration process in Hilbert space and employing Ishikawa technique established the following interesting results.

Theorem 1.3.8 :

Suppose (i) $T : C \to C$ where C is convex subset of Hilbert space H, (ii) T is demicontractive with contraction coefficient k, (iii) set F(T) of fixed points of T in C is nonempty, (iv) $\sum \alpha_n(1-\alpha_n)$ diverges and (v) $\alpha_n \to < 1-k$.

Then $\underline{\lim} |x_n-Tx_n| = 0$ for each $x \in C$, where x_{n+1} is defined by (1.3.7).

S.R.Joshi and S.N.Kasralikar [20] have proved fixed point theorem for non-expansive mappings.

Theorem : 1.3.9 :

Let T be a non-expansive mapping of S into itself, where S is compact and convex subset of X Banch space X. Then T has a fixed point is S.

On the principle of contraction mapping Bryant established the following.

<u>Theorem 1.3.10</u> : [9]

Let (X,d) be a complete metric space, and T be a self mapping on X such that

$$d(T^n x, T^n y) \leq \alpha\, d(x,y) \text{ for all } x,y \in X,\ 0<\alpha<1.$$

and n ∈ N; then T has a unique fixed point in X. More recently Khan (1976), Chatterjee (1981), A.N.Gupta and A.Saxena have discussed a number of interesting results related to above theorem in [34] Simeon Reich has proved the following result for a pair of mapping in complete metric space.

<u>Theorem 1.3.11</u> :

Let X be a complete metric space and T be a self map on X satisfy the property,

$$d(Tx,Ty) \leq a\, d(x,Tx) + b\, d(y,Ty) + c\, d(x,y)$$

where a,b,c ≥ 0 real numbers such that a+b+c < 1 then T has a unique fixed point.

Rhoades (1977) has shown that "For a mapping T satisfying certain contractive definitions; if the sequences of Mann iterates converges, then it converges to a fixed point of T". He also compares various definitions of contractive mappings studied by different authores. (Rhoades-1977). [29]

Mann iteration process associated by T a self map of Banach space X is defined as follows :

let x_0 be in X, set $x_{n+1} = (1-c_n)x_n + c_n T x_n$, for $n \geq 0$, where c_n satisfies (i) $c_0 = 1$, (ii) $0 \; c_n < 1$ for $n > 0$ (1.3.12)
(iii) $\lim_{n \to \infty} c_n = h > 0$.

Banach's contraction principle was extended by D.S.Jaggi and Bal Kishan Dass [18] through rational expression which is given as below.

<u>Theorem 1.3.12</u> :

Let f be a self map defined on a metric space (X,d) satisfying :

(i) for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$,

$$d(fx,fy) \leq \frac{d(x,fx).d(y,fy)}{d(x,fy)+d(y,fx)+d(x,y)} + \beta\, d(x,y), \quad x,y \in X, \; x \neq y$$

(ii) there exists $x_0 \in X$: $\{f^n x_0\} \supset \{f^{n_k} x_0\}$ with $\lim_{k \to \infty} f^{n_k} x_0 \in X$ Then f has unique fixed point $u = \lim_{k \to \infty} f^{n_k}(x_0)$

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