CHAPTER - II

FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

INTRODUCTION :

This chapter has been divided into two sections. Section first consists some theorems on common fixed points for a pair of mappings in complete metric spaces. While in the section second, we represent fixed point theorems, for a sequence of mappings in complete metric spaces. Further we extend this work of fixed point theorems for family of mappings and also for non-expansive mappings.

SECTION - I

In this section we prove some theorems on pair of mappings in complete metric space.

Here we have extended the result of P.L.Sharma and A.K.Yuel [16] and established a theorem on common fixed points for a pair of mappings in complete metric space. It is further shown that the results of P.L.Sharma and A.K.Yuel are the special cases of our theorem. Consequentely, the results of Banach, Kannan, Fisher and Jaggi are the particular cases of our theorem.

Banach contraction principle states that, a contraction mapping on complete metric space has unique fixed point.

Kannan [12] has generalized this principle as follows.

Theorem A -

If T is a mapping of complete metric space X into itself such that,

$$d(Tx,Ty) \leq \beta \{ d(x,Tx) + d(y,Ty) \}$$

for all $x,y \in X$, where $0 \leq \beta < \frac{1}{2}$,

Then T has a unique fixed point.

B. Fisher [7] in 1975 proved the following theorems.

Theorem B -

If T is a mapping of the complete metric space X into itself such that,

$$d(Tx,Ty) \leq \gamma \{ d(x,Ty) + d(y,Tx) \}$$

for all $x,y \in X$, where $0 \leq \gamma < \frac{1}{2}$

Then T has a unique fixed point.

Theorem C -

If T is a mapping of the complete metric space X into itself such that,

$$d(Tx,Ty) \leq \alpha\, d(x,y) + \beta \{ d(x,Tx) + d(y,Ty) \} + \gamma \{ d(x,Ty) + d(y,Tx) \}$$

for all $x,y \in X$ where $0 \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \leq 1$,

$$\beta + \gamma < 1, \quad \alpha + 2\gamma < 1, \quad \gamma \geq 0,$$

Then T has unique fixed point.

Afterwards, Jaggi (1977) [10] proved the following theorem.

Theorem D -

If T is a mapping of complete metric space X into itself such that,

$$d(Tx,Ty) \leq \alpha \left\{ \frac{d(x,Tx)\ d(y,Ty)}{d(x,y)} \right\} + \beta\, d(x,y),$$

for all $x,y \in X$, where α, $\beta \in (0,1)$ with $\alpha + \beta < 1$, Then T has a unique fixed point.

At last P.L.Sharma and A.K.Yuel in [16] (1980) generlized this result through the following experssion.

Theroem E -

If T is a mapping of the complete metric space X into itself such that,

$$d(Tx,Ty) \leq \alpha \left\{ \frac{d(x,Tx)\ d(y,Ty)}{d(x,y)} \right\} + \beta \{ d(x,Tx) + d(y,Ty) \}$$

$$+ \gamma \{ d(x,Ty) + d(y,Tx) \} + \delta\, d(x,y),$$

for all $x,y \in X$, where $0 < \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} < 1$,

$$\beta + \gamma < 1, \quad 2\gamma + \delta < 1, \quad \gamma \geq 0.$$

Then T has a unique fixed point.

Main Results :

Now we wish to establish a theorem which includes all above five theorems as a special cases of the following theorem.

Theorem : 2.1 :

Let T_1 and T_2 be two continuous self mappings of a complete metric space (X,d) such that.

$$d(T_1x,T_2y) \leq \alpha \left\{\frac{d(x,T_1x)\ d(y,T_2y)}{d(x,y)}\right\} + \beta\ \{d(x,T_1x) + d(y,T_2y)\}$$

$$+ \gamma\{ d(x,T_2y) + d(y,T_1x) \} \ + \delta\ d(x,y) \ldots (2.1.1)$$

for all $x,y \in X$, where $0 \leq \dfrac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} < 1$

$\beta + \gamma < 1, \qquad 2\gamma + \delta < 1, \qquad \delta \geq 0.$

Then T_1 and T_2 have a unique common fixed point.

Proof :

Let x_0 be any arbitrary point in X, and define a sequence $\{x_n\}$ as $x_{2n+1} = T_1x_{2n}$, $x_{2n+2} = T_2x_{2n+1}$, $n = 0,1,2,\ldots$

$$x_1 = T_1x_0,\ x_2 = T_2x_1,\ x_3 = T_1x_2,\ x_4 = T_2x_3 \ldots$$

then by(2.1.1) we have

$$d(x_{2n+1}, x_{2n+2}) = d(T_1x_{2n}, Tx_{2n+1})$$

$$\leq \alpha\left\{\frac{d(x_{2n}, T_1x_{2n})\ d(x_{2n+1}, T_2x_{2n+1})}{d(x_{2n}, x_{2n+1})}\right\}$$

$$+ \beta\{d(x_{2n}, T_1x_{2n}) + d(x_{2n+1}, T_2x_{2n+1})\}$$

$$+ \gamma\{d(x_{2n}, T_2x_{2n}) + d(x_{2n+1}, T_1x_{2n})\} + \delta d(x_{2n}, x_{2n+1}),$$

i.e. $d(x_{2n+1}, x_{2n+2}) \leq \alpha\{ \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} \}$

$+ \beta\{ d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) \}$

$+ \gamma\{d(x_{2n}, x_{2n+2}) + d(x_{1n+1}, x_{2n+1})\} + \delta\, d(x_{2n}, x_{2n+1})$

$\leq \alpha\{ d(x_{2n+1}, x_{2n+2})\} + \beta\, d(x_{2n}, X_{2n+1}) + d(x_{2n+1}, X_{2n+2}))\}$

$+ \gamma\{ d(x_{2n}, x_{2n+2})\} + \delta\, d(x_{2n}, X_{2n+1})$

$\leq \alpha \quad d(x_{2n+1}, X_{2n+2}) + \beta\, d(x_{2n}, x_{2n+1}) + \beta\, d(x_{2n+1}, X_{2n+2})$

$+ \gamma\, d(x_{2n}, x_{2n+1}) + \gamma\, d(x_{2n+1}, x_{2n+2}) + \delta\, d(x_{2n}, x_{2n+1})$

$\leq \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma}\, d(x_{2n}, x_{2n+1})$

where $0 < \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} = q < 1$,

so $d(x_{2n+1}, x_{2n+2}) \leq q \quad d(x_{2n}, x_{2n+1})$ ...(2.1.2)

Again $d(x_{2n}, x_{2n+1}) = d(T_1 x_{2n}, T_2 x_{2n-1})$

$\leq \alpha\{\frac{d(x_{2n}, Tx_{2n})\, d(x_{2n-1}, T_2 x_{2n-1})}{d(x_{2n}, x_{2n-1})}\}$

$+ \beta\{ d(x_{2n}, T_1 x_{2n}) + d(x_{2n-1}, T_2 x_{2n-1})\}$

$+ \gamma\{ d(x_{2n}, T_2 x_{2n-1}) + d(x_{2n-1}, T_1 x_{2n})\} + \delta\, d(x_{2n}, x_{2n-1})$

$\leq \alpha\{ \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n-1}, x_{2n})}{d(x_{2n}, x_{2n-1})}\} + \beta\{d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n})\}$

$+ \gamma\{d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})\} + \delta\, d(x_{2n}, x_{2n-1})$

i.e. $d(x_{2n},x_{2n+1}) \leq \alpha\, d(x_{2n},x_{2n+1})+\beta d(x_{2n},x_{2n+2})$

$+\beta\, d(x_{2n-1},x_{2n})+\gamma d(x_{2n-1},x_{2n})+\gamma d(x_{2n},x_{2n+1})$

$+\delta\, d(x_{2n},x_{2n-1})$

$\frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}\, d(x_{2n},x_{2n-1})$

where $0 \leq \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} = q < 1$

so $d(x_{2n}, x_{2n+1}) \leq q\ \ d(x_{2n},x_{2n-1})$ ...(2.1.3)

Hence from(2.1.2) and(2.1.3), we get

$d(x_{n+1},x_n) \leq q\ d(x_n x_{n-1})$ for all $n \geq 1$

$\leq q^2 d(x_{n-1},x_{n-2})$

$\leq$ ....

$\leq q^n d(x_1 x_0)$,

for $m > n$, we have

$d(x_m,x_n) \leq d(x_m\ ,x_m x)+d(x_{m-1},x_{m-2})+..+d(x_{m-n+1}, d_{m-n})$

$\leq \ldots qd^m(x_1,x_0)+q^{m-1}d(x_1,x_1)+\ldots+q^{m-n+1}d(x_1,x_0)$

$\leq (q^m+q^{m-1}+\ldots+q^{m-n+1})\ d(x_1,x_0)$

$< \frac{q^m}{1-q}\ d(x_1,x_0)$

$\rightarrow$ as $n \rightarrow \infty$, since $q < 1$.

.: $\{x_n\}$ is a Cauchy sequence since, X is complete, there exists $z \in X$, such that $z = \lim_{n\to\infty} x_n$.

Clearly, $x_{2n+1} \to z$ and $x_{2n+2} \to z$ for all n.

Now we show that z is fixed point of T_1 and T_2. If possible let $z \neq T_1 z$.

Now

$$d(z,T_1z) \leq d(z,T_2x_{2n+1}) + d(T_2x_{2n+1},T_1z)$$

$$\leq d(z,x_{2n+2}) + d(T_1z,T_2x_{2n+1}) \qquad \ldots(2.1.4)$$

using(2.1.1) we have

$$d(T_1z,T_2x_{2n+1}) \leq \alpha\left\{ \frac{d(z,T_1z)\,d(x_{2n+1},T_2x_{2n+1})}{d(z,x_{2n+1})} \right\}$$

$$+ \beta\{ d(z,T_1z) + d(x_{2n+1},T_2x_{2n+1}) \}$$

$$+ \gamma\{ d(z,T_2x_{2n+1}) + d(x_{2n+1},T_1z) \} + \delta\, d(z,x_{2n+1}) \qquad \ldots(2.1.5)$$

from(2.1.4) and(2.1.5) we have -

$$d(z,T_1z) \leq d(z,x_{2n+2}) + \alpha\left\{ \frac{d(z_1T_1z)\ d(x_{2n+1},x_{2n+2})}{d(z,x_{2n+1})} \right\}$$

$$+ \beta\{ d(z,T_1z) + d(x_{2n+1},x_{2n+2}) \}$$

$$+ \gamma\{ d(z,x_{2n+2}) + d(x_{2n+1},T_1z) \} + \delta\, d(z,x_{2n+1})$$

$$d(z,T_1z) \leq d(z,x_{2n+2}) + \alpha\left\{ \frac{d(z,T_1z)\ d(x_{2n+1},x_{2n+2})}{d(z,T_1x_{2n})} \right\}$$

$$+ \beta\{ d(z,T_1z) + d(x_{2n+1},x_{2n+2}) \}$$

$$+ \gamma\{ d(z,x_{2n+2}) + d(x_{2n+1},T_1z) \} + \delta\, d(z,x_{2n+1})$$

on letting $n \to \infty$.

we have

$$d(z,T_1z) \leq (\beta + \gamma)\, d(z,T_1z)$$

Since $(\beta + \gamma) < 1$

so $d(z,T_1z) < d(z,T_1z)$

which is contradiction

hence $d(z,T_1z) = 0$, therefore

it follows that $T_1z = z$,

i.e. z is fixed point T_1.

Similarly we can show that z is fixed point T_2.

In order to prove z is unique fixed point of T_1 and T_2, let w be another fixed point of T_1 and T_2 such that $z \neq w$.

Then we have by (2.1.1)

$$d(z,w) = d(T_1z,\ T_2w)$$

$$\leq \alpha\left\{\frac{d(z,T_1z)\ d(w,T_2w)}{d(z,w)}\right\} + \beta\{d(z,T_1z) + d(w,T_2w)\}$$

$$+\gamma\{\ d(z,T_2w) + d(w,T_1z)\} + \delta\ d(z,w)$$

$$\leq \gamma\{\ d(z,w) + d(w,z)\} + \delta\, d(z,w)$$

$$\leq (2\gamma + \delta)\ d\ (z,w)$$

since $(2\gamma + \delta) < 1$,

it follows that

$$d(z,w) = 0$$

$\Rightarrow z = w$

Hence z is the unique common fixed point of T_1 and T_2. This completes the proof.

Remark :

If T_1 and T_2 are not necessarily continuous, then we have following observations

(a) When $T_1=T_2$, we get P.L.Sharma and A.K.Yuel[16] theorem (E).

(b) If $T_1=T_2$ and $\alpha=\beta=\gamma=0$ then $0\leq\delta<1$, we get Banach contraction principle.

(c) When $T_1=T_2$ and $\alpha=\gamma=\delta=0$ then $0\leq\beta\leq\frac{1}{2}$, we have theorem due to Kannan (1968).

(d) When $T_1=T_2$ and $\alpha=\beta=\delta=0$, then $0\leq\gamma<\frac{1}{2}$, the resulting theorem (B) is of Fisher (1975).

(e) When $\alpha=0$, regarding $T_1=T_2$ we get the theorem (C) of Fisher (1975).

(f) When $\beta=\gamma=0$ and $T_1=T_2$ then $0\leq\alpha, \delta<1$, and we get the theorem (D) of Jaggi [1977].

We now extend our theorem (2.1) for a pair of mappings T_1^p, T_2^q, where p,q are some positive integers; in the follows theorem.

Theorem : 2.2 :

Let T_1 and T_2 be two continuous self-mappings of metric space (X,d) such that,

$$d(T_1^{p}x,T_2^{q}y)\leq\alpha\left\{\frac{d(x,T_1^{p}x)\ d(y,T_2^{q}y)}{d(x,y)}\right\}+\beta\{d(x,T_1^{p}x)+d(y,T_2^{q}y)\}$$

$$+\gamma\{d(x,T_2^{q}y)+d(y,T_1^{p}x)\}+\delta\ d(x,y)\qquad\ldots(2.2.1)$$

for all $x,y \in X$, where

$$0\leq\frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}<1,\quad(\alpha+\gamma)<1,\ (2\gamma+\delta)<1,\quad\gamma\geq 0,$$

and p,q are some positive integers

Then T_1 and T_2 have unique fixed point.

Proof :

Let $x_0 \in X$ be any arbitrary point.

Define sequence $\{x_n\}$ as

$$x_{2n+1}=T_1^{p}x_{2n},\qquad x_{2n+2}=T_2^{q}x_{2n+1},\qquad n=0,1,\ldots\ldots$$

By theorem (2.1) T_1^{p} and T_2^{q} have unique fixed point $z \in X$.

Now $T_1^{p}z=z$ and $T_2^{q}z=z$.

Hence $T_1^{p}(T,z) = T_1(T_1^{p}z) = T_1z$

i.e. T_1z is fixed point of T_1^{p}.

But z is unique fixed point of T_1^{p}.

therefore $T_1z = z$.

Again $T_2^{q}(T_2z) = T_2(T_2^{q}z) = T_2z$

i.e. T_2z is a fixed point of T_2^{q}.

Since z in unique fixed point of $T_2^{q} = T_2z$.

Therefore z is a fixed point of T_1 and T_2.

To prove uniqueness, let $z \neq \omega$ be another fixed point of T_1 and T_2;

$$d(z,w) = d(T_1^{p}z,\ T_2^{q}z)$$

$$\leq \alpha\left\{\frac{d(z,T_1^{p}z)\ d(w,T_2^{q}w)}{d(z,w)}\right\} + \beta\{\ d(z,T_1^{p}z)+d(w,T_2^{q}w)\ \}$$

$$+\ \gamma\{d(z,T_2^{q}w)+d(w,T_1^{p}z)\}\ +\ \delta\ d(z,w)$$

$$\leq\ (2\gamma+\delta)\ \ d(z,w)$$

Since $2\gamma + \delta < 1$

therefore it follows that $z = w$

i.e. z in unique fixed point of T_1 and T_2

Hence the theorem.

Remark : On taking $p=q=1$, $T_1=T_2$, $\alpha=\beta=\gamma=0$, we get following result due Edelstein (1962) [6] as a corollary to our theorem (2.2).

Carallary :

If X be a complete metric space such that $d(Tx,Ty) < d(x,y)$ for all $x \neq y \in X$ and if for some $x_0 \in X$, sequence $\{x_n\}$ be defined as

$$x_n = T^n x_0$$

then z is the unique fixed point of T.

SECTION - II

In this section we prove some theorems on common fixed points for a sequence of mappings in complete metric spaces.

P.L.Sharma and A.K.Yuel have [16] proved the existance of fixed point of an operator T mapping a metric space (X,d) into itself by using the condition :

$$d(Tx,Ty) \leq \alpha\left\{\frac{d(x,Tx)\; d(y,Ty)}{d(x,y)}\right\} + \beta\{d(x,Tx) + d(y,Ty)\}$$

$$+\gamma\{ d(x,Ty) + d(y,Tx)\} + \delta\, d(x,y)$$

for x,y, $\in$ X, where $0 \leq \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} < 1$, $(\beta+\gamma)<1$, $(2\gamma+\delta)<1$, $\gamma \geq 0$.

Baidyanath Ray [1] obtained a following theorem in complete metric space into itself.

Theorem (A) -

Let $\{T_i\}$ be a sequence of maps each mapping a complete metric space (X,d) into itself such that,

(i) for any two distinct maps Ti, Tj
$d(T_i x, T_j y) \leq r d(x,y)$,
Where $0 \leq r < 1$ for all $x \neq y$,

(ii) There is a point x_0 in X such that any two consencutive members $(x_n = T_n x_{n-1})$ are distinct.

Then T_k has a unique common fixed point.

Here we generalize this result and show that it will be special case of our result.

Main theorem : 2.3 :

Let $\{T_n\}$ be a sequence of maps, each mapping a complete metric space (X,d) into itself such that,

(i) for any two maps T_i, T_j.

$$d(T_i x, T_j y) \leq \alpha\left\{\frac{d(x,T_i x)\, d(y,T_j y)}{d(x,y)}\right\} + \beta\{d(x,T_i x) + d(y,T_j y)\}$$

$$+\gamma\{d(x,T_j y) + d(y,T_i x)\} + \delta\, d(x,y) \qquad \ldots(2.3.1)$$

Where $0 \leq \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} < 1$, $(\beta+\gamma) < 1$, $(2\gamma+\delta) < 1$, $\gamma \geq 0$,

for $x \neq y$ in X.

(ii) There is a point $x_0 \in X$ such that any two distinct consecutive members $(x_n = T_n x_{n-1})$.

Then $\{T_n\}$ has a unique common fixed point.

Proof :

First we show that $\{x_n\}$ is cauehy sequence defined as $x_1 = T_1 x_0$, $x_2 = T_2 x_1$ ....

then by (2.3.1) we have

$$d(x_1,x_2) = d(T_1 x_0, T_2 x_1)$$

$$\leq \alpha\left\{\frac{d(x_0,T_1 x_0)\, d(x_1,T_2 x_1)}{d(x_0 x_1)}\right\} + \beta\{d(x,T_1 x_0) + d(x_1,T_2 x_1)\}$$

$$+ \gamma\{ d(x_0,T_2x_1) + d(x_1,T_1x_0)\} \quad + \delta\ d(x_0,x_1)$$

i.e. $$d(x_1 x_2) \leq \alpha\{ \frac{d(x_0,x_1)d(x_1,x_2)}{d(x_0,x_1)}\} + \beta\{d(x_0,x_1)+d(x_1,x_2)\}$$

$$+\gamma\{ d(x_0,x_2)+d(x_1,x_1)\}+\delta d(x_0,x_1)$$

$$\leq \alpha\ d(x_1,x_2)+\beta d(x_0 x_1)+\beta d(x_1 x_2)$$

$$+\gamma d(x_0,x_1)+\gamma d(x_1,x_2)+\delta d(x_0,x_1)$$

$$\leq \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}\ d(x_0,x_1) \qquad \ldots(2.3.2)$$

i.e $d(x_1,x_2) \leq rd(x_0,x_1)$ where $r = \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} < 1$

Again $d(x_2,x_3) = d(T_1x_2,\ T_2x_1)$

$$\leq \alpha\ \{ \frac{d(x_2,T_1x_2)d(x_1,T_2x_1)}{d(x_2,x_1)}\} + \beta\{d(x_2,T_1x_2)+d(x_1,T_2x_1)\}$$

$$+ \gamma\{ d(x_2,Y_2x_1)+d(x_1,T_1x_2)\} \quad + \delta\ d(x_2,x_1)$$

$$\leq \alpha\ \{ \frac{d(x_2,x_3)d(x_1,x_2)}{d(x_2,x_1)}\} + \beta\{d(x_2,x_3)+d(x_1 x_2)\}$$

$$+ \gamma\{ d(x_2,x_2)+d(x_1 x_3)\} \quad + \delta\ d(x_2,x_1)$$

$$\leq \alpha\ d(x_2,x_3)+\beta d(x_2,x_3)+\beta d(x_1 x_2)+\gamma d(x_1,x_3)+\delta d(x_2,x_1)$$

$$\leq \alpha\ d(x_2,x_3)+\beta d(x_2,x_3)+\beta d(x_1,x_2)+\gamma d(x_1,x_2)+\gamma d(x_2,x_3) +\delta d(x_2,x_1)$$

$$\leq \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma}\ d(x_1,x_2)$$

$$0 \leq \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} < 1 \implies r < 1$$

Thus $d(x_2,x_3) \leq \quad r\, d(x,x_2)$ ...(2.3.3)

from(2.3.2) and(2.3.3) we have

$d(x_2,x_3) \leq r^2 d(x_0,x_1)$

continueing the same process we have

$d(x_n,x_{n+1}) \leq r^n\, d(x_0,x_1)$ where $r = \dfrac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} < 1$

Further for any $p > 0$, we have

$$d(x_{n+p},x_n) \leq d(x_n,x_{n+1}) + \ldots\ldots + d(x_{n+p-1},x_{n+P})$$

$$\leq (r^n + r^{n+1} + \ldots\ldots + r^{n+p-1})\, d(x_0,x_1)$$

$$\leq \frac{r^n}{1-r}\, d(x_0,x_1) \qquad \ldots(2.3.4)$$

as $p \to \infty$, the bracketed quantity will be a sum of infinite G.P. with first r^n and common ratio r.

on letting $n \to \infty$ in(2.3.4) we have

$$d(x_{n+p},x_n) \to 0$$

Hence $\{x_n\}$ is a cauchy's sequence. Since X is complete. Then $\{x_n\}$ must be converge to some point u in X.

i.e. $\lim\limits_{n\to\infty} x_n = u$

Now we show that u is fixed point of T_n for fixed n, consider.

$$d(u,T_m u) \leq d(u,x_n) + d(x_n,T_m u)$$

$$\leq d(u,x_n) + d(T_n x_{n-1},T_m u)$$

$$\leq d(u,x_n) + r d(x_{n-1},u) \qquad \ldots(2.3.5)$$

Since $\lim_{n\to\infty} x_n = u$

therefore(2.3.5) becomes

$$d(u,T_m u) = 0$$

Thus $T_m u=u$ for all m

Hence u is common fixed point of $\{T_n\}$ n = 1,2.....

For uniqueness, let v is another fixed point such that $u \neq v$.

Consider

$$d(u,v) = d(T_i u, T_j v)$$

$$\leq \alpha\left\{\frac{d(u,T_i v)d(v,T_j v)}{d(u,v)}\right\} + \beta\{d(u,T_i u)+d(v,T_i v)\}$$

$$+ \gamma \{d(u,T_j v)+d(v,T_i v)\} + \delta\, d(u,v)$$

$$\leq (2\gamma+\delta)\, d(u,v)$$

$$< d(u,v)$$

which is contraction.

Thus u = v

Hence u is unique common fixed point of $\{T_n\}$.

This completes the proof.

Remark -

If we put $\alpha = 0 = \beta = \gamma$ in theorem (1) then we get the result of Baidyanath Ray [1].

Theorem : 2.4 :

Let $F = \{T_n\}$ is a family of maps each mapping a complete metric space (X,d) into itself such that

(i) $$d(T_p x, T_q y) \leq \alpha\left\{\frac{d(x,T_p x)d(y,T_q y)}{d(x,y)}\right\} + \beta\{d(x,T_p x)+d(y,T_q y)\}$$

$$+\gamma\{d(x,T_q y)+d(y,T_p x)\} + \delta\, d(x,y)$$

where $0 \leq \frac{\beta+\gamma+\delta}{1-\alpha-\beta-\gamma} < 1$, $(\beta+\gamma) < 1$, $(2\gamma+\delta) < 1$, $\gamma \geq 0$, for any $x \neq y \in X$.

(ii) Corresponding to each countable subfamily $\{T_i\}$ of F. There is a point x_0 in X such that any two consecutive members are distinct.

Then there is a unique fixed point common to each member of F.

Proof :

Consider a countable subfamily $\{T_i\}$ of F. apply theorem (1) we get a unique fixed point u common to $\{T_i\}$, i = 1, 2 .... suppose $T_p = F\setminus\{T_i\}$, and now consider T and T_i as countable subfamilies of F. By theorem (1) u' is common fixed point for T_p and T_i.

But u is unique fixed point for T_i which implies that

$$u = u'$$

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Hence u is common fixed point for the family F.

Hence the proof.

Recently Khan (1976) [12], Chatterjee (1981) [3], have discussed a number of interesting results related to the theorem of Bryant as follows :

Theorem (B) -

Let (X,d) be a complete metric space, and T be a self mapping on X such that

$d(T^n x, T^n y) \leq \alpha\, d(x,y)$; for all $x, y, \in X$,

$0 < \alpha < 1$ and $n \in N$; then T has a unique fixed point in X.

In (1988) Y.C.Paliwal [14] proved a theorem for a pair of mapping, which is the extension of the theorem of Jaggi and Bal Kishan Dass [11] as follows

Theorem (C) -

Let T_1 and T_2 be two continuous self mapping of a metric space (X,d) such that,

$$d(T_1^r x, T_2^s y) < \frac{\alpha\, d(x, T_1^r x)\; d(y, T_2^s y)}{d(x, T_2^s y) + d(y, T_1^r x) + d(x,y)} + \beta\, d(x,y)$$

for all $x, y \in X$, $x \neq y$, where $r > 0$, $s > 0$ are integers and α, β are non-negative real numbers such that $(\alpha + \beta) = 1$. If for some $x_0 \in X$, the sequence $\{x_n\}$

consisting of points.

$$x_{2n+1} = T_1{}^r x_{2n},\ x_{2n+2} = T_2{}^r x_{2n+1}$$

has a subsequence $\{ x_{n_k} \}$ converging to a point u; then T_1 and T_2 have a unique common fixed point u.

Following this formalism, we prove the following for sequence of continuous mapping in complete metric space X.

Our theoram :

Theorem : 2.5 :

Let $\{ T_n \}$ be the sequence of continus self mapping of complete metric space (X,d) such that any two distinct maps T_i, and T_j satisfies

$$d(T_i{}^p x, T_j{}^q y) \le \frac{a_1 [1+d(x,T_i{}^p x)][1+d(y,T_2{}^q y)]}{1+d(T_i{}^p x, T_j{}^q y)}$$

$$+ \frac{a_2 [d(x,T_i{}^p x) d(x,T_j{}^q y) + d(y,T_j^q y) d(y,T_j^p x)]}{d(x,T_j{}^q y) + d(y,T_i{}^q x)}$$

$$+ a_3\, d(x,y) - a_1 \qquad \ldots(2.5.1)$$

for all x, y $\in$ X, a_1, a_2, a_3 are positive real numbers such that $(a_1+a_2+a_3) < 1$ and p,q are positive integers, then the sequence $\{ T_n \}$ has unique common fixed point in X.

Proof -

For some $x_0 \in X$, define a sequence $\{x_n\}$ such that $x_{2n+1} = T_i^p x_{2n}$, $x_{2n+2} = T_j^q x_{2n+1}$, $n=0,1,2,\ldots$ then by(2.5.1) we have

$$d(x_1,x_2)=d(T_i^p x_0,T_j^q x_1) \leq \frac{a_1[1+d(x_0,T_i^p x_0)][1+d(x_1,T_j^q x_1)]}{1+d(T_i^p x_0, T_j^q x_1)}$$

$$+ \frac{a_2[d(x_0,T_i^p x_0)d(x_0,T_j^q x_1)+d(x_1,T_j^q x_1)d(x_1,T_i^p x_0)]}{d(x_0,T_j^q x_1)+d(x_1,T_i^p x_0)}$$

$$+ a_3\, d(x_0,x_1) - a_1$$

$$\leq \frac{a_1[1+d(x_0,x_1)][1+d(x_1,x_2)]}{1+d(x_1,x_2)}$$

$$+ \frac{a_2[d(x_0,x_1)d(x_0,x_2)+d(x_1x_2)d(x_1,x_1)]}{d(x_0,x_2)+d(x_1,x_1)}$$

$$+ a_3\, d(x_0,x_1) - a_1$$

$$\leq \frac{a_1[1+d(x_0,x_1) +d(x_1,x_2)+d(x_0,x_1)d(x_1,x_2)]}{1+d(x_1,x_2)}$$

$$+ \frac{a_2[d(x_0,x_1)\ d(x_0,x_2)]}{d(x_0,x_2)} + a_3 d(x_0,x_1) - a_1$$

$$\leq \frac{a_1[1+d(x_1,x_2)+d(x_0,x_1)\quad 1+d(x_1,x_2)\quad]}{1+d(x_1,x_2)}$$

$$+ a_2\, d(x_0,x_1) + a_3(x_0,x_1) - a_1$$

$$\leq \frac{a_1[\ \{1+d(x_1,x_2)\}\ \{1+d(x_0,x_1)\}\ ]}{\{1+d(x_1,x_2)\}}$$

$+ a_2 d(x_0,x_1)+a_3 d(x_0,x_1) - a_1$

$\leq a_1[1+d(x_0,x_1)]+a_2 d(x_0,x_1)+a_2 d(x_0,x_1)-a_1$

$\leq a_1+a_1 d(x_0,x_1)+a_2 d(x_0,x_1)+a_3 d(x_0,x_1)-a_1$

$\leq (a_1+a_2+a_3)d(x_0,x_1)$

$d(x_1,x_2) \leq rd(x_0,x_1)$ where $r = a_1+a_2+a_3 < 1$

continuing this process weget

$d(x_n,x_{n+1}) \leq r^n d(x_0,x_1)$

In general, for $m > n \in N$

$$d(x_n,x_m) \leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+ \dots + d(x_{m-1},x_m)$$

$$\leq r^n d(x_0,x_1)+r^{n+1}d(x_0,x_1)+ \dots + r^{m-1}d(x_0,x_1)$$

$$\leq (r^n+r^{n+1}+\dots+r^{m-1})d(x_0,x_1)$$

where $r^n, r^{n+1} \dots + r^{m-1}$ is G-p.

with common ratio r

$$\therefore \quad d(x_n,x_m) \leq \frac{r^n}{1-r} d(x_0,x_1)$$

$$\to 0 \quad \text{as } n \to \infty$$

Thus $\{x_n\}$ is cauchys sequence and from completeness of X it must be convergent and converges to some element inXi.e.

$$\lim_{n \to \infty} x_n = u \in X$$

Now we have to show that is fixed point T_i^p & T_j^q let $u = T_i^p$.

$$d(u,T_i{}^pu) \leq d(u,x_{2m+2}) + (x_{2n+2},T_i{}^pu) \leq d(u,x_{2n+2}) + d(T_i^pu,x_{2n+1})$$

Now $d(T_i{}^pu,T_j{}^qx_{2n+1}) \leq \dfrac{a_1[1+d(u,T_i{}^pu)][1+d(x_{2n+1},T_j^qx_{2n+1})]}{1+d(T_i{}^pu,T_j{}^qx_{2n+1})}$

$$+ \frac{a_2[d(u,T_i{}^pu)d(u,T_j{}^qx_{2n+1})+d(x_{m+1},T_j{}^qx_{2n+1})d(x_{2n+1}T_i^pu)]}{d(u,T_j{}^qx_{m+1})+d(x_{2n+1},T_i{}^pu)}$$

$$+ a_3d(u,x_{2n+1})-a_1$$

$$\leq \frac{a_1[1+d(u,T_i{}^pu)][1+d(x_{2n+1},x_{2n+2})]}{1+d(t_i{}^pu,x_{2n+2})}$$

$$+ \frac{a_2[d(u,T_i^pu)d(u,x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},T_i^pu)]}{d(u,x_{2n+2})+d(x_{2n+1},T_i{}^pu)}$$

$$+ a_3d(u,x_{2n+1}) - a_1$$

i.e. $d(u,T_i{}^pu) \leq d(u,x_{2n+2})$

$$+ \frac{a_2[d(u,T_i{}^pu)d(u,x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},T_i^pu)]}{d(u,x_{2n+2})+d(x_{2n+1},T_i{}^pu)}$$

$$+ a_3d(u,x_{2n+1}) - a_1$$

on letting $n \to \infty$

$$d(u,T_i^pu) \leq \frac{a_1[1+d(u,T_i{}^pu)][1+d(u,u)]}{1+d(t_i{}^pu,u)} + d(u,u)$$

$$+ \frac{a_2[d(u,T_i{}^pu)d(u,u)+d(u,u)d(u,T_i{}^pu)]}{d(u,u)+d(u,T_i{}^pu)}$$

$$+ a_3d(u,u) - a_1$$

$$\leq \frac{a_1[1+d(u,T_i^{p}u)]}{[1+d(T_i^{p}u,u)]} + 0 + \frac{a_2[0 + 0]}{[0+d(u,T_i^{p}u)]} + 0 - a_1$$

$\leq a_1 - a_1$

≤ 0

$\Rightarrow d(u,T_i^{p}u) = 0$

$\Rightarrow T_i^{p}u = u$

$\therefore$ u is fixed point of T_i^{p}.

Similarly we can show that u is a fixed point of T_j^{q}.

For uniqueness, let $v \neq u$ be another fixed point of T_i^{p} and T_j^{q}.

then $d(u,v) = d(T_i^{p}u, T_j^{q}v)$

$$\leq \frac{a_1[1+d(u,T_i^{p}u)][1+d(v,T_j^{q}v)]}{1+d(T_i^{p}u,T_i^{q}v)}$$

$$\leq \frac{a_2[d(u,T_i^{p}u)d(u,T_i^{q}u)+d(v,T_j^{q}v)d(v,T_i^{p}u)]}{d(u,T_j^{q}v)+d(v,T_i^{p}u)}$$

$$+ a_3 d(u,v) - a_1$$

$$\leq \frac{a_1}{1+d(u,v)} + a_2[0]+a_3 d(u,v) - a_1$$

$$\leq \frac{a_1+a_3 d(u,v)+a_3[d(u,v)]^2-a_1 d(u,v) - a_1}{1+d(u,v)}$$

$$\leq \frac{a_3 d(u,v)+a_3[d(u,v)]^2-a_1 d(u,v)}{1+d(u,v)}$$

$$[d(u,v)]^2 \leq \frac{a_3-a_1-1}{1-a_3} d(u,v)$$

let $d(u,v) \neq 0$

then it follows $d(u,v) = \frac{a_3-a_1-1}{1-a_3}$

$$= \frac{-(1-a_3)-a_1}{1-a_3}$$

$$\leq - k,$$

where $k = [\frac{(1-a_3)-a_1}{1-a_3}]$

which follows $d(u,v)$ 0

Hence it is contiadiction.

$\therefore$ $d(u,v) = 0$

$u = v$

$\therefore$ u is unique common fixed point of T_i^p and T_j^q.

This completes the proof.

A fixed point theorem : For non-expansive mapping.

Belluce and Kirk[2]proved their theorem on fixed point for a certain class of non-expansive mappings. Dunford N. and Schwartz [5] had given a useful note on linear operators. Robert H. Martin J. [15] have studied non linear operators and differential equantions in Banach spaces.

Following the work of Belluce and Kirk we prove a theorem on fixed point for non-expansive mappings in complete metric space as below.

Before it we consider a definition of non-expansive mappings in metric space.

Definition :

Let X be a metric space. A mapping $T : X \to X$ is said to be non-expansive if for each pair $x, y \in X$ and $x \neq y$ such that

$$d(Tx,Ty) \leq d(x,y)$$

we prove one lemma which we require for our theorem

Lemma : 2.6 :

Let T be a non-expansive mapping on a subset S of X, where S is compact and convex. Further G be a mapping on S defind by

$$G(x) = ax_0+b(Tx), \text{ for all } x \in S \qquad \ldots(2.6.1)$$

where $x_0 \in S$ is a fixed point in S, and a, b are two possitive numbers such that

$$(a + b) = 1$$

Then G has a unique fixed point in S.

Proof :

Since x_0, $Tx \in S$ and $a+b=1$, it follows from convexity of S that G maps S into itself.

Let $u, v \in S$, then we have from the definition of G and T.

$$\begin{aligned} d[G(u),G(v)] &= d[ax_0+b(Tu),\ ax_0+b(Tv)] \\ &= d[b(Tu),\ b(Tv)] \\ &= bd(Tu,\ Tv) \\ &\leq bd(u,v) \qquad \text{by def}^n \end{aligned}$$

Note that S is complete and $b<1$.

{ Because $S \subset X$ and $(a+b)=1$ }

Hence we conclude from the contraction mapping principle that, G has a unique fixed point in S, for every pair (a,b) of two positive numbers such that $(a + b) = 1$. and the proof is complete.

Main theorem is as follows :

Theorem : 2.7 :

Let X be a complete metric space and let, T be a non-expansive mapping of S into itself where S is compact and convex subset of complete metric space X. Then T has a fixed point in S.

Proof -

Let x_0 be a fixed point of S and $\{a_n\}$, $\{b_n\}$ be two real possitive sequences, such that,

$$(a_n+b_n)=1 \text{ and } a_n \to 0 \text{ as } n \to \infty.$$

For each pair of such sequences, there exists, by the above lemma [2.6] a unique fixed point x_n of G is S, where G is defined by (2.6.1)

Hence we have

$$x_n = G(x_n) = a_n x_0 + b_n (Tx_n).$$

$$\left[\begin{array}{l} x_1 = G(x_1) = a_1 x_0 + b_1 (Tx_1) \\ x_2 = G(x_2) = a_2 x_0 + b_2 (Tx_2) \end{array}\right]$$

. . . .

Further it implies that

$$\begin{aligned} x_n - Tx_n &= a_n x_0 + b_n (Tx_n) - Tx_n \\ &= a_n x_0 + (b_n - 1) Tx_n \\ &= a_n x_0 - a_n Tx_n \qquad \left\{\because a_n + b_n = 1\right\} \\ &= a_n (x_0 - Tx_n) \qquad \ldots (2.7.1) \end{aligned}$$

From (2.7.1) we conclude that $x_n - Tx_n \to 0$.

$$\left\{ \lim_{n \to \infty} a_n \, (x_0 - Tx_n) \quad 0 \quad \because \; a_n \to 0 \;\; \text{as } n \to \infty \right\}$$

This shows, by virtue of compactness of S, that there exists a subsequence of $\{ x_n \}$ converging to some $x \in S$, and this x is fixed point of T. This completes the proof

R E F E R E N C E S

1. Baidyanath Ray and A.P. Baisnal : on simultataneous fixed points, *Bull.Cal.Math.Soc. 65, 78-84 (1973)*.

2. Bellace and Kirk : Fixed point theorem for certain class of non-expensive mappings, *Proc. Amer.Math.Soc. (20) (1) (1969), 141-146.*

3. Chatterjee (1981) : *Bull.Cal.Math.Soc. 73, 325-327.*

4. Dass B.K., Gupta (1975) : *Indian J. pure. Appl Maths. 6(12), 1455-58.*

5. Dunford N. and J.T. Schwartz : Linear operators I, General theory Pure and Appl. Math. Vol.7, New York, (1958).

6. Edelstein M. (1962) : *Jr.Lond.Math.Soc. 37, 74-79.*

7. Fisher B. (1975) : A fixed point theorem : *Math.Mag. 48, 223.*

8. Fisher B. (1975) : Fixed point mappings, : *Atti.Della,Acad.Naz.De Lincei, LIX, Fasc.5.*

9. Haimabati Chatterji : Fixed points for a pair of mappings : *Ind.Jr.pure and Appl.Maths. 10(7) : 886-889, July (1979).*

10. Jaggi D.S. (1977) : Some unique fixed points *Bull.Cal.Math.Soc. 60-71.*

11. Jaggi and Bal Kishan Dass : A extension of

Banach fixed point theorem through a rational expression. *Bull.Cal.Math.Soc. 72, 261-262 (1980).*

12. Kannan R. : Some Results on fixed points. *Bull.Cal.Math.Soc.Vol. 60 No (12), 1968.*

13. Khan M.S. (1976) : Estrao da. Rend. lst di *Mateum. Univ. Triesti (VII) fasc 1,1.*

14. Paliwal Y.C. : Fixed point theorems in metric spaces. *Bull.Cal.Maths.Soc. 80, 205-208 (1988).*

15. Robert H. Martin : Jr. Non-linear operators and differential equation in Banach spaces. John Willey and Sons. (1976).

16. Sharma and A.K.Yuel : A unique fixed point theorem in metric space; *Bull. Cal. Math. Soc. 76, 153-156 (1984).*

17. Singh S.P. : On fixed point theorem in metric space, *Ann.Soc.Sci.Brux. 85, 117-123.*