C H A P T E R - I

I N T R O D U C T I O N

CHAPTER - I

INTRODUCTION

1.1 Integral Transforms :

Many functions in analysis can be expressed as Lebesgue integrals or improper Riemann integrals of the form

$$F(S) = \int_{0 \text{ or } -\infty}^{\infty} K(S,x)\, F(x)\,dx \qquad \ldots (1.1\text{-}1)$$

A function F defined by an equation of this type (in which S may be real or complex) is called an integral transform of F. The function K which appears in the integrand is known as the kernel of the transformation. It is assumed that the infinite integral in the equation (1.1-1) is convergent.

Integral transformations are employed very extensively in both pure and applied mathematics. Certain boundary value problems and certain types of integral equations can be solved with the help of different form of kernel K(S, x) and the range of integration. The important aspect of the integral transformation is its inversion theorem.

The problems involving several variables can be solved by applying integral transformations successively with regard to several variables.

There are several problems which can be solved by the repeated application of Laplace and K transforms. The K transformations can be extended to certain generalized functions by combining some of the techniques used in generalizing the Laplace and Hankel transformations.

The K type transform of a suitably restricted function f(t) is defined by the integral

$$F(S) = \int_0^\infty f(t)\ \sqrt{st}\ K_\mu\ (st)dt$$

C.S.Meijer was the first to investigate the K transformation. In 1940, he [10] formulated the transform,

$$K_\nu(f) = F(S) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sqrt{st}\ K_\nu(st)dt \quad \ldots(1.1\text{-}2)$$

which is known as the Meijer-Bessel transform.

If we construct an integral transform for which the kernel is the product of two certain type kernels then such integral transform is termed as two dimensional integral transform.

P.N.Rathie studied two dimensional Meijer-Bessel transform.

If f(x,y) is a suitably restricted function on $0 < x < \infty$, $0 < y < \infty$ then its Meijer-Bessel type transform F(p,q) is definedby an integral

$$F(p,q) = K_{\nu,\mu}(f) = \frac{2}{\pi}\, pq \int_0^\infty \int_0^\infty \sqrt{pqxy}\, K_\nu(px)\ X$$

$$K_\mu\ (qy) f(x,y)\ dxdy$$

$$Re(p,q) > 0. \qquad \ldots (1.1\text{-}3)$$

where $K_\alpha(z)$ is the modified Bessel function of third kind of order α and α is real.

The inversion formula for the above Meijer-Bessel transform is given by

$$F(x,y) = \lim_{\substack{R \to \infty \\ R' \to \infty}} \frac{1}{2\pi i^2} \int_{6-iR}^{6+iR} \int_{6-iR'}^{6+iR'} F(p,q)\ X$$

$$\left(\frac{x}{p}\right)^{1/2} \left(\frac{y}{q}\right)^{1/2} I_\nu(px)\ I_\mu(qy)\, dpdq \qquad \ldots (1.1\text{-}4)$$

In the present work, the attempt has been made to study some classical as well as the distributional properties of the transform defined in (1.1.3).

1.2 Some Classical Results

Let Z be a complex variable and μ, a fixed complex number. $I_\mu(z)$ denotes the modified Bessel function of first kind of order μ and $K_\mu(z)$, the modified Bessel function of third kind of order μ [7].

These functions are analytic on the Z-plane except possibly for branch points at Z = 0 and Z = ∞. The series and asymptotic expansions of $I_\mu(z)$ and $K_\mu(z)$ are as follows:

$$I_\mu(z) = \sum_{K=0}^{\infty} \frac{z^{2K+\mu}}{K!\, 2^{2K+\mu}\, \Gamma(K+\mu+1)}, \text{ for arbitrary } \mu \quad \ldots(1.2\text{-}1)$$

Also for any $\mu \neq 0, 1, 2 \ldots.$

$$K_\mu(z) = \frac{\pi}{2\sin \mu\pi} \left[\sum_{K=0}^{\infty} \frac{z^{2K-\mu}}{K!\, 2^{2K-\mu}\, \Gamma(K+1-\mu)} - \sum_{K=0}^{\infty} \frac{z^{2K+\mu}}{K!\, 2^{2K+\mu}\, \Gamma(K+1+\mu)} \right] \quad \ldots (1.2\text{-}2)$$

If n = 0, 1, 2 ....

$$K_n(z) = \frac{1}{2} \sum_{K=0}^{n-1} \frac{(-1)^K (n-K-1)!}{K!} \left(\frac{z}{2} \right)^{2K-n} + (-1)^n \cdot \sum_{K=0}^{\infty} \frac{(z/2)^{n+2K}}{(K\ (n+K)} \left[-\log \left(\frac{e}{2} \right)^{\nu z} + \frac{1}{2} \left(\sum_{p=1}^{K} \frac{1}{p} + \sum_{p=1}^{n+K} \frac{1}{p} \right) \right] \quad \ldots (1.2\text{-}3)$$

where γ is an Euler's constant.

For any fixed $\epsilon > 0$ and for $|Z| \to \infty$

$$\sqrt{Z}\, K_\mu(z) = \sqrt{\frac{\pi}{2}}\, e^{-z}\, [1 + 0(|z|^{-1})],$$

$$-\frac{3\pi}{2} + \epsilon < \arg z < \frac{3\pi}{2} - \epsilon \qquad \ldots (1.2\text{-}4)$$

$$\sqrt{Z}\, I_\mu(z) = \frac{1}{\sqrt{2\pi}} (e^{z} + ie^{-z+i\mu\pi})[1+0(|z|^{-1})], -$$

$$-\frac{\pi}{2} + \epsilon < \arg Z < \frac{3\pi}{2} - \epsilon \qquad \ldots (1.2\text{-}5)$$

$$\sqrt{Z}\, I_\mu(z) = \frac{1}{\sqrt{2\pi}} (e^{z} - ie^{-z-i\mu\pi})[1+0(|z|^{-1})],$$

$$-\frac{3\pi}{2} + \epsilon < \arg Z < \frac{\pi}{2} - \epsilon \qquad \ldots (1.2\text{-}6)$$

uniformly for arg Z.

If $S_{\mu,t} = t^{-\mu-1/2}\, D_t\, t^{2\mu+1}\, D_t\, t^{-\mu-1/2}$ ... (1.2-7)

where $D_t = \frac{\partial}{\partial t}$ then for any $K = 0, 1, 2 \ldots$

$$S^{k}_{\mu,t} \sqrt{st}\, K_\mu(st) = s^{2k} \sqrt{st}\, K_\mu(st) \qquad \ldots (1.2\text{-}8)$$

$$S^{k}_{\mu,t} \sqrt{st}\, I_\mu(st) = s^{2k} \sqrt{st}\, I_\mu(st) \qquad \ldots (1.2\text{-}9)$$

We also use the following result :

$$\int t I_\mu(zt)\, K_\mu(st)\,dt = \frac{t}{z^2 - s^2}\left[\, z\, I_{\mu+1}(zt) K_\mu(st) + s\, I_\mu(zt)\, K_{\mu+1}(st) \,\right] \quad \ldots (1.2\text{-}10)$$

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