CHAPTER - THREE

3.1)

3.1.1 Definition -

The conventional Hankel transformation $h_{\mu,\lambda,\alpha}$ for $\mu > -\frac{1}{2}$ is defined as

$$\phi(y) = (h_{\mu,\lambda,\alpha}\phi)(x) = \lambda \int_0^\infty (xy)^{\lambda-1/2} J_\mu(x^\lambda y^\lambda)\phi(x)\,dx.$$

3.1.2. Now we define two linear differential operators $N_{\mu,\lambda,\alpha}$ and $M_{\mu,\lambda,\alpha}$ and one linear integral operator $N^{-1}_{\mu,\lambda,\alpha}$ as follows

1] $N_{\mu,\lambda,\alpha}\,\phi(x) = x^{\lambda\mu+1/2} D_x x^{-\lambda\mu-\lambda+1/2}\phi(x)$

2] $M_{\mu,\lambda,\alpha}\phi(x) = x^{-\lambda\mu-\lambda+1/2} D_x x^{\lambda\mu+1/2}\phi(x)$

3] $N^{-1}_{\mu,\lambda,\alpha}\,\phi(x) = x^{\lambda\mu+\lambda-1/2} \int_\infty^x x^{-\lambda\mu-1/2}\phi(t)\,dt$

[3]

3.3.2 Theorem 3.2.1 -

For $\mu > -1/2$, the conventional Hankel transformation $h_{\mu,\lambda,\alpha}$ is a continuous linear mapping from the space $(H_{\mu,\lambda,\alpha,A})$ to $(H^{2\alpha,(2e)^{2\alpha}B^2}_{\mu,\lambda})$

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Proof -

Let P be a bounded set in $H_{\mu,\lambda,\alpha,A}$.

Let $\phi \in P$. Then ϕ satisfies the inequality

$$\left| x^{m\lambda} \, (x^{1-2\lambda}D)^{k} \, x^{-\lambda\mu-+1/2} \, \phi(x) \right|$$

$$< \; A_{k\delta} \; (B + \delta)^{m\lambda} \; (m\lambda)^{m\lambda\alpha}.$$

for m, k = o, 1, 2 ....

where constant $A_{k\delta}$ is independent of ϕ,

Let $\phi(y) = (h_{\mu,\lambda,\alpha}\phi)(x)$

Then for any pair of non-negative integers m and k and using the result

$$N_{\mu+k+m-1,\lambda,\alpha} \text{----} N_{\mu,\lambda,\alpha} \; x^{\lambda k} \; \phi(x) =$$

$$x^{\lambda k} \; N_{\mu+m-1,\lambda,\alpha} \text{----} N_{\mu,\lambda,\alpha}\phi(x).$$

We have,

$$(-y)^{m\lambda} \; N_{\mu+k-1,\lambda,\alpha} \text{----} N_{\mu,\lambda,\alpha} \; \phi(y) =$$

$$= h_{\mu+m+k,\lambda,\alpha} \left\{ (-x)^{\lambda k} \; N_{\mu+m-1,\lambda,\alpha} \text{----} N_{\mu,\lambda,\alpha}\phi(x) \right\}$$

$$= \int_0^\infty (-x)^{\lambda k} \quad N_{\mu+m-1,\lambda,\alpha} \text{----} N_{\mu,\lambda,\alpha}\phi(x) \;\; (xy)^{\lambda-\frac{1}{2}} \, J_{\mu+m+k}(x^{\lambda}y^{\lambda}) \, dx. \qquad \ldots (1)$$

Now we have two results which are true by an induction on k and m. They are as follows :

$$N_{\mu+k-1,\lambda,\alpha} \cdots N_{\mu,\lambda,\alpha}\, \not{\phi}(y) =$$

$$= y^{\lambda\mu+\lambda(k+1)-1/2}\, (y^{1-2\lambda}D)^k\, y^{-\lambda\mu-\lambda+1/2}\, \not{\phi}(y) \quad \ldots (2.)$$

and

$$N_{\mu+m-1,\lambda,\alpha} \cdots N_{\mu,\lambda,\alpha}\, \phi(x)$$

$$= x^{\lambda\mu+\lambda(m+1)-1/2}\, (x^{1-2\lambda}D)^k x^{-\lambda\mu-\lambda+1/2}\, \phi(x) \quad \ldots (3)$$

Therefore from (1) we have

$$(-y)^{m\lambda}\, y^{\lambda\mu+\lambda(k+1)-1/2}\, (y^{1-2\lambda}D)^k\, y^{-\lambda\mu-\lambda+1/2}\, \not{\phi}(y)$$

$$= \int_0^\infty (-x)^{\lambda k}.\, x^{\lambda\mu+\lambda(m+1)-1/2}\, (x^{1-2\lambda}D)^k\, x^{-\lambda\mu-\lambda+1/2}\, \phi(x)$$

$$\times (xy)^{\lambda-1/2}\, J_{\mu+m+k}(x^\lambda y^\lambda)\, dx.$$

$$\left| y^{m\lambda}(y^{1-2\lambda}D)^k\, y^{-\lambda\mu-\lambda+1/2}\, \not{\phi}(y) \right|$$

$$= \int_0^\infty \frac{x^{\lambda k+\lambda\mu+\lambda(m+1)-1/2+\lambda-1/2}}{y^{\lambda\mu+\lambda k+\lambda-1/2-\lambda+1/2}}\, (x^{1-2\lambda}D)^k\, x^{-\lambda\mu-\lambda+1/2}\, \phi(x)$$

$$\times J_{\mu+m+k}(x^\lambda y^\lambda)\, dx$$

$$< \int_0^\infty x^{2\lambda\mu+2\lambda k-1+\lambda(m+2)} (x^{1-2\lambda}D)^k x^{-\lambda\mu-\lambda+1/2} \phi(x) \times$$

$$\times \frac{J_{\mu+m+k}(x^\lambda y^\lambda)}{(xy)^{\lambda\mu+\lambda k}} dx.$$

Taking a positive integer ν greater than $2\lambda\mu-1$ we write.

$$\left| y^{m\lambda}(y^{1-2\lambda}D)^k y^{-\lambda\mu-\lambda+1/2} \phi(y) \right|$$

$$< \int_0^\infty x^{\nu+2\lambda k+\lambda(m+2)} (x^{1-2\lambda}D)^k x^{-\lambda\mu-\lambda+1/2} \phi(x)$$

$$\times \frac{J_{\mu+m+k}(x^\lambda y^\lambda)}{(x.y)^{\lambda\mu+\lambda k}}$$

Since the quotient

$\dfrac{J_{\mu+m+k}(x^\lambda y^\lambda)}{(xy)^{\lambda\mu+\lambda k}}$ is bounded by

say $A'_{k\delta}$.

$\therefore$ By definition of space $H_{\mu,\lambda,\alpha,A}$ since

$\phi \in H_{\mu,\lambda,\alpha,A}$. We have

$$\left| y^{m\lambda}(y^{1-2\lambda}D)^k y^{-\lambda\mu-\lambda+1/2} \phi(y) \right|$$

$$< A_{k\delta} [1\phi + (B+\delta)^{\nu+1+2\lambda k+\lambda(m+2)} (\nu+1+2\lambda k+\lambda(m+2))^{(\nu+1+2\lambda k+\lambda(m+2))\alpha}$$

$$< A_{k\delta\epsilon} (B^2 + \delta^1)^k (1+\epsilon')^k (2ek)^{2\alpha k}$$

By [2]

$$< A_{k\delta} \left\{ (2e)^{2\alpha} B^2 + \delta'' \right\}^k k^{2\alpha k} \qquad \ldots (4)$$

Here the constants $A_{k\delta}$ in independent of ϕ in P.

Also $\Phi(y) = (h_{\mu,\lambda,\alpha}\phi)(x)$ lies in the space $H_{\mu,\lambda}^{2\alpha, (2e)^{2\alpha}B^2}$

From (4) $\phi \longrightarrow h_{\mu,\lambda,\alpha}\phi$ maps a bounded set in $H_{\mu,\lambda,\alpha,A}$ into a bounded set in $H_{\mu,\lambda}^{2\alpha, (2e)^{2\alpha}B^2}$

Hence the mapping $h_{\mu,\lambda,\alpha}$ is continuous.

3.3 The generalised Hankel Transformation -

In this section μ lies in the interval $-\frac{1}{2} < \mu < \infty$.

3.3.1 The generalised Hankel transformation $h'_{\mu,\lambda,\alpha}$ is defined on the dual spaces $H'_{\mu,\lambda,\alpha,A}$ and $(H^{B,C}_{\mu,\lambda,\alpha,B})'$

For arbitrary $f \in H'_{\mu,\lambda,\alpha,A}$

We define $F = h'_{\mu,\lambda,\alpha} f$, by

$$< F, \Phi > = < f, \phi >$$

where $F = h'_{\mu,\lambda,\alpha} f$.

$$\Phi = h_{\mu,\lambda,\alpha}\, \phi$$

$\phi \in H_{\mu,\lambda,\alpha,A}$

$f \in H'_{\mu,\lambda,\alpha,A}$

OR we can also define it as

$$< h'_{\mu,\lambda,\alpha} f, \phi > = < f, h_{\mu,\lambda,\alpha} \phi > .$$

We shall prove the following theorem.

Theorem 3.3.1 -

For $\mu > -1/2$, the generalised Hankel transformation $h'_{\mu,\lambda,\alpha}$ is a continuous linear mapping from the dual space $(H^{2\alpha,(2e)^{2\alpha}B^2}_{\mu,\lambda})'$ to $(H_{\mu,\lambda,\alpha,A})'$

Proof -

Let $h_{\mu,\lambda,\alpha}$ be a continuous linear mapping from $(H_{\mu,\lambda,\alpha,A})$ to $(H^{2\alpha,(2e)^{2\alpha}B^2}_{\mu,\lambda})$
[This is proved previously]

Now F is a member of $H'_{\mu,\lambda,\alpha,A}$.

For

Let $\phi, \psi \in H_{\mu,\lambda,\alpha,A}$ and α, β be any complex numbers.

Then

$$< F, \alpha \phi + \beta \psi > \;=\; < f, \alpha \phi + \beta \psi >$$

$$= < f, \alpha \phi > + < f, \beta \psi >$$

$$= \alpha < f, \phi > + \beta < f, \psi >$$

$$= \alpha < F, \phi > + \beta < F, \psi >$$

Thus F is a linear functional on $H_{\mu,\lambda,\alpha,A}$.

For continuity -

Let $\{\phi_\gamma\}$ converges to zero in $H_{\mu,\lambda,\alpha,A}$. Then as $\gamma \to \infty$.

$h_{\mu,\lambda,\alpha} \phi_\gamma \longrightarrow 0$ [Since $h_{\mu,\lambda,\alpha}$ is continuous]

$$\therefore < F, \phi_\gamma > = < h'_{\mu,\lambda,\alpha} f, \phi_\gamma >$$

$$= < f, h_{\mu,\lambda,\alpha} \phi_\gamma >$$

$$\longrightarrow 0 \quad \text{as } \gamma \to \infty$$

[since f is continuous]

$\therefore$ F is a continuous functional on $H_{\mu,\lambda,\alpha,A}$.

Thus F is a continuous linear functional on the space $H_{\mu,\lambda,\alpha,A}$.

$\therefore$ By definition F is a member of the dual space $H'_{\mu,\lambda,\alpha,A}$.

Now we shall prove the linearity and continuity of the mapping $h'_{\mu,\lambda,\alpha} \longrightarrow \phi$

For Linearity -

Let f and g $\in$ $(H^{2\alpha,(2e)^{2\alpha}B^2}_{\mu,\lambda})'$ and α, β be any two complex numbers. Then

$$< h'_{\mu,\lambda,\alpha}(\alpha f+\beta g), \phi > = < \alpha f + \beta g, h_{\mu,\lambda,\alpha}\phi >$$

$$= \alpha < f, h_{\mu,\lambda,\alpha}\phi > + \beta < g, h_{\mu,\lambda,\alpha}\phi >$$

$$= \alpha < h'_{\mu,\lambda,\alpha}f, \phi > + \beta < h'_{\mu,\lambda,\alpha}g, \phi >$$

$$= < \alpha h'_{\mu,\lambda,\alpha}f, \phi > + < \beta h'_{\mu,\lambda,\alpha}g, \phi >$$

$$= < \alpha h'_{\mu,\lambda,\alpha}f + \beta h'_{\mu,\lambda,\alpha}g , \phi >$$

Thus $h'_{\mu,\lambda,\alpha}$ is linear.

For continuity -

Let f_ν converges to zero in $(H^{2\alpha,(2e)^{2\alpha}B^2}_{\mu,\lambda})'$

Then

$$< h'_{\mu,\lambda,\alpha}f_\nu, \phi > = < f_\nu, h_{\mu,\lambda,\alpha}\phi > .$$

$$\longrightarrow 0 \text{ as } \nu \longrightarrow \infty.$$

$\therefore$ $h'_{\mu,\lambda,\alpha}$ is continuous.

Thus $h'_{\mu,\lambda,\alpha}$ is a continuous linear mapping from

$(H^{2\alpha,\ (2e)^{2\alpha}B^2}_{\mu,\lambda})'$ to $(H_{\mu,\lambda,\alpha,A})'$.

3.4 Non-Trivility -

3.4.1 The non-triviality of the space $H_{\mu,\lambda,\alpha}$-

For any real number α, the space $H_{\mu,\lambda,\alpha}$ is non-trivial.

Case (1) - Let $\alpha > 0$

Let ϕ be a smooth function with compact support on $(0, \infty)$.

The Taylor's expansion of ϕ near origin is

$$\phi(x) = x^{-\lambda\mu-\lambda+1/2} (a_o + a_2x^2 \ldots a_{2k} x^{2k} + R_{2k}(x)) \ldots (1)$$

where $k = 0, 1, 2 \ldots$

and

$$a_{2k} = \lim_{x \to 0+} \frac{(x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x)}{2^k \, k!}$$

and $R_{2k}(x) = 0(x^{2k+2})$

that i.e. $R_{2k}(x)$ is a function of slow growth.

Let $L = \operatorname{Sup} \{ x / x \in \text{support of } \phi \}$ Then from (1) we can write

$$\left| x^{m\lambda} (x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right|$$

$$< \sup_{0 < x < L} \left| x^{m\lambda} (x^{1-2\lambda} D)^k \; x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right|$$

Let $\sup_{0 < x < L} \left| (x^{1-2\lambda} D)^k \; x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right| = C_{\phi k}$

Then

$$\left| x^{m\lambda} \, (x^{1-2\lambda} D)^k \; x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right|$$

$$< C_{\phi k} \cdot \frac{L^{m\lambda}}{A^{m\lambda} \, m^{m\lambda\alpha}} \quad x \, A^{m\lambda} \, m^{m\lambda\alpha} \qquad \ldots (2)$$

Let $C = \max \left\{ \frac{L}{A}, \left(\frac{L}{A2^{\alpha}} \right)^2, \ldots \left(\frac{L}{Ak_o^{\alpha}} \right)^{k_o} \right\}$

where $K_o = \left[\frac{L}{A} \right]^{1/\alpha} + 1$

where $\left[\frac{L}{A} \right]$ denotes the greatest integer not exceeding $\frac{L}{A}$

Clearly $\left(\frac{L}{AK^{\alpha}} \right) \leq 1$ iff $k \geq \left(\frac{L}{A} \right)^{1/\alpha}$

$$\therefore \left| x^{m\lambda} (x^{1-2\lambda} D)^k \; x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right|$$

$$< C_{\phi k} \cdot C \cdot A^{m\lambda} \, m^{m\lambda\alpha}$$

$$< C \cdot C_{\phi k} \, A^{m\lambda} \, m^{m\lambda\alpha}$$

This implies by definition that $\phi \in H_{\mu,\lambda,\alpha}$

Thus $H_{\mu,\lambda,\alpha}$ is non-trivial.

Case (2) -

If $\alpha = 0$. Then we know that, for $\alpha = 0$

$$H_{\mu,\lambda,0,A} \subseteq B_{\mu,A}$$

and since $B_{\mu,A}$ is non-trivial.

$\therefore$ $H_{\mu,\lambda,0}$ is non trivial.

Thus the space $H_{\mu,\lambda,\alpha}$ is non-trivial for any real number α.

3.4.2 For any $\beta > 0$, the space $H^{\beta}_{\mu,\lambda}$ is non-trivial.

Case (1) $\mu > -1/2$

The conventational Hankel transformation $h_{\mu,\lambda,\alpha}$ maps the space $H_{\mu,\lambda,\alpha,A}$ to the space $H^{2\alpha(2e)^{2\alpha}B^2}_{\mu,\lambda}$,

$h_{\mu,\lambda,\alpha}$ is defined as

$$\tilde{\phi}(y) = (h_{\mu,\lambda,\alpha}\phi)(x) = \lambda \int_0^{\infty} (xy)^{\lambda-1/2} J_\mu(x^\lambda y^\lambda)\ \phi(x)\ dx$$

Its inverse is defined as

$$h^{-1}_{\mu,\lambda,\alpha}\tilde{\phi} = \lambda \int_0^{\infty} (xy)^{\lambda-1/2} J_\mu(x^\lambda y^\lambda)\ \tilde{\phi}(y)\ dy$$

Note that

for $\mu > -1/2$, $h_{\mu,\lambda,\alpha} = h^{-1}_{\mu,\lambda,\alpha}$

Hence the mapping $h_{\mu,\lambda,\alpha}$ is one-to-one for $\mu > -1/2$.

Since $H_{\mu,\lambda,\alpha}$ is non-trivial. Therefore the space $H^{\beta}_{\mu,\lambda}$ is non-trivial.

Case (2) - $\mu < -\frac{1}{2}$

We define

$$h_{\mu,m,\lambda,\alpha}\phi(x) = (-1)^m\, y^{-m\lambda}\, h_{\mu+m,\lambda,\alpha} \left\{ N_{\mu+m-1,\lambda,\alpha} \text{----} \right.$$

$$\left. \text{----}\; N_{\mu,\lambda,L}\;\; \phi(x) \right\}$$

and

$$h^{-1}_{\mu,m,\lambda,\alpha}\,\Phi(y) = (-1)^m\, N^{-1}_{\mu,\lambda,\alpha} \text{----} N^{-1}_{\mu+m-1,\lambda,\alpha} \cdot y^m\, \Phi\,(y)$$

where m is any positive integer greater than $-\mu - \frac{1}{2}$

Then by applying $h_{\mu,m,\lambda,\alpha}$ instead of $h_{\mu,\lambda,\alpha}$, we can say that $H^{\beta}_{\mu,\lambda}$ is non-trivial for $\mu < -\frac{1}{2}$

Thus $H^{\beta}_{\mu,\lambda}$ is non-trivial.

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