CHAPTER - I

PRELIMINARIES

This chapter consists of all the basic definitions and results which will be useful in the second chapter.

1.1 SOME CATEGORICAL NOTIONS:

Definition (1.1.1)

A category is a class $\mathcal{A}$ together with a class μ which is a disjoint union of the form

$$\mu = \bigcup_{(A,B) \in \mathcal{A} \times \mathcal{A}} [A,B]_{\mathcal{A}}$$

To avoid logical difficulties we postulate that each $[A,B]_{\mathcal{A}}$ is a set (possibly void. When there is danger of no confusion we shall write $[A,B]$ in place of $[A,B]_{\mathcal{A}}$). Furthermore, for each triple (A,B,C) of members of $\mathcal{A}$ we are to have, a function from $[B,C] \times [A,B]$ to $[A,C]$. The image of the pair (β, α) under this function will be called the composition of β by α, and will be denoted by $\beta\alpha$. The composition functions are subject to two axioms.

(i) Associativity : Whenever the compositions make sense we have $(\gamma\beta)\alpha = \gamma(\beta\alpha)$.

(ii) Existence of identities : For each $A \in \mathcal{A}$ we have an element $1_A \in [A,A]$ such that $1_A.\alpha = \alpha$ and $\beta.1_A = \beta$ whenever the composition make sense.

The members of $\mathcal{A}$ are called objects and the members of μ are called morphisms. If $\alpha \in [A,B]$ we shall call A the domain of α and B the codomain, and we shall say "α is a morphism from A to B". This last statement is represented by "$\alpha : A \longrightarrow B$", or sometimes "$A \xrightarrow{\alpha} B$".

Definition (1.1.2)

A category $\mathcal{A}'$ is called a subcategory of a category $\mathcal{A}$ under the following conditions :

(i) $\mathcal{A}' \subseteq \mathcal{A}$

(ii) $[A,B]_{\mathcal{A}'} \subseteq [A,B]_{\mathcal{A}}$ for all $(A,B) \in \mathcal{A}' \times \mathcal{A}$

(iii) The composition of any two morphisms in $\mathcal{A}'$ is the same as their composition in $\mathcal{A}$

(iv) 1_A is the same in $\mathcal{A}'$ as in $\mathcal{A}$ for all $A \in \mathcal{A}'$

If furthermore $[A,B]_{\mathcal{A}'} = [A,B]_{\mathcal{A}}$ for all $(A,B) \in \mathcal{A}' \times \mathcal{A}'$ we say that $\mathcal{A}'$ is a full sub-category of $\mathcal{A}$.

Definition (1.1.3) :

A morphism $f : A \longrightarrow B$ is an isomorphism if there exists a morphism $g : B \longrightarrow A$ such that $gf = 1_A$ and $fg = 1_B$.

If there exists an isomorphism $f : A \longrightarrow B$, then A and B are said to be isomorfic objects.

Definition (1.1.4) :

A morphism $f : A \longrightarrow B$ is called a monomorphism if $fg = fh$ implies that $g = h$ for all pairs of morphisms f, g with codomain A.

Definition (1.1.5) :

A morphism $f : A \longrightarrow B$ is called an epimorphism if $gf = hf$ implies that $g = h$ for all pairs of morphisms f, g with domain B.

Remark :

An isomorphism is both, a monomorphism and an epimorphism. The converse is not true i.e. a morphism which is both monomorphism and epimorphism may not be an isomorphism.

Definition (1.1.6) :

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A covariant functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an assignment of an object $F(A) \in \mathcal{B}$ to each object $A \in \mathcal{A}$ and a morphism $F(\alpha) : F(A) \longrightarrow F(A')$ to each morphism $\alpha : A \longrightarrow A'$ in $\mathcal{A}$ subject to the following conditions :

(i) Preservation of composition: If $\alpha'\alpha$ is defined in $\mathcal{A}$ then $F(\alpha'\alpha) = F(\alpha').F(\alpha)$.

(ii) Preservation of identities : For each $A \in \mathcal{A}$ we have $F(1_A) = 1_{F(A)}$.

We shall call the category $\mathcal{A}$ the domain of F and the category $\mathcal{B}$ the codomain and we shall say that T has values in $\mathcal{B}$

Remarks (1) :

(1) If we replace the conditions $\alpha : A \longrightarrow A'$ implies that $F(\alpha) : F(A) \longrightarrow F(A')$ and $F(\alpha'\alpha) = F(\alpha').\ F(\alpha)$ by the conditions $\alpha : A \longrightarrow A'$ implies that $F(\alpha) : F(A') \longrightarrow F(A)$ and $F(\alpha'\alpha) = F(\alpha).\ F(\alpha')$ in the above definition we obtain the definition of a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$

(2) The unqualified term "functor" will usually mean covariant functor.

The forgetful functor $F : \mathcal{G} \longrightarrow \mathcal{S}$ from the category of abelian groups to the category of sets is the functor which forgets the abelian group structure on the objects of

That is, if G is an abelian group, then F(G) is the underlying set G of G and if α is a group morphism, $F(\alpha) = \alpha$.

Definition (1.1.7) :

An object $u \in \mathcal{A}$ is called an initial object of $\mathcal{A}$ if the set $[u,A]_{\mathcal{A}}$ contains precisely one morphism for each $A \in \mathcal{A}$

Definition (1.1.8) :

Given two morphisms $f,\ g : A \longrightarrow B$, a morphism

Definition (1.1.8) :

Given two morphisms f, g : A ⟶ B, a morphism μ : B ⟶ K is called a coequalizer for f and g if the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "\mu"] \\
B \arrow[r, "\mu"'] & K
\end{tikzcd}$$

commutes, and if, whenever there is μ' : B ⟶ K' making the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\mu'"] \\
B \arrow[r, "\mu'"'] & K'
\end{tikzcd}$$

commutative, then there exists a unique morphism γ : K ⟶ K' such that

$$\begin{tikzcd}
B \arrow[r, "\mu"] \arrow[dr, "\mu'"'] & K \arrow[d, "\gamma"] \\
 & K'
\end{tikzcd}$$

commutes.

Definition (1.1.9) :

Let $\{A_\alpha\}_{\alpha \in I}$ be a family of objects in a category $\mathcal{A}$. A coproduct for this family is a family of morphisms $\{f_\alpha : A_\alpha \longrightarrow A\}$ with the property that, for any $\{g_\alpha : A_\alpha \longrightarrow B\}$ there is a unique morphism $f : A \longrightarrow B$ such that the following diagram commutes for every $\alpha \in I$:

$$\begin{tikzcd}
A_\alpha \arrow[r, "f_\alpha"] \arrow[dr, "g_\alpha"'] & A \arrow[d, "f"] \\
 & B
\end{tikzcd}$$

Definition : (1.1.10) :

Let I be a directed set and $\mathcal{A}$ is a category. A direct system in $\mathcal{A}$ with index set I is a functor $F : I \longrightarrow \mathcal{A}$

That is, for each $\alpha \in I$, there is an object F_α and whenever $\alpha, \beta \in I$ satisfy $\alpha \leq \beta$, there is a morphism $\phi_{\beta\alpha} : F_\alpha \longrightarrow F_\beta$ such that

(i) $\phi_{\alpha\alpha} : F_\alpha \longrightarrow F_\alpha$ is the identity for every $\alpha \in I$;

(ii) If $\alpha \leq \beta \leq \gamma$, there is a commutative diagram

$$\begin{tikzcd}
F_\alpha \arrow[rr, "\phi_{\gamma\alpha}"] \arrow[dr, "\phi_{\beta\alpha}"'] & & F_\gamma \arrow[dl, "\phi_{\gamma\beta}"] \\
 & F_\beta &
\end{tikzcd}$$

Definition (1.1.11) :

Let $F = \{F_\alpha, \phi_{\beta\alpha}\}$ be a direct system in $\mathcal{A}$ The direct limit of this system, denoted by $\varinjlim F_\alpha$, is an object and a family of morphisms $f_\alpha : F_\alpha \longrightarrow \varinjlim F_\alpha$ with $f_\alpha = f_\beta . \phi_{\beta\alpha}$ whenever $\alpha \leq \beta$ satisfying the following universal mapping problem :

$$\begin{tikzcd}
\varinjlim F_\alpha \arrow[rr, dashed, "\phi"] & & X \\
& F_\alpha \arrow[ul, "f_\alpha"'] \arrow[ur, "g_\alpha"] \arrow[d, "\phi_{\beta\alpha}"] & \\
& F_\beta \arrow[uul, "f_\beta"] \arrow[uur, "g_\beta"'] &
\end{tikzcd}$$

for every object X and every family of morphisms $g_\alpha : F_\alpha \longrightarrow X$ with $g_\alpha = g_\beta \phi_{\beta\alpha}$. Whenever $\alpha \leq \beta$, there is a unique morphism $\phi : \varinjlim F_\alpha \longrightarrow X$ making the above diagram commute.

Definition (1.1.12) :

A diagram scheme $\underline{\Sigma}$ is a triple (I, M, D) where I is a set whose elements are called vertices, M is a set whose elements are called arrows, and D is a function from M to I x I. If $m \in M$ and $D(m) = (\alpha, \beta)$ we call α the origin of m and β the extremity of m.

A diagram in a category $\mathcal{A}$ over the scheme $\underline{\Sigma}$ is a function $\mathcal{D}$ which assigns to each vertex $\alpha \in I$ an object $\mathcal{D}_\alpha \in \mathcal{A}$ and to each arrow m with origin α and extremity β a morphism $\mathcal{D}(m) = [\mathcal{D}_\alpha, \mathcal{D}_\beta]_{\mathcal{A}}$

If $\mathcal{D}$ is a diagram in $\mathcal{A}$ over $\Sigma = (I,M,D)$, we call a family of morphisms $\{\mathcal{D}_\alpha \longrightarrow X\}_{\alpha \in I}$ a co-compatible family for $\mathcal{D}$ if for every arrow $m \in M$ the diagram.

$$\begin{tikzcd}
X & \mathcal{D}_\alpha \arrow[l, dash] \arrow[d, "\mathcal{D}(m)"] \\
 & \mathcal{D}_\beta \arrow[ul]
\end{tikzcd}$$

is commutative. The family is called a colimit for $\mathcal{D}$ if it is cocompatible and if for every cocompatible family $\{\mathcal{D}_\alpha \longrightarrow Y\}_{\alpha \in I}$ there is a unique morphism $X \longrightarrow Y$ such that for each $\alpha \in I$, the diagram

$$\begin{tikzcd}
X \arrow[r] & Y \\
 & \mathcal{D}_\alpha \arrow[ul] \arrow[u]
\end{tikzcd}$$

is commutative.

Definition (1.1.13) :

A category $\mathcal{A}$ is called co-complete if every diagram in $\mathcal{A}$ over every diagram scheme has a colimit.

Remarks :

(1) Colimits are unique upto isomorphisms.

(2) Initial objects, coequalizers, coproducts and direct limits are the special types of colimits.

(3) The dual notions of co-compatible family, limit, complete category and inverse limit can be obtained by reversing the arrows in the definitions of

cocompatible family, colimit, cocomplete category and direct limit respectively. Since we are not dealing with these notions in the present work we avoided to define these notions in detail. In the second chapter we are primarily dealing with colimits.

Proposition (1.1.1) :

If a category $\mathcal{A}$ has a finite coproducts and direct limits, then it has arbitrary coproducts.

Proposition (1.1.2) :

A category $\mathcal{A}$ is complete if it has coproducts and coequalizers.

Remark :

A category of algebras and modules are complete.

1.2 THE DERIVATION MODULES :

Convention :

R denotes a commutative ring with unit and A denotes a commutative unitary R-algebra.

Definition (1.2.1) :

Let M be an A-module. An R-derivation d from A into M is an R linear mapping $d : A \longrightarrow M$ satisfying the condition

$$d(ab) = a\,d(b) + b\,d(a) \text{ for all } a, b \in A.$$

Definition (1.2.2) :

An R-derivation module is an ordered triple (A,M,d) where A is commutative unitary R-algebra M is unitary A-module and d : A $\longrightarrow$ M is an R-derivation.

Definition (1.2.3) :

An R-derivation module (A, N, δ) is said to be a derivation A-submodule of a derivation module (A,M,d) if N is A-submodule of M and d restricted to N is δ.

Definition (1.2.4) :

A derivation module (A, M, d) is called simple if it does not contain any proper derivation A-submodule.

Remarks :

(1) Let (A, M, d) be an R-derivation module and let N be the A-submodule of M generated by $dA = \{da/a \in A\}$ Then (A, N, d) is a derivation A-submodule of (A,M,d) and thus every derivation A-module contains a derivation A-submodule.

(2) From the above remark and the definition of simple derivation module, it follows that (A, M, d) is simple if and only if M is generated by dA as an A-module.

Definition (1.2.5) :

Let (A,M,d) and (B,N,δ) be two derivation modules. An order pair ($\emptyset$,f) is called a derivation module homomorphism of (A,M,d) into (B,N,δ) and written as

$(\emptyset, f) : (A,M,d) \longrightarrow (B,N,\delta)$, if

(i) $\emptyset : A \longrightarrow B$ is an R-algebra homomorphism;

(ii) $f : M \longrightarrow N$ is an R-module homomorphism;

(iii) $f(am) = \emptyset(a).f(m)$, $a \in A$, $m \in M$ and

(iv) the diagram

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "\emptyset"'] & M \arrow[d, "f"] \\ B \arrow[r, "\delta"'] & N \end{tikzcd}$$

commutes.

Instead of saying that

$(\emptyset, f) : (A,M,d) \longrightarrow (B,N,\delta)$ is a derivation module homomorphism, sometimes it will be said that

$f : (A,M,d) \longrightarrow (B,N,\delta)$ is a $\emptyset$-derivation module homomorphism.

Definition (1.2.6) :

A derivation module homomorphism $(\emptyset, f) : (A,M,d) \longrightarrow (A,N,\delta)$ is called an A-derivation module homomorphism if $\emptyset = I_A$. In this case, $fd = \delta$ and such a derivation module homomorphism will be denoted by simply f.

Remark

(1) Let (A,M,d) be a simple derivation module. If there exists an A-derivation module homomorphism

$f : (A,M,d) \longrightarrow (A,N,\delta)$, then f is unique.

(2) If (A,N,δ) is simple, then any A-derivation module homomorphism

$$f : (A,M,d) \longrightarrow (A,N,\delta)$$

is an epimorphism.

Proposition (1.2.1) :

The class of all R-derivation modules and the derivation module homomorphisms forms a category.

This category will be denoted by R-DM.

Proposition (1.2.2) :

The class of all derivation modules of an R-algebra A and A-derivation module homomorphisms forms a category.

This category will be denoted by A-DM.

Definition (1.2.7) :

An initial object in the category A-DM is called a universal A-derivation module.

Obviously it is unique upto A-derivation module isomorphism.

A universal derivation module (A,M,d) can be constructed in the following way :

Let $u = A\otimes_R A/J$ where J is the A-submodule of $A\otimes_R A$ generated by all $1 \otimes ab - b\otimes a - a\otimes b$, $a, b \in A$.

Define $d : A \longrightarrow u$ by $d(a) = \gamma\ (1 \otimes a)$, $a \in A$ where $\gamma = A \otimes_R A \longrightarrow u$ is the natural A-module homomorphism. The (A,u,d) is universal derivation module of A.

Remark :

Let (A,u,d) be universal A-derivation module and $\emptyset : A \longrightarrow B$ is any unitary algebra homomorphism, then there exists a unique $\emptyset$ - derivation module homomorphism

$$f : (A,u,d) \longrightarrow (B,M,\delta).$$

Proposition (1.2.3) :

For any unitary commutative R-algebra A, there exists a universal derivation module of A.

Proposition (1.2.4) :

A category of derivation module is complete and cocomplete.

1.3 FIELD EXTENSIONS :

Theorem (1.3.1) :

If k is a field and K $(x_1, x_2 \ldots x_n)$ is an algebraic extension of k, then

$$K(x_1, x_2 \ldots\ldots x_n) = \overset{n}{\underset{1}{\otimes}} k(x_i)$$

Theorem (1.3.2) :

Let K be a modular inseparable extension of the field k with finite exponent (not necessarily finite extension), then K is the coproduct of simple extension $\{K(b)\}_{b \in B}$ where B is a modular base for K [22].