CHAPTER - II

COLIMITS OF DERIVATION MODULES AND UNIVERSAL DERIVATION MODULES

2.1 COLIMITS IN THE CATEGORY OF DERIVATION MODULES :

Let $(\phi, f) : (A,M,d) \longrightarrow (B,N,\delta)$, and

$(\psi, g) : (A,M,d) \longrightarrow (B,N,\delta)$ be the two derivation module homomorphisms in R-DM. Let B' be the ideal of B generated by $(\phi - \psi)(A) = \{\phi(a) - \psi(a) / a \in A\}$ and Let N' be the B - submodule of N generated by $(f-g)(M) \cup B'N$.

Set $\bar{B} = B/B'$ and $\bar{N} = N/N'$.

Then B-module $\bar{N}$ can be considered as $\bar{B}$-module via the natural R-algebra epimorphism $\mu : B \longrightarrow \bar{B}$;

If $\bar{b} = b + B'$ and $\bar{n} = n + N'$ are in $\bar{B}$ and $\bar{N}$ respectively, define $\bar{b}.\ \bar{n} = bn + N'$. This multiplication is well defined.

Next, consider the diagram

$$\begin{array}{ccc} B & \xrightarrow{\delta} & N \\ \mu \downarrow & & \downarrow \gamma \\ \bar{B} & & \bar{N} \end{array}$$

Where $\gamma : N \longrightarrow \bar{N}$ is a natural module homomorphism.

Consider $x \in B'$, such that

$x = b(\phi - \psi)(a)$, Then

$$\delta(x) = \delta(b.(\phi - \psi)(a))$$
$$= b'\ \delta b + b.\ (\delta\phi - \delta\psi)(a)$$
where $b' = (\phi - \psi)(a) \in B'$.
$$= b'\delta b + b\ (fd - gd)\ (a)$$
Since $(\phi, f) : (A,M,d) \longrightarrow (B,N,\delta)$

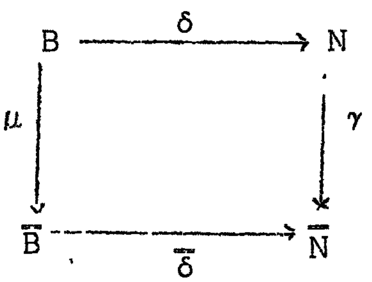
and $(\psi, g) : (A,M,d) \longrightarrow (B,N,\delta)$

are derivation module homomorphisms.

$= b'\delta b + b\ (f - g)(da)$ are both in N', $\delta(x) \in N'$

Similarly it can be proved that
$\delta((\phi - \psi)(a_1).(\phi - \psi)(a_2)) \in N'$ for all $a_1, a_2 \in A'$

Since B' is the ideal generated by $(\phi - \psi)(A)$ in B, it follows from the above discussion that $\delta(B') \subseteq N'$. Therefore, there exists a unique mapping $\bar{\delta} : \bar{B} \longrightarrow \bar{N}$ making the following diagram commutative



This $\bar{\delta}$ can be proved to be a derivation and so $(\bar{B}, \bar{N}, \bar{\delta})$ is a derivation module. Moreover, commutativity of the above diagram implies that

$(\mu, \gamma) : (B,N,\delta) \longrightarrow (\bar{B}, \bar{N}, \bar{\delta})$ is a derivation module homomorphism and from these constructions, follow

$\mu\phi = \mu\psi$ and $\gamma f = \gamma g$ i.e. the diagram

$$\begin{array}{ccc} (A,M,d) & \xrightarrow{(\phi,f)} & (B,N,\delta) \\ {\scriptstyle(\psi,g)}\downarrow & & \downarrow{\scriptstyle(\mu,\gamma)} \\ (B,N,\delta) & \xrightarrow[(\mu,\gamma)]{} & (\bar{B},\bar{N},\bar{\delta}) \end{array}$$

commutes.

Now, suppose $(\Theta,h) : (B,N,\delta) \longrightarrow (C,F,\Delta)$ is another derivation module homomorphism such that the following diagram

$$\begin{array}{ccc} (A,M,d) & \xrightarrow{(\phi,f)} & (B,N,\delta) \\ {\scriptstyle(\psi,g)}\downarrow & & \downarrow{\scriptstyle(\Theta,h)} \\ (B,N,\delta) & \xrightarrow[(\Theta,h)]{} & (C,F,\Delta) \end{array}$$

commutes.

The Θ maps B' to zero and h maps N' to zero. Hence there exists a unique algebra homomorphism $\bar{\Theta} : \bar{B} \longrightarrow C$ and a unique module homomorphism $\bar{h} : \bar{N} \longrightarrow F$ making the diagrams

$$\begin{array}{ccc} B & \xrightarrow{\eta} & \bar{B} \\ & {\scriptstyle\Theta}\searrow & \downarrow{\scriptstyle\bar{\Theta}} \\ & & C \end{array} \qquad\qquad \begin{array}{ccc} N & \xrightarrow{\gamma} & \bar{N} \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle\bar{h}} \\ & & F \end{array}$$

commutative, Further

$$
\begin{aligned}
\bar{h}\,\bar{\delta}\,(\bar{b}) &= \bar{h}\,\bar{\delta}\,(\mu(b)) \\
&= \bar{h}\,\gamma\delta\,(b) \\
&= h\,\delta\,(b) \\
&= \Delta\,\Theta\,(b) \\
&= \Delta\,\bar{\Theta}\,\mu(b) \\
&= \Delta\,\bar{\Theta}\,(\bar{b})
\end{aligned}
$$

and so $(\bar{\Theta}, \bar{h}) : (\bar{B}, \bar{N}, \bar{\delta}) \longrightarrow (C, F, \Delta)$ is a derivation module homomorphism making the diagram

$$
\begin{array}{ccc}
(B,N,\delta) & \xrightarrow{(\mu,\gamma)} & (\bar{B},\bar{N},\bar{\delta}) \\
 & \searrow^{(\Theta,h)} & \downarrow (\bar{\Theta},\bar{h}) \\
 & & (C,F,\Delta)
\end{array}
$$

commutative.

Moreover, such derivation module homomorphism is unique since μ and γ are epimorphisms.

We have the following

<u>Lemma</u> (2.1.1) :

The derivation module homomorphism (μ,γ) : $(B,N,\delta) \longrightarrow (\bar{B},\bar{N},\bar{\delta})$ is a coequalizer of the derivation modules $(\emptyset,f) : (A,M,d) \longrightarrow (B,N,\delta)$, and $(\psi, g) : (A,M,d) \longrightarrow (B,N,\delta)$

<u>Corollary</u> (2.1.1) :

If the derivation module homomorphism (μ,γ) is the coequalizer of $(\emptyset,f):(A,M,d) \longrightarrow (B,N,\delta)$ and (ψ, g) :

$(A,M,d) \longrightarrow (B,N,\delta)$, then the algebra homomorphism μ is the coequalizer of the two algebra homomorphisms $\emptyset : A \longrightarrow B$ and $\psi : A \longrightarrow B$.

i.e. the forgetful functor S : R-Dm $\longrightarrow$ CAlg. Where CAlg be the category of commutative unitary R-algebras, which sends a derivation module (A,M,d) to a commutative unitary R-algebra A preserves coequalizer.

Proof :

By the construction of the ideal B', generated by $(\emptyset - \psi)(A)$ and by the algebra homomorphism μ in the Lemma(2.1.1) it is obevious that the algebra homomorphism $\mu : B \longrightarrow \overline{B}$ is the coequalizer of the two algebra homomorphism $\emptyset : A \longrightarrow B$ and $\psi : A \longrightarrow B$.

Lemma (2.1.2) :

Let A and B be commutative unitary R-algebras. If M is an A-module and N is a B-module then $M \otimes_R N$ is an $A \otimes_R B$ -module with scalar operations that satisfy

$$(a \otimes b)(m \otimes n) = am \otimes bn.$$

Proof :

For $a \in A$, $b \in B$, let λ_a and λ_b be the left multiplication endomorphisms of M and N corresponding to a and b. ($\lambda_a(m) = am$, $\lambda_b(n) = bn$). Then $\lambda_a \otimes \lambda_b \in E_R(M \otimes_R N)$ satisfies $(\lambda_a \otimes \lambda_b)(m \otimes n) = am \otimes bn$.
Moreover, $(a,b) \longrightarrow (\lambda_a \otimes \lambda_b)$ is a bilinear mapping of $A \times B$ into $E_R(M \otimes_R N)$ such that

$$\emptyset(a \otimes b) = \lambda_a \otimes \lambda_b.$$

Define a mapping

$$(A \otimes_R B) \times (M \otimes_R N) \longrightarrow M \otimes_R N \text{ by}$$

$$(a\otimes b,\ m\otimes n) \longmapsto (\emptyset\ (a\otimes b))\ (m\otimes n)$$

Since $\emptyset$ is a homomorphism of R-modules and $\emptyset(a\otimes b) = \lambda_a \otimes \lambda_b \in E_R\ (M\otimes_R N)$, the mapping is bilinear i.e. a scalar multiplication on $M \otimes_R N$ and by construction

$$\begin{aligned}(a \otimes b)\ (m \otimes n) &= \emptyset\ (a \otimes b)\ (m \otimes n)\\ &= (\lambda_a \otimes \lambda_b)\ (m \otimes n)\\ &= am \otimes bn.\end{aligned}$$

Lemma (2.1.3) :

Let A, B, C be commutative unitary R-algebras. If $\emptyset_1 : A \longrightarrow C$ and $\emptyset_2 : B \longrightarrow C$ are algebra homomorphisms then there exists unique algebra homomorphism

$\emptyset : A \otimes_R B \longrightarrow C$ that satisfies $\emptyset(a\otimes b) = \emptyset_1(a) . \emptyset_2(b)$ and $\emptyset j_A = \emptyset_1$, $\emptyset j_B = \emptyset_2$ where $j_A : A \longrightarrow A\otimes_R B : a \longrightarrow a\otimes 1$ and $j_B : B \longrightarrow A \otimes_R B : b \longrightarrow 1\otimes b$ are natural injections.

Remarks :

(1) Since the algebra A can be considered as an A-module and B can be considered as B-module. $A\otimes N$ and $M\otimes B$ are both $A \otimes B$ modules.

(2) Lemma (2.1.3) means in categorical language, that $A\otimes_R B$ is the coproduct of the algebras A and B in the category of R-algebras.

Lemma (2.1.4) :

Let (A,M,d) and (B,N,δ) be derivation modules. The mapping

$$d \otimes I_B : A \otimes B \longrightarrow M \otimes_R B \text{ is an } R\text{-derivation.}$$

Proof :

Clearly $d\otimes I_B$ is R-linear since d is R-linear. The product rule is also satisfied. For a, a' $\in$ A, b, b' $\in$ B then

$$\begin{aligned}(d\otimes I_B)((a\otimes b)(a'\otimes b')) &= (d\otimes I_B)(aa'\otimes bb') \\ &= d(aa') \otimes bb' \\ &= (a\, d\, a' + a'\, d\, a) \otimes b\, b' \\ &\quad (\text{since } d : A \longrightarrow M \text{ is an } R\text{-derivation}) \\ &= (a\, d\, a' \otimes b\, b') + (a'\, da \otimes b\, b') \\ &= (a\otimes b)\ (da'\otimes b') + (da\otimes b)\ (a'\otimes b') \\ &= (a\otimes b)\ ((d\otimes I_B)(a'\otimes b')) + ((d\otimes I_B)(a\otimes b)).(a'\otimes b').\end{aligned}$$

Lemma (2.1.5) :

Let (A, M, d) and (B, N, δ) be derivation modules. Then $(A \oplus B, M \oplus N, d \oplus \delta)$ is a derivation module.

Proof :

The scalar multiplication defined by

$$(a, b)\ (m, n) = (am, bn)$$

assigns an $A \oplus B$ - module structure on $M \oplus N$. Further $d \oplus \delta : A \oplus B \longrightarrow M \oplus N$, defined by $(d \oplus \delta)\ (a,b) = (da, \delta b)$, $a \in A$, $b \in B$ is an R-derivation. Hence, the proposition.

Remark :

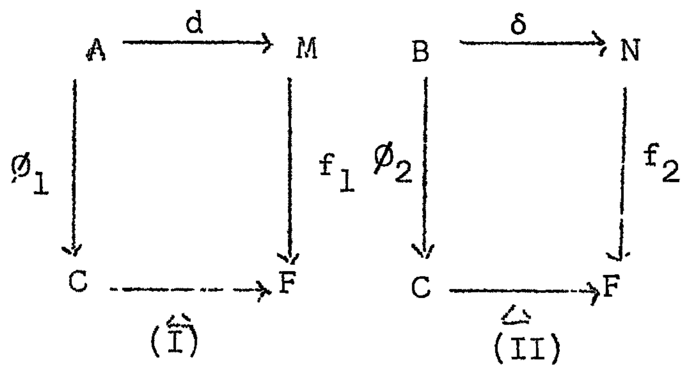
Since $M \otimes B$ is an $A \otimes_R B$ - module and $d \otimes I_B$: $A \otimes B \longrightarrow M \otimes B$ is an R-derivation. $(A \otimes_R B,\ M \otimes_R B,\ d \otimes I_B)$ is a derivation module. Similarly $(A \otimes_R B,\ A \otimes_R N,\ I_A \otimes \delta)$ is also a derivation module. Therefore, by the direct sum of these two

i.e. $(A \otimes_R B,\ M \otimes_R B \oplus A \otimes_R N,\ d \otimes I_B \oplus I_A \otimes \delta)$ is also a derivation module.

Let $(\phi_1, f_1) : (A, M, d) \longrightarrow (C, F, \triangle)$ and

$(\phi_2, f_2) : (B, N, \delta) \longrightarrow (C, F, \triangle)$

are derivation module homomorphismsi.e. the following diagram commute,



then it can be proved that there exists a unique derivation module homomorphism

$(\phi, f) : (A \otimes B,\ M \otimes B \oplus A \otimes N,\ d \otimes I_B \oplus I_A \otimes \delta) \longrightarrow (C, F, \triangle)$

making the following diagram commutative

$$\begin{array}{ccc}
(A,\ M,\ d) & & \\
\downarrow{\scriptstyle (j_A,\, j_M)} & \searrow^{(\emptyset_1,\, f_1)} & \\
(\ A\otimes B,\ M\otimes B\oplus A\otimes N,\ d\otimes I_B \oplus I_n\otimes\delta\ ) & \xrightarrow{(\emptyset,\, f)} & (C,\Gamma,\Delta) \\
\uparrow{\scriptstyle (j_B,\, j_N)} & \nearrow_{(\emptyset_2,\, f_2)} & \\
(B,\ N,\ \delta) & &
\end{array}$$

(III)

Define $j_A : A \longrightarrow A\otimes B$ and $j_M : M \longrightarrow M\otimes B\oplus A\otimes N$ by $j_A(a) = a \otimes 1$ and $j_M(m) = (m\otimes 1,\ 0)$. Then j_A is an R-algebra homomorphism and j_M is an R-module homomorphism. Moreover, since

$$\begin{aligned}
(d\otimes I_E \oplus I_A\otimes\delta)\ j_A(a) &= (d\otimes I_B \oplus I_A\otimes\delta)(a\otimes 1) \\
&= (da\otimes 1,\ o),
\end{aligned}$$

the following diagram commutes.

$$\begin{array}{ccc}
A & \xrightarrow{\quad d \quad} & M \\
{\scriptstyle j_A}\downarrow & & \downarrow{\scriptstyle j_M} \\
A\otimes B & \xrightarrow[d\otimes I_B\oplus I_A\otimes\delta]{} & M\otimes B\oplus A\otimes N
\end{array}$$

Thus

$$(j_A, j_M) : (A,\ M,\ d) \longrightarrow (A\otimes B,\ M\otimes B\oplus A\otimes N,\ d\otimes I_B\oplus I_A\otimes\delta)$$

is a derivation module homomorphism.

Similarly,

$$(j_B, j_N) : (B, N, \delta) \longrightarrow (A\otimes B,\ M\otimes B\oplus A\otimes N,\ d\otimes I_B\oplus I_A\otimes\delta)$$

is also a derivation module homomorphism.

Now, since $A \otimes B$ is the coproduct of the algebras A and B, there exists a unique algebra homomorphism

$$\phi : A \otimes B \longrightarrow C \text{ such that}$$

$$\phi j_A = \phi_1 \text{ and } \phi j_B = \phi_2$$

and this ϕ satisfies

$$\phi(a \otimes b) = \phi_1(a).\ \phi_2(b).$$

Further, since $f_1 : M \longrightarrow F$ and $\phi_2 : B \longrightarrow C$ are R-module homomorphisms, the mapping

$$(m,\ b) \longrightarrow \phi_2(b).f_1(m) \quad \text{of}$$

$M \times B \longrightarrow F$ is bilinear and, therefore, by definition of tensor product, there exists a unique R-module homomorphism

$$f' : M \otimes B \longrightarrow F \text{ such that}$$

$$f'(m \otimes b) = \phi_2(b).\ f_1(m).$$

This f' is a ϕ-homomorphism since

$$\begin{aligned} f'((a'\otimes b')(m\otimes b)) &= f'(a'm\otimes b'b) \\ &= \phi_2(b'b).f_1(a'm) \\ &= \phi_2(b')\phi_2(b).\phi_1(a').f_1(m) \\ &\quad \text{(since } f_1 \text{ is a } \phi\text{-homomorphism)} \\ &= \phi_1(a').\phi_2(b').\phi_2(b).f_1(m) \\ &= \phi(a'\otimes b').\ f'(m\otimes b). \end{aligned}$$

Similarly, we have

$$f'' : A \otimes N \longrightarrow F, \text{ such that}$$

$f''(a\otimes n) = \phi_1(a).f_2(n)$, which is also a ϕ - homomorphism.

Let $f = f' + f''$. Then

$f : M \otimes B \oplus A \otimes N \longrightarrow F$ is a ϕ - homomorphism and

$f(m \otimes b, a \otimes n) = f'(m \otimes b) + f''(a \otimes n)$.

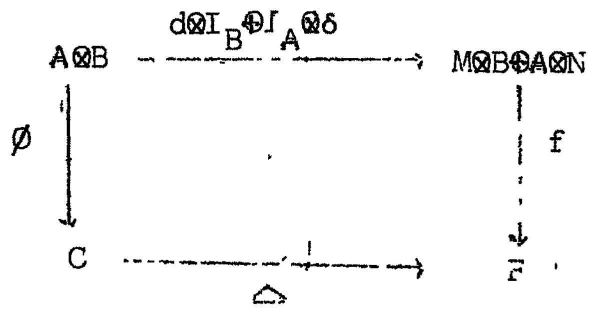
Finally, since

$$\begin{aligned} f(d \otimes I_B \oplus I_A \otimes \delta)(a \otimes b) &= f(da \otimes b, a \otimes \delta b) \\ &= f'(da \otimes b) + f''(a \otimes \delta b) \\ &= \phi_2(b).f_1 d(a) + \phi_1(a).f_2 \delta(b) \\ &= \phi_2(b) \triangle \phi_1(a) + \phi_1(a). \triangle \phi_2(b); \end{aligned}$$

(by the commutativity of diagrams (I) and (II).)

$$\begin{aligned} \text{and } \triangle \phi(a \otimes b) &= \triangle(\phi_1(a).\phi_2(b)), \\ &= \phi_1(a). \ \phi_2(b) + \phi_2(b) \ \phi_1(a), \end{aligned}$$

$f.(d \otimes I_B \oplus I_A \otimes \delta) = \triangle \phi$ i.e. the following diagram commutes



Hence,

$$(\phi, f) : (A \otimes B, \quad M \otimes B \oplus A \otimes N, \ d \otimes I_B \oplus I_A \otimes \delta) \to (C, F, \triangle)$$

is an R-derivation module homomorphism. It is obvious that $\phi j_A = \phi_1$, $\phi j_B = \phi_2$ and $f j_m = f_1$, $f j_n = f_2$; and so

$$(\phi, f)(j_A, j_M) = (\phi_1, f_1)$$

$$(\phi, f)(j_B, j_N) = (\phi_2, f_2).$$

Thus the diagram III commutes. Therefore, we have

Lemma (2.1.6) :

$(A\otimes B,\ M\otimes B\oplus A\otimes N,\ d\otimes I_B\oplus_A\otimes\delta)$ is the coproduct of the derivation modules (A, M, d) and (B, N, δ) in R-DM.

Corollary : (2.1.2) :

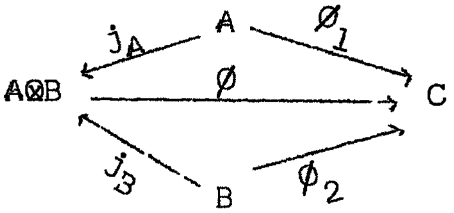
If $(A\otimes B,\ M\otimes B\oplus A\otimes N,\ d\otimes I_B\oplus I_A\otimes\delta)$ is the coproduct of the derivation modules (A, M, d) and (B, N, δ) in R-DM, then $A\otimes B$ is the coproduct of the commutative unitary R-algebras A and B in CAlg. (CAlg is the category of commutative unitary R-algebras).

i.e. the forgetful functor S : R-DM $\longrightarrow$ CAlg which sends a derivation module (A, M, d) to a commutative unitary R-algebra A preserves a finite coproduct.

Proof :

By the construction of the derivation module $(A\otimes B,\ M\otimes B\oplus A\otimes N,\ d\otimes I_B\oplus I_A\otimes\delta)$ from the derivation modules (A, M, d) and (B, N, δ) and from the commutativity of diagram (III) in Lemma (2.1.6) we can easily say that $A\otimes B$ is the coproduct of the two algebras. A and B in CAlg.

i.e. the following diagram commutes.



Let I be a directed set and $(A_\alpha, M_\alpha, d_\alpha)_{\alpha \in I}$ be a family of derivation modules indexed by I. For each $\alpha, \beta \in I$ with $\alpha \leq \beta$, let

$$(\phi_{\beta\alpha}, f_{\beta\alpha}) : (A_\alpha, M_\alpha, d_\alpha) \longrightarrow (A_\beta, M_\beta, d_\beta) \text{ be}$$

derivation module homomorphisms. Then$((A_\alpha, M_\alpha, d_\alpha), (\phi_{\beta\alpha}, f_{\beta\alpha}))_{\alpha \in I}$ is a direct system in R-DM; $(A_\alpha, \phi_{\beta\alpha})_{\alpha \in I}$ is a direct system of R-algebras; and $(M_\alpha, f_{\beta\alpha})_{\alpha \in I}$ is a direct system of A_α, -modules.

Let $A = \varinjlim A_\alpha$, $M = \varinjlim M_\alpha$. Then

$A = U_\alpha A_\alpha / \equiv$ where ' $\equiv$ ' is the equivalence relation on $U_\alpha A_\alpha$ defined as follows -

Let a, b be any two elements in $U_\alpha A_\alpha$ such that $a \in A_\alpha$ and $b \in A_\beta$. Then $a \equiv b$ if and only if there exists a $\delta \in I$ such that $\delta \geq \alpha$, β and $\phi_{\delta\alpha}(a) = \phi_{\delta\beta}(b)$.

Similar is the construction of M. Also, it is known that, if

$$\phi_\alpha : A_\alpha \longrightarrow A \text{ and } f_\alpha : M_\alpha \longrightarrow M$$

are natural homomorphisms, then f_α is a ϕ_α-homomorphism; A is an R-algebra; and M is an A-module with respect to the following operations :

Addition :

For $x, y \in M$ with $x = f_\alpha(x_\alpha)$, $y = f_\beta(y_\beta)$ define

$$x + y = f_\delta(f_{\delta\alpha}(x_\alpha) + f_{\delta\beta}(y_\beta))$$

$$\delta \geqslant \alpha, \beta \text{ in } I.$$

Scalar Multiplication :

For $a = \emptyset_\beta(a_\beta)$ and $x = f_\alpha(x_\alpha)$ in A and M respectively, define

$$ax = f_\delta\,(\emptyset_{\delta\beta}(a_\beta).f_{\delta\alpha}(x_\alpha))$$

$$\delta \geqslant \alpha, \beta \text{ in } I.$$

Now, since the following diagram commutes

$$\begin{array}{ccc} A_\alpha & \xrightarrow{d_\alpha} & M_\alpha \\ {\scriptstyle \emptyset_{\beta\alpha}}\downarrow & & \downarrow{\scriptstyle f_{\beta\alpha}} \\ A_\beta & \xrightarrow[d_\beta]{} & M_\beta \end{array}$$

for each pair α, β in I with $\alpha \leqslant \beta$, (d_α) $(\alpha \in I)$ is a direct system of mappings.

Let $\delta_\alpha = f_\alpha d_\alpha$. Then $\delta_\alpha : A_\alpha \longrightarrow M$ and for $\alpha \leqslant \beta$.

$$\begin{aligned} \delta_\beta \emptyset_{\beta\alpha} &= f_\beta d_\beta \emptyset_{\beta\alpha} \\ &= f_\beta\, f_{\beta\alpha}\, d_\alpha \quad \text{(by the commutativity of the above diagram)} \\ &= f_\alpha\, d_\alpha \\ &= \delta_\alpha \end{aligned}$$

Hence, there exists a unique mapping $d : A \longrightarrow M$ making the diagram

$$\begin{array}{ccc} A_\alpha & \xrightarrow{d_\alpha} & M_\alpha \\ \downarrow{\scriptstyle \phi_\alpha} & & \downarrow{\scriptstyle f_\alpha} \\ A & \xrightarrow[d]{} & M \end{array}$$

commutative i.e. $d\phi_\alpha = f_\alpha d_\alpha$ for each $\alpha \in I$. Clearly d is R-linear. Moreover, if $a, b \in A$, then $a = \phi_\alpha(a_\alpha)$ and $b = \phi_\beta(b_\beta)$ for some $a_\alpha \in A_\alpha$ and $b_\beta \in A_\beta$; and $ab = \phi_\delta(\phi_{\delta\alpha}(a_\alpha) . \phi_{\delta\beta}(b_\beta))$

$\delta \geqslant \alpha, \beta$ in I.

Hence,

$$\begin{aligned} d(ab) &= d(\phi_\delta(\phi_{\delta\alpha}(a_\alpha) . \phi_{\delta\beta}(b_\beta)) \\ &= f_\delta d_\delta(\phi_{\delta\alpha}(a_\alpha) . \phi_{\delta\beta}(b_\beta)) \\ &= f_\delta(\phi_{\delta\alpha}(a_\alpha) . d_\delta\phi_{\delta\beta}(b_\beta) + \phi_{\delta\beta}(b_\beta)\ d_\delta\phi_{\delta\alpha}(a_\alpha)) \\ &\qquad (\text{since } f_\delta \text{ is } \phi_\delta \text{ homomorphism}) \\ &= \phi_\alpha(a_\alpha)\ d\phi_\beta(b_\beta) + \phi_\beta(b_\beta) .\ d\phi_\alpha(a_\alpha) \\ &= a . db + b . da \end{aligned}$$

Thus $d : A \longrightarrow M$ is an R-derivation and, therefore, (A, M, d) is a derivation module.

Let $\{ (\Theta_\alpha, g_\alpha) : (A_\alpha, M_\alpha, d_\alpha) \longrightarrow (B, N, \delta) \}_{\alpha \in I}$

be any family of derivation module homomorphisms such that

$$(\Theta_\alpha, g_\alpha) = (\Theta_\beta, g_\beta)(\phi_{\beta\alpha}, f_{\beta\alpha})$$

whenever $\alpha \leq \beta$.

Then $\Theta_\alpha : A_\alpha \longrightarrow B$ is such that

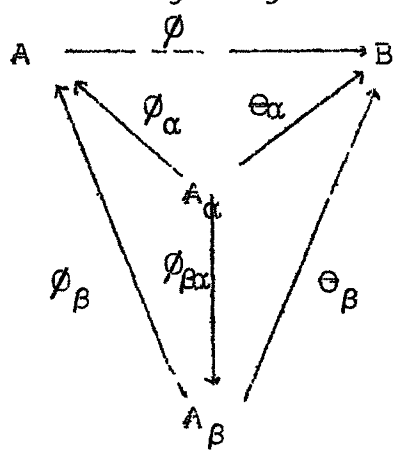
$$\Theta_\beta \cdot \phi_{\beta\alpha} = \Theta_\alpha \quad \text{for each } \alpha \in I.$$

Since $A = \lim A_\alpha$, there exists a unique R-algebra homomorphism

$\phi : A \longrightarrow B$ such that

$\Theta_\alpha = \phi \, \phi_\alpha$ ———— (I)

i.e. making the following diagram commutative :



Similarly, since $M = \varinjlim M_\alpha$, there exists a unique R-module homomorphism

$f : M \longrightarrow N$ such that

$g_\alpha = f \, f_\alpha$ ———— (II)

This f is a $\emptyset$ - homomorphism. For,
if $a \in A$ and $m \in M$, then
$a = \emptyset_\alpha(a_\alpha)$; $m = f_\beta(m_\beta)$ where $a_\alpha \in A_\alpha$ and
$m_\beta \in M_\beta$; $am = f_\delta\ (\emptyset_{\delta\alpha}(a_\alpha).f_{\delta\beta}(m_\beta))$.

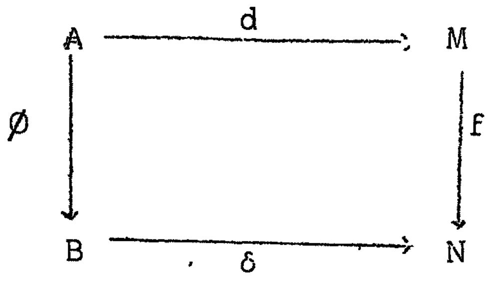
Therefore,

$$
\begin{aligned}
f(am) &= f'\ (f_\delta(\emptyset_{\delta\alpha}(a_\alpha)\ .f_{\delta\beta}(m_\beta)) \\
&= g_\delta(\emptyset_{\delta\alpha}(a_\alpha)\ .f_{\delta\beta}(m_\beta)) \quad \text{(using II)} \\
&= \Theta_\delta\ \emptyset_{\delta\alpha}\ (a_\alpha)\ .\ g_\delta f_{\delta\beta}(m_\beta) \\
&\qquad \text{(since } g_\delta \text{ is } \Theta_\delta \text{ - homomorphism)} \\
&= \emptyset\emptyset_\delta\emptyset_{\delta\alpha}(a_\alpha).ff_\delta f_{\delta\beta}(m_\beta) \\
&\qquad \text{(by (I) and (II))} \\
&= \emptyset\ \emptyset_\alpha(a_\alpha).ff_\beta(m_\beta) \\
&\qquad \text{(since } \delta \geq \alpha,\ \beta) \\
&= \emptyset(a).f(m).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\delta\emptyset\emptyset_\alpha &= \delta\ \Theta_\alpha \\
&= g_\alpha\ d_\alpha \quad (\text{ since } (\Theta_\alpha, g_\alpha):(A_\alpha, M_\alpha, d_\alpha) \rightarrow (B,N,\delta) \\
&\qquad \text{is a derivation module homomorphism}) \\
&= ff_\alpha d_\alpha \\
&= fd\emptyset_\alpha
\end{aligned}
$$

Since ϕ_α is an epimorphism this implies that $\delta\,\phi = fd$ i.e. the following diagram commutes



Hence, $(\phi, f) : (A, M, d) \longrightarrow (B, N, \delta)$ is a derivation module homomorphism.

Therefore, we have the following :

<u>Lemma</u> (2.1.7) :

The derivation module (A, M, d) is the direct limit of the direct system $((A_\alpha, M_\alpha, d_\alpha), (\phi_{\beta\alpha}, f_{\beta\alpha}))$ in R-DM.

<u>Corollary</u> (2.1.3) :

If the derivation module (A, M, d) is the direct limit of the direct system $((A_\alpha, M_\alpha, d_\alpha), (\phi_{\beta\alpha}, f_{\beta\alpha}))_{\alpha\in I}$ in R-DM, then an algebra A is the direct limit of the direct system $(A_\alpha, \phi_{\beta\alpha})_{\alpha\in I}$ in CAlg.

i.e. The forgetful functor

$$S : \text{R-DM} \longrightarrow \text{CAlg}.$$

which sends a derivation module (A, M, d) to an algebra A preserves direct limit.

Proof :

By the construction of the derivation module (A, M, d) from the direct system $((A_\alpha, M_\alpha, d_\alpha), (\emptyset_{\beta\alpha}, f_{\beta\alpha}))_{\alpha\in I}$ and by Lemma (2.1.7) we know that this derivation module (A, M, d) is the direct limit of the above system. From this we can easily verify that an algebra A is the direct limit of the direct system $(A_\alpha, \emptyset_{\beta\alpha})_{\alpha\in I}$ in CAlg.

Theorem (2.1.1) :

The forgetful functor

$S : R\text{-}DM \longrightarrow CAlg.$ which sends a derivation module (A, M, d) to a commutative unitary R-algebra A preserves colimits.

Proof :

By the corollary of Lemma (2.1.1) the forgetful functor $S : R\text{-}DM \longrightarrow CAlg$ preserves coequalizer. Also by the corollaries of Lemma (2.1.6) and Lemma (2.1.7) the forgetful functor $S : R\text{-}DM \longrightarrow CAlg$ preserves coproduct and direct limit respectively. Therefore, we can say that the forgetful functor -

$$S : R\text{-}DM \longrightarrow CAlg$$

preserves colimits.

2.2 SIMPLE DERIVATION MODULES :

Proposition (2.2.1) :

Let (A, M, d) and (B, N, δ) be two simple R-derivation modules. If the derivation module homomorphism $(\mu,\gamma) : (B,N,\delta) \longrightarrow (\bar{B}, \bar{N}, \bar{\delta})$ is the coequalizer of the derivation module homomorphisms.

$(\phi, f) \;:\; (A, M, d) \longrightarrow (B, N, \delta)$ and

$(\psi, g) \;:\; (A, M, d) \longrightarrow (B, N, \delta)$ then the

R-derivation module $(\bar{B}, \bar{N}, \bar{\delta})$ is simple.

Proof :

Since (A, M, d) and (B, N, δ) be the two R-derivation simple modules, the module M is generated by $dA = \{ da \mid a \in A \}$ as an A-module and the module N is generated by $\delta B = \{ \delta b / b \in B \}$ as an B-module. Hence, any element $m \in M$ and any element $n \in N$ can be written as $m = \sum \alpha_i da_i$ and $n = \sum \beta_i \delta b_i$ respectively, where $\alpha_i, a_i \in A$ and $\beta_i, b_i \in B$. $(\mu, \gamma) : (B, N, \delta) \longrightarrow (\bar{B}, \bar{N}, \bar{\delta})$ is the coequalizer of

$(\phi, f) \;:\; (A, M, d) \longrightarrow (B, N, \delta)$ and

$(\psi, g) \;:\; (A, M, d) \longrightarrow (B, N, \delta)$

consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\quad \delta \quad} & N \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \gamma} \\ \bar{B} & \xrightarrow[\quad \bar{\delta} \quad]{} & \bar{N} \end{array} \qquad \text{------ (I)}$$

Here by Lemma (2.1.1), we have

$\bar{B}$ = B/B', B' is the ideal generated by $(\phi - \psi)(A)$

$\bar{N}$ = N/N', N' is the B - submodule of N generated by $(f-g)(M) \cup B'N$.

The mappings

$$\mu : B \longrightarrow \bar{B} \text{ and } \gamma : N \longrightarrow \bar{N}$$

$$\mu(b) = \bar{b} \text{ and } \gamma(n) = \bar{n}, \qquad b \in B, \bar{b} \in \bar{B}, \; n \in N, \bar{n} \in \bar{N}$$

are canonical homomorphisms.

Let $\bar{n} \in \bar{N}$ then there exists $n \in N$ such that $\bar{n} = \gamma(n)$

Since (B, N, δ) is simple.

$$n = \sum \beta_i \, \delta b_i \qquad \text{for some } \beta_i, b_i \in B$$

Therefore $\bar{n} = \gamma(n)$

$$= \gamma(\sum \beta_i \delta b_i)$$

$$= \sum \gamma(\beta_i \delta b_i)$$

$$= \sum \mu(\beta_i).\gamma\delta(b_i) \qquad \text{(Since } \gamma \text{ is } \mu\text{-homomorphism)}$$

$$= \sum \bar{\beta}_i \, \bar{\delta}\mu(b_i) \qquad \text{(as diagram I commutes)}$$

$$= \sum \bar{\beta}_i \, \bar{\delta}(\bar{b}_i)$$

Thus any arbitrary element $\bar{n} \in \bar{N}$ can be written as $\sum \bar{\rho}_i \, \bar{\delta} \, \bar{b}_i$ where $\bar{\rho}_i, \bar{b}_i \in \bar{B}$. Therefore, the module $\bar{N}$ is generated by $\bar{\delta} \, \bar{B} = \{\bar{\delta} \, \bar{b} \, / \, \bar{b} \in \bar{\beta}\}$ as an $\bar{B}$ - module. Hence R-derivation module $(\bar{B}, \bar{N}, \bar{\delta})$ is simple.

<u>Proposition</u> (2.2.2) :

The coproduct of the two simple R-derivation modules (A, M, d) and (B, N, δ) is simple.

<u>Proof</u> :

By the Lemma (2.1.6) we know that the R-derivation module

$$(A \otimes B, \; M \otimes B \oplus A \otimes N, \; d \otimes I_B \oplus I_A \otimes \delta)$$

is the coproduct of the given derivation modules (A, M, d) and (B, N, δ).

Since the derivation modules (A, M, d) and (B, N, δ) are simple, the module M is generated by $dA = \{da / a \in A\}$ as an A-module and the module N is generated by $\delta B = \{\delta b / b \in B)\}$ as a B-module. Hence any element $m \in M$ and any element $n \in N$ can be written as

$$m = \sum \alpha_i da_i \quad \text{and } n = \sum \rho_i \delta b_i \quad \text{where}$$
$$\alpha_i, \; a_i \in A \text{ and } \rho_i, \; b_i \in B.$$

Let $(m \otimes b, \; a \otimes n)$ be any generating element of the derivation module $(M \otimes B \oplus A \otimes N)$ where $m \in M$, $n \in N$, $a \in A$ and $b \in B$.

As this element can be written as the linear combination of the elements

$(m\otimes 1,\ 0)$ and $(0,\ 1\otimes n)$

we can consider

$$\{(m\otimes 1,\ 0)\ /\ m \in M\} \cup \{(0,\ 1\otimes n)\ /\ n \in N\}$$

as the generators of the $A\otimes B$ - module $M\otimes B\oplus A\otimes N$.

Since $m = \sum \alpha_i da_i$,

$$m \otimes 1 = (\sum \alpha_i da_i) \otimes 1$$

$$= \sum (\alpha_i da_i \otimes 1).$$

$$= \sum (\alpha_i \otimes 1)(da_i \otimes 1), \quad \text{where } \alpha_i \otimes 1 \in A\otimes B.$$

Similarly

$$1\otimes n = \sum (1\otimes \beta_i)(1\otimes \delta b_i), \quad \text{where } 1\otimes\beta_i \in A \otimes B.$$

Therefore, we can consider

$$\{(da\otimes 1,\ 0)\ /\ a \in A\} \cup \{(0,\ 1\otimes\delta b)\ /\ b \in B\}$$

as the $A\otimes B$ - generators of the module $M\otimes B\oplus A\otimes N$.

As $(d\otimes I_B \oplus I_A \otimes \delta)(a\otimes 1) = (da\otimes 1,\ 0)$ and

$(d\otimes I_B \oplus I_A \otimes \delta)(1\otimes b) = (0,\ 1\otimes \delta b)$.

Therefore,

$$\{(d\otimes I_B \oplus I_A\otimes\delta)(a\otimes 1)\ /\ a\in A\} \cup \{(d\otimes I_B\oplus I_A\otimes\delta)(1\otimes b)\ /\ b\in B\}$$

is the set of generators of the module $M\otimes B \oplus A \otimes N$.

That is, module $M\otimes B\oplus A\otimes N$ is generated by

$(d\otimes I_B\oplus I_A\otimes\delta)(A\otimes B) = \{(d\otimes I_B\oplus I_A\otimes\delta)(a\otimes b)\ /\ a\in A,\ b\in B\}$ as an $A\otimes B$-module.

Therefore, the coproduct derivation module ($A\otimes B$, $M\otimes B\oplus A\otimes N$, $d\otimes I_B\oplus I_A\otimes\delta$) is simple.

Proposition (2.2.3) :

Let $((A_\alpha, M_\alpha, d_\alpha), (\phi_{\beta\alpha}, f_{\beta\alpha}))_{\alpha,\beta\in I}$ be the direct system over a directed set I, where $(A_\alpha, M_\alpha, d_\alpha)$ is the simple derivation module for each $\alpha\in I$.
If $\{(\phi_\alpha, f_\alpha) : (A_\alpha, M_\alpha, d_\alpha) \longrightarrow (A, M, d)\}_{\alpha\in I}$ is the direct limit of the above direct system, then the derivation module (A, M, d) is simple.

Proof :

Since the derivation module $(A_\alpha, M_\alpha, d_\alpha)$ is simple, the module M_α is generated by $d_\alpha A_\alpha = \{d_\alpha a_\alpha ; a_\alpha \in A_\alpha\}$ as an A_α - module. Therefore any arbitrary element $m_\alpha \in M_\alpha$ can be written as $m_\alpha = \sum q_{\alpha i} d_\alpha a_{\alpha i}$, where $q_{\alpha i}$, $a_{\alpha i} \in A_\alpha$.

Since $(\phi_\alpha, f_\alpha) : (A_\alpha, M_\alpha, d_\alpha) \longrightarrow (A, M, d)$ is the derivation module homomorphism, and $f_\alpha : M_\alpha \longrightarrow M$ is a natural module homomorphism, for some $m \in M$ there exists $m_\alpha \in M_\alpha$ such that

$$m = f_\alpha(m_\alpha).$$

As $(A_\alpha, M_\alpha, d_\alpha)$ is simple

$$m_\alpha = \sum q_{\alpha i} d_\alpha a_{\alpha i} \quad \text{for some } q_{\alpha i}, a_{\alpha i} \in A_\alpha.$$

Therefore,

$$m = f_\alpha(m_\alpha)$$

$$= f_\alpha(\sum q_{\alpha i} d_\alpha a_{\alpha i})$$

$$= \sum \phi_\alpha(q_{\alpha i}) \cdot f_\alpha d_\alpha a_{\alpha i} \quad \text{(since } f_\alpha \text{ is } \phi_\alpha \text{ - homomorphism)}$$

$$= \sum \phi_\alpha(q_{\alpha i}) d\phi_\alpha(a_{\alpha i}) \quad \text{(Since } (\phi_\alpha, f_\alpha):(A_\alpha, M_\alpha, d_\alpha) \to (A,M,d) \text{ is a derivation module homomorphism.}$$

$$= \sum q_i \; d(a_i) \;, \quad q_i,\; a_i \in A.$$

Thus any arbitrary element $m \in M$ can be written as $m = \sum q_i d(a_i)$. That is, the module M is generated by $dA = \{da \,/\, a \in A\}$ as an A-module. Therefore, the derivation module (A, M, d) is simple.

From Propositions (2.2.1, 2.2.2, 2.2.3) the following result is immediate :

<u>Theorem</u> (2.2.1) :

If $((A_\alpha, M_\alpha, d_\alpha), (\phi_{\beta\alpha}, f_{\beta\alpha}))$ is a diagram in Diag., where each $(A_\alpha, M_\alpha, d_\alpha)$ is simple, and $(\phi_\alpha, f_\alpha):(A_\alpha, M_\alpha, d_\alpha) \to (A,M,d)$ is the direct limit of the above diagram, then (A,M,d) is simple.

Let CAlg be the category of commutative unitary R-algebra.

Let $(A_\alpha, \emptyset_{\beta\alpha})$ be a fixed diagram in CAlg over a diagram scheme $\sum = (I, M, D)$. Here for each $\alpha \in I$ we assign a commutative unitary R-algebra A_α and for each $m \in M$ for which $D(m) = (\alpha, \beta)$ we assign an algebra homomorphism

$$\emptyset_{\beta\alpha} \; : \; A_\alpha \longrightarrow A_\beta$$

Let $\{ \emptyset_\alpha : A_\alpha \longrightarrow A \}$ be the colimit of the diagram in CAlg.

Let Diag denote the subcategory of the category of diagrams in R-DM over $\sum$ obtained as follows :

If $\mathcal{D}$ is an object in Diag then $\mathcal{D}_\alpha$ is an R-derivation module $(A_\alpha, M_\alpha, d_\alpha)$ and for each $m \in M$ for which $D(m) = (\alpha, \beta)$

$$\mathcal{D}(m) = (\emptyset_{\beta\alpha}, f_{\beta\alpha}) \; : \; (A_\alpha, M_\alpha, d_\alpha) \longrightarrow (A_\beta, M_\beta, d_\beta)$$

The morphism in Diag is the family

$$\{ \Psi_\alpha : \mathcal{D}_{(\alpha)} \longrightarrow \mathcal{D}'_{(\alpha)} \}_{\alpha \in I}$$

Where (i) for each $\alpha \in I$,

$$\Psi_\alpha : (I_\alpha, f_\alpha) : (A_\alpha, M_\alpha, d_\alpha) \longrightarrow (A_\alpha, M'_\alpha, d'_\alpha)$$

and (ii) for each $m \in M$ for which $D(m) = (\alpha, \beta)$, the following diagram commutes.

$$\begin{array}{ccc} (A_\alpha, M_\alpha, d_\alpha) & \xrightarrow{(I_\alpha, f_\alpha)} & (A_\alpha, M'_\alpha, d'_\alpha) \\ \Big\downarrow (\emptyset_{\beta\alpha}, f_{\beta\alpha}) & & \Big\downarrow (\emptyset_{\beta\alpha}, f'_{\beta\alpha}) \\ (A_\beta, M_\beta, d_\beta) & \xrightarrow[(I_\beta, f_\beta)]{} & (A_\beta, M'_\beta, d'_\beta) \end{array}$$

In view of the colimit preserving functor S : R-DM $\longrightarrow$ CAlg each object $\mathcal{D}$ in Diag corresponds to the unique derivation module (A, M, d) given by the colimit of $\mathcal{D}$. This defines a functor F: Diag. $\longrightarrow$ R-DM where morphisms are mapped in the obvious way. Hereafter A is assumed to be the algebra in the colimit of $(A_\alpha, \emptyset_{\beta\alpha})$.

<u>Theorem</u> (2.3.1) :

The functor F : Diag $\longrightarrow$ R-DM is onto.

<u>Proof</u> :

Let (A, M, d) be an R-derivation module. To each $\alpha \in I$, M can be made into A_α - module by defining the multiplication as

$$a_\alpha\, x = \emptyset_\alpha(a_\alpha)\, x \qquad \text{where } a_\alpha \in A_\alpha,\ x \in M$$

and $\emptyset_\alpha : A_\alpha \longrightarrow A$ is an algebra homomorphism.

We shall denote $M = M_\alpha$ when M is treated as an A_α - module.

Define $d_\alpha : A_\alpha \longrightarrow M_\alpha$ as

$$d_\alpha (a_\alpha) = d\, \phi_\alpha (a_\alpha) \text{ where } a_\alpha \in A_\alpha. \quad \text{—— (I)}$$

Since d and ϕ_α are R-linear, so is d_α.

Moreover, for a_α, $a'_\alpha \in A_\alpha$, we have

$$\begin{aligned} d_\alpha(a_\alpha . a'_\alpha) &= d\, \phi_\alpha(a_\alpha . a'_\alpha) \\ &= d\, (\phi_\alpha(a_\alpha) . \phi_\alpha(a'_\alpha)) \quad (\text{as } \phi_\alpha \text{ is an algebra homo.}) \\ &= \phi_\alpha(a_\alpha) . d(\phi_\alpha(a'_\alpha)) + \phi_\alpha(a'_\alpha) . d(\phi_\alpha(a_\alpha)) \\ &\quad (\text{Since } d : A \longrightarrow M \text{ is an R-derivation}) \\ &= \phi_\alpha(a_\alpha) . d_\alpha(a'_\alpha) + \phi_\alpha(a'_\alpha) . d_\alpha(a_\alpha) \\ &= a_\alpha . d_\alpha(a'_\alpha) + a'_\alpha . d_\alpha(a_\alpha) \end{aligned}$$

(by the definition of multiplication).

Thus $d_\alpha : A_\alpha \longrightarrow M_\alpha$ is an R - derivation.

Hence $(A_\alpha, M_\alpha, d_\alpha)$ is an R-derivation module.

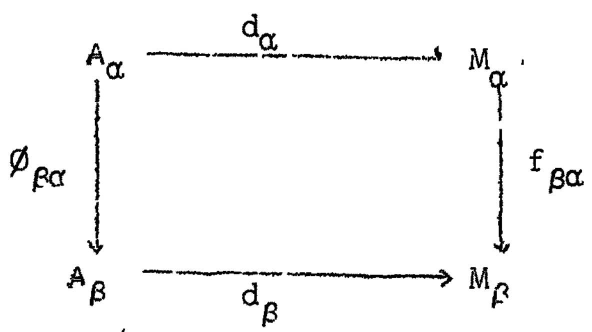
Now for each $\alpha, \beta \in I$, define

$$f_{\beta\alpha} : M_\alpha \longrightarrow M_\beta \text{ by } f_{\beta\alpha} = \text{identity on } M_\alpha.$$

Then

$$
\begin{aligned}
d_\gamma \phi_{\gamma\alpha} &= d\,\phi_\gamma\, \phi_{\gamma\alpha} \\
&= d\,\phi_\alpha \\
&= d_\alpha \\
&= f_{\gamma\alpha}\, d_\alpha \quad (\text{Since } f_{\gamma\alpha} \text{ is an identity on } M_\alpha)
\end{aligned}
$$

Therefore, the following diagram commutes.



Moreover, for each $a_\alpha \in A_\alpha$, $m_\alpha \in M_\alpha$

$$
\begin{aligned}
f_{\gamma\alpha}(a_\alpha m_\alpha) &= a_\alpha m_\alpha \\
&= \phi_\alpha(a_\alpha)\cdot m_\alpha \\
&= \phi_\gamma \phi_{\gamma\alpha}(a_\alpha)\cdot m_\alpha \\
&= \phi_{\gamma\alpha}(a_\alpha)\cdot m_\alpha \quad (\text{By the definition of multiplication.}) \\
&= \phi_{\gamma\alpha}(a_\alpha)\cdot f_{\gamma\alpha}(m_\alpha) \quad (\text{as } f_{\gamma\alpha} \text{ is an identity map}).
\end{aligned}
$$

thus $f_{\gamma\alpha}$ is a $\phi_{\gamma\alpha}$ - homomorphism.

Hence, $(\emptyset_{\beta\alpha}, f_{\beta\alpha}) : (A_\alpha, M_\alpha, d_\alpha) \longrightarrow (A_\beta, M_\beta, d_\beta)$ is an R - derivation module homomorphism.

Define $f_\alpha : M_\alpha \longrightarrow M$ as an identity map.

Since $f_\alpha\ (a_\alpha m_\alpha) = a_\alpha m_\alpha$

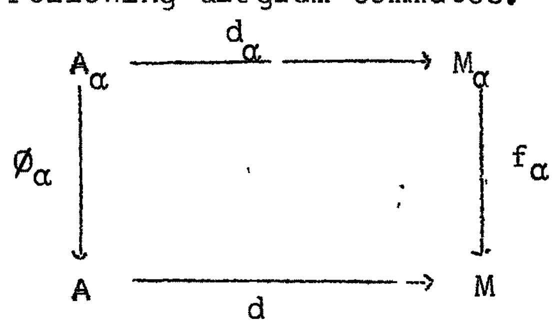
$$= \emptyset_\alpha(a_\alpha) \cdot f_\alpha(m_\alpha) \quad \text{(by the definition of multiplication)}$$

f_α is a $\emptyset_\alpha$ - homomorphism.

Moreover

$$f_\alpha \cdot d_\alpha = d_\alpha = d\ \emptyset_\alpha \qquad \text{by (I)}$$

That is the following diagram commutes.



Therefore, $(\emptyset_\alpha, f_\alpha) : (A_\alpha, M_\alpha, d_\alpha) \longrightarrow (A, M. d)$ is a derivation module homomorphism.

Clearly $((A_\alpha, M_\alpha, d_\alpha)\ (\emptyset_{\beta\alpha}, f_{\beta\alpha}))$ is a diagram in Diag whose colimit is

$$\left\{ (\emptyset_\alpha, f_\alpha) \ : \ (A_\alpha, M_\alpha, d_\alpha) \longrightarrow (A, M, d) \right\}$$

Let $(I, h) : (A, M, d) \longrightarrow (A, M', d')$ be an R-derivation module homomorphism.

Define $h_\alpha : M_\alpha \longrightarrow M'_\alpha$ by setting

$$h_\alpha = h \text{ on } M_\alpha \quad \text{and } M'_\alpha = A_\alpha - \text{module } M'$$

For $a_\alpha \in A_\alpha$

$$
\begin{aligned}
h_\alpha d_\alpha(a_\alpha) &= h\, d\phi_\alpha(a_\alpha) && (\text{Since } h_\alpha = h \text{ and } d_\alpha = d\,\phi_\alpha \\
&= h\, d(a) && (\text{as } \phi_\alpha(a_\alpha) = a \in A). \\
&= d'(a) && (\text{Since } hd = d'I)
\end{aligned}
$$

Also
$$
\begin{aligned}
d'_\alpha(a_\alpha) &= d'\,\phi_\alpha(a_\alpha) \\
&= d'(a).
\end{aligned}
$$

Hence, the following diagram commutes

$$
\begin{array}{ccc}
A_\alpha & \xrightarrow{\quad d_\alpha \quad} & M_\alpha \\
{\scriptstyle I_\alpha}\downarrow & & \downarrow{\scriptstyle h_\alpha} \\
A_\alpha & \xrightarrow[\quad d'_\alpha \quad]{} & M_\alpha
\end{array}
$$

i.e. $h_\alpha d_\alpha = d'_\alpha$

Moreover $h_\alpha(a_\alpha m_\alpha) = h_\alpha(\phi_\alpha(a_\alpha).m_\alpha)$ (by definition of multiplication by elements of A_α)

$= h\,(\phi_\alpha(a_\alpha).m_\alpha)$

$= \phi_\alpha(a_\alpha).h(m_\alpha)$ (Since h is a module homomorphism)

$= a_\alpha\, h_\alpha\,(m_\alpha)$ (Since $h_\alpha = h$ on M_α)

$= I_\alpha(a_\alpha).\ h_\alpha(m_\alpha)$

Therefore $(I_\alpha, h_\alpha) : (A_\alpha, M_\alpha, d_\alpha) \longrightarrow (A_\alpha, M'_\alpha, d'_\alpha)$ is an R-derivation module homomorphism.

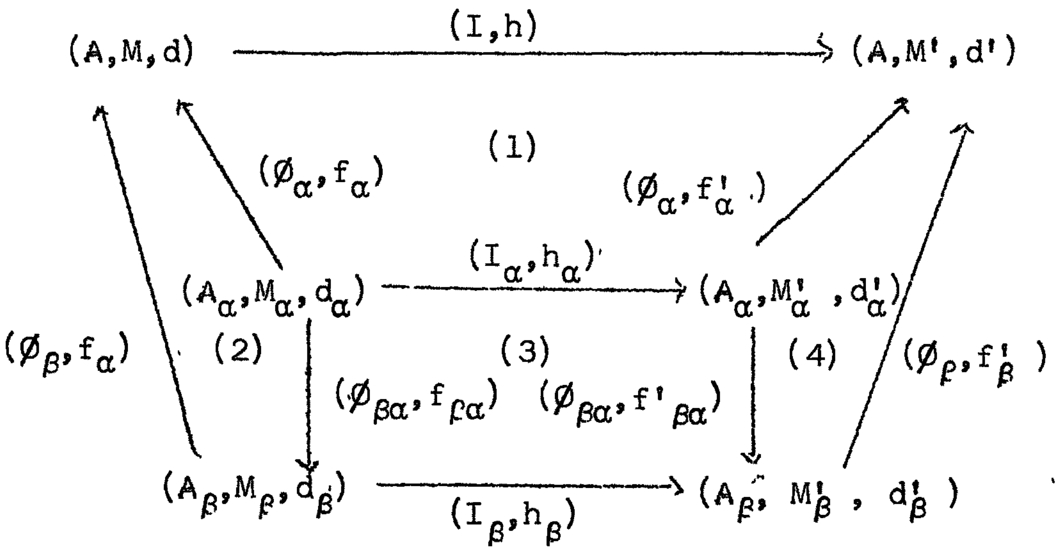
We may easily verify that $f'_{\beta\alpha}\, h_\alpha = h_\beta\, f_{\beta\alpha}$.

Hence, the following diagram commutes

$$
\begin{array}{ccc}
(A_\alpha, M_\alpha, d_\alpha) & \xrightarrow{(I_\alpha, h_\alpha)} & (A_\alpha, M'_\alpha, d'_\alpha) \\
{\scriptstyle (\phi_{\beta\alpha}, f_{\beta\alpha})}\downarrow & & \downarrow{\scriptstyle (\phi_{\beta\alpha}, f'_{\beta\alpha})} \\
(A_\beta, M_\beta, d_\beta) & \xrightarrow[(I_\beta, h_\beta)]{} & (A_\beta, M'_\beta, d'_\beta)
\end{array}
$$

(II)

Consider the following diagram.



Since (A, M, d) and (A, M'. d') are colimits of $((A_\alpha, M_\alpha, d_\alpha),$ $(\emptyset_{\beta\alpha}, f_{\beta\alpha}))$ and $((A_\alpha, M'_\alpha, d'_\alpha), (\emptyset_{\beta\alpha}, f'_{\beta\alpha}))$ respectively diagrams (2) and (4) are commutative. We have seen in (II) that the diagram (3) is commutative. By the definitions of f_α and f'_α, the diagram (1) commutes.

Therefore, the derivation module homomorphism $(I,h) : (A, M, d) \longrightarrow (A, M', d')$ is the image of the derivation module homomorphism

$$(I_\alpha, h_\alpha) : ((A_\alpha, M_\alpha, d_\alpha), (\emptyset_{\beta\alpha}, f_{\beta\alpha})) \longrightarrow ((A_\alpha, M'_\alpha, d'_\alpha), (\emptyset_{\beta\alpha}, f'_{\beta\alpha}))$$

under the functor F. Hence, the result.

Theorem (2.3.2) :

If $(A_\alpha, u_\alpha, \delta_\alpha)$ $(\emptyset_{\beta\alpha}, \Psi_{\beta\alpha}))$ is a diagram in R-DM over a scheme Σ = (I, M, d) such that $(A_\alpha, u_\alpha, \delta_\alpha)$ is the universal derivation module for each $\alpha \in I$, then its colimit (A, u, δ) is the universal derivation module.

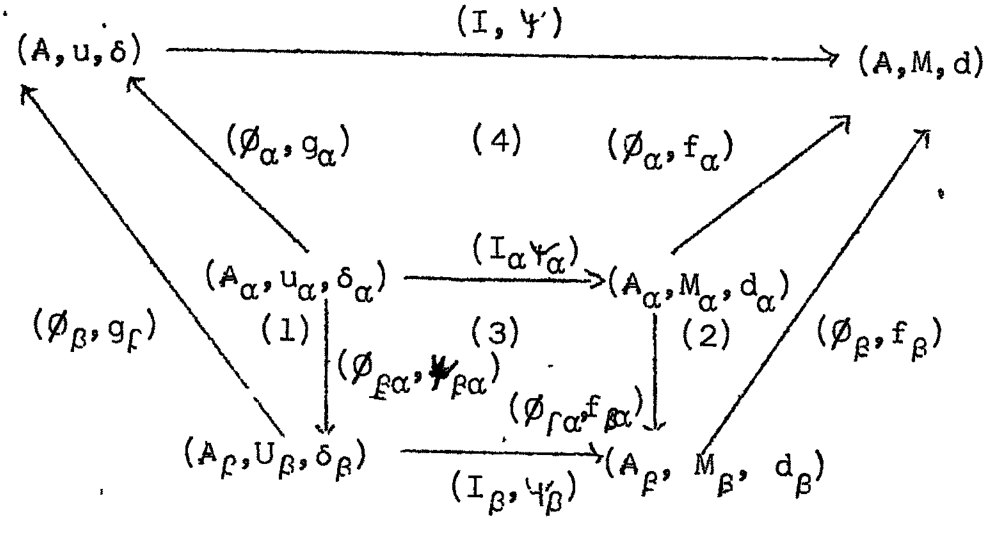
Proof :

Let $((A_\alpha, u_\alpha, \delta_\alpha), (\emptyset_{\beta\alpha}, \Psi_{\beta\alpha}))$ be a diagram in Diag such that each $(A_\alpha, u_\alpha, \delta_\alpha)$ is a universal derivation module. Let (A, u, δ) be the R-derivation module offered by the colimit of the diagram. Since $(A_\alpha, u_\alpha, \delta_\alpha)$ is simple for each $\alpha \in I$, (A, u, δ) is a simple derivation module.

If (A, M, d) is any R-derivation module, let $((A_\alpha, M_\alpha, d_\alpha), (\emptyset_{\beta\alpha}, f_{\beta\alpha}))$ be its pre-image in Diag. By the universality of $(A_\alpha, u_\alpha, \delta_\alpha)$ there is a unique R-derivation module homomorphism

$$(I_\alpha, \Psi_\alpha) : (A_\alpha, u_\alpha, \delta_\alpha) \longrightarrow (A_\alpha, M_\alpha, d_\alpha).$$

Consider the following diagram.



Since (A, u, δ) and (A, M, d) are colimits of $((A_\alpha, u_\alpha, \delta_\alpha), (\emptyset_{\beta\alpha}, \psi_{\beta\alpha}))$ and $((A_\alpha, M_\alpha, d_\alpha), (\emptyset_{\beta\alpha}, f_{\beta\alpha}))$ respectively diagrams (1) and (2) are commutative. As seen in Theorem (2.3.1), the diagram (3) is commutative. Hence, there exists unique $(I, \psi) : (A, u, \delta) \longrightarrow (A, M, d)$ making the diagram (4) commutative.

This establishes the existance of an R-derivation module homomorphism

$$(I, \psi) : (A, u, \delta) \longrightarrow (A, M, d)$$

The uniqueness of this R-derivation module homomorphism follows the fact that (A, u, δ) is simple.

Since the derivation module (A, M, d) is arbitrary the derivation module (A, u, δ) is the universal derivation module.

Corollary (2.3.1) :

If $A_1, A_2, \ldots\ldots A_n$ is a finite family of R-algebras, then the universal derivation module over $\overset{n}{\underset{1}{\otimes}} A_i$ is

$$\left(\overset{n}{\underset{1}{\otimes}} A_i, \overset{n}{\underset{1}{\otimes}} u_i, d_1 \otimes I_2 \otimes \ldots\ldots \otimes I_n + J_1 \otimes d_2 \otimes I_3 \otimes \ldots\ldots \otimes I_n + \ldots\ldots + J_1 \otimes J_2 \otimes \ldots\ldots \otimes J_{n-1} \otimes d_n \right)$$

where $(A_i, u_i\ d_i)$ is the universal derivation module; J_i

is the main involution of u_i and I_i is the identity on U_i

U_i ($1 \leq i \leq n$)

Proof :

This result follows from the fact that the derivation module

$$\left(\bigotimes_1^n A_i, \ \bigotimes_1^n u_i, \ d_1 \otimes I_2 \otimes \ldots \otimes I_n + J_1 \otimes d_2 \otimes I_3 \otimes \ldots \otimes I_n + \ldots + J_1 \otimes J_2 \otimes \ldots J_{n-1} \otimes d_n \right)$$

is the coproduct of the family (A_i, u_i, d_i).

$i = 1$ to n.

In particular if k is a field and (A_i, u_i, d_i) is the universal derivation module, where A_i is the commutative unitary k-algebra, then the derivation module

$$\left(\bigotimes_1^n A_i, \ \bigotimes_1^n u_i \ , \ d_1 \otimes I_2 \otimes \ldots I_n \oplus J_1 \otimes d_2 \otimes I_3 \otimes \ldots I_n + \ldots + J_1 \otimes \ldots \otimes J_{n-1} \otimes d_n \right).$$

gives the universal derivation module over the algebraic extension $k(x_1, \ldots x_n)$ as

$$k(x_1, \ldots x_n) = \bigotimes_1^n k(x_i).$$

By Theorem (1.3.2) of Chapter I, we have the following

Corollary (2.3.2) :

Let K be a modular inseparable extension of the field **k with** finite exponent (not necessarily finite extension) then a universal derivation module over K as k-algebra is the coproduct, object of the family

$$\{(A_b,\ u_b,\ d_b)\}_{b \in B} \quad \text{where } (A_b,\ u_b,\ d_b)$$

is the universal derivation module over k-algebra k(b) of simple extension of k and B is a modular base for K over k.