CHAPTER I.

CHAPTER I

INTRODUCTION

Nevanlinna Theory.

Let f(z) be a function meromorphic and non-constant in the complex plane C. Nevanlinna theory gives the idea of how densely the roots of the equation

$$f(z) = a \quad (z \in C;\ a \in C \cup \{\infty\})$$

are distributed over C; it also studies the mean approximation of the function f(z) to the value a along large concentric circles around the origin z = 0, a problem which turns out to be equivalent to the former.

Nevanlinna theory originates from a general formula

$$\log |f(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(R e^{i\phi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi +$$

$$+ \sum_{i=1}^{m} \log \left| \frac{R(z-a_i)}{R^2 - \bar{a}_i z} \right| - \sum_{j=1}^{n} \log \left| \frac{R(z-b_j)}{R^2 - \bar{b}_j z} \right|$$

where a function f(z) is meromorphic in $|z| \leq R$, $0<R<\infty$ with $z=re^{i\theta}$ $0<r<R$, such that $f(z) \neq 0, \infty$ and that a_i' s, the zeros and b_j' s are the poles of f(z) in $|z| < R$. This formula is due to F and R Nevanlinna [17], by which they were developing a general method for the investigation of meromorphic functions. This formula includes both the poisson formula and the Jensen formula as special cases, and in its most important form it expresses the logarithm of the modulus of an arbitrary meromorphic function by the boundary values of the function along a concentric circle around the origin and the zeros and poles of the function inside this circle. Nevanlinna theory created in 1924 when Rolf Nevannlina gave the formula an ingenious interpretation. The most general result of Nevanlinna theory can be summarized by saying that the distribution of the solutions to the equation f(z)= a is extremely uniform for almost all values of a except for a small minority of values which the function takes relatively rarely and these values are known as exceptional values. The main task of the value distribution theory in the sense of Nevanlinna theory is to investigate these exceptional values.

The earlier value distribution theory before Nevanlinna can be traced back to the year 1876 when K.Weierstrass [30] proved that a meromorphic function f(z) approaches to every value closely in the vicinity of its isolated essential singularity. But the actual study of exceptional values for entire functions started with the famous theorems of E.Picard and Borel. Picard's theorem states that if f(z) is an entire function, then f(z)-a has infinity of zeros except possible for one value of a. Further in 1879 Picard [19] even proved the surprising fact that a meromorphic function takes in the vicinity of an isolated essential singularity every finite or infinite value a with 2 exceptions at the most. These exceptional values are called as Picard execptional values of the function. The results which were found after by the mathematicians E.Laguerre, H.Poincare, J. Hadamard, E.Borel and others revealed that inspite of the possible existence of Picard exceptional values the distribution of zeros or, more generally, the distribution of a-points of an entire function is controlled, at least in some sense, by the growth behaviour of the maximum modulus function

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

And hence it is quite natural to define the order of an entire function by

$$\limsup_{r \to \infty} \frac{\log \log M(r,f)}{\log r}.$$

Since for an entire function f, M(r,f) satisfies the double inequality $\frac{M(r,f)-|f(0)|}{r} \leq M(r,f') \leq \frac{M(R,f)}{R-r}$

for all $0 < r < R$, it follows that the order of an entire function f is the same as the order of its derivative f'. Infact it is also known that $\rho_{f+g} \leq \max(\rho_f, \rho_g)$ and $\rho_{fg} \leq \max(\rho_f, \rho_g)$ where ρ_f denotes the order of f. In discussing meromorphic functions f(z) we can no longer use the maximum modulus function as a convenient tool for expressing the rate of growth of the function as the earlier approach to the value distribution theory breaks down, since M(r,f) becomes infinite if f(z) has a pole on the circle $|z|=r$. E.Borel [3] had tried to include meromorphic functions in this frame-work, but he was not very successful. Rolf Nevanlinna replaced the roll of log M(r,f) by an increasing real valued function $T(r,f)$, which is called the "Nevanlinna characteristic function" of $f(z)$ and plays a cardinal role in the whole Nevanlinna theory.

For an entire function $f(z)$, the Nevanlinna characterstic function $T(r,f)$ is connected by $\log M(r,f)$ by the following inequality

$$T(r,f) \leqslant \log M(r,f) \leqslant \frac{R+r}{R-r} T(R,f),$$

where $0 \leqslant r < R$. Using the above inequality, it is easy to show that for an entire function f,

$$\limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} = \limsup_{r \to \infty} \frac{\log \log M(r,f)}{\log r}$$

This motivated the following

Definition. The order ρ of a meromorphic function $f(z)$ is defined by

$$\rho = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r}$$

This definition also gives similar relations regarding order of meromorphic function viz. $\rho_{f+g} \leqslant \max(\rho_f, \rho_g)$ etc. Let us note however that though the order being defined by either the characteristic function or the logarithmic function give the same value, the functions $T(r,f)$ and $\log M(r,f)$ are not the same. Infact if f is an entire function of order ρ having exceptional value Borel (defined on p. 5) then $\log \frac{M(r,f)}{T(r,f)} \to \pi$ as $r \to \infty$. A great deal of work had been done in establishing the relationship between distribution of values and growth when Rolf Nevanlinna created his epoque making theory. This theory, which applies to entire functions, as well as to meromophic functions, even improved tremendously the earlier value distribution theory of entire functions. Picard's and Borel's theorems have been proved in a more general context of meromorphic functions by R. Nevanlinna. The second fundamental theorem of Nevanlinna furnishes a very simple proof of Picard and Borel theorems. Borel's theorem states that if f(z) is an entire function of finite order then $\rho_1(a) = \rho$ except possible for one value of a and if this exception occurs then ρ must be an integer $\rho_1(a)$ denoting the exponent of convergence of the zeros of f(z)-a and given by

$$\rho_1(a) = \limsup_{r \to \infty} \frac{\log^+ n(r,a)}{\log r} \quad \text{where}$$

n(r,a) denotes the number of zeros of f(z)-a in $|z| \leq r$, and ρ denotes the order of the function, For functions of finite order, the theorem of Borel includes the theorem of Picard as a particular case. Borel later generalized his theorem for infinite order. He introduces a variable order $\eta(r)$ and showed that for certain categories of entire functions,

$$\limsup_{r \to \infty} \frac{\log^+ n(r,a)}{\eta(r) \log r} = 1$$

except possibly for one value of a in which case the left hand side of the above equality is less than 1. Valiron has entended Borel's theorem in a sector and proved that if f(z) is a mermorphic function, then there exists a direction D (which he calls Borel direction) such that in every angle containing that line in its interior, the exponent of convergence of the zeros of f(z)-a is equal to the order of the function for all values of a except possibly two (for a=∞, the zeros of f(z)-a are to be replaced by the poles of f(z)).

Many have attempted to extend the Nevanlinna theory in sevaral directions. One of these, known as the theory of holomorphic or meromorphic curves, which was initiated by H. and J.Weyl [31] in 1938; the most difficult problem of this extension, the proof of the defect relation for holomorphic curves, was solved by L.Ahlfors [1] ; recently a very modern treatment of this theory was given by H.Wu[33] . In its simplest form this theory investigates the distribution of the zeros of linear combinations $A_0f_0(z) + - - - +A_nf_n(z)$ of finitely many integral functions $w_j = f_j(z)$ for different systems of constant multipliers $A= (A_0,\ldots,A_n)$ or, in other words, the theory analyses the position of a non-degenerate meromorphic curve $C \to P^n$ relative to the hyperplanes $A_0w_0+\ldots+ A_nw_n = o$ in the complex project - . space P^n. This theory by Weyl-Ahlfors was further extended to a higher dimensional in a most general way by W.Stoll [29] .Then there have been many attempts to extend the theory of holomorphic curves in different direction, stressing Hermitian geometric aspects, by S.S. Chern [6] , R.Bott and S.S.Chevn [4] and other authors. Again in 1972, giving very interesting new ideas, this theory was extended in a different direction, stressing to algebraic geometry,by J.Carlson and P.Griffiths[5] to equidimensional holomorphic mappings $C^m \longrightarrow V_m$, where V_m is a projective algebraic veriety. This theory was further generalized in the same direction by P.Griffiths and J.King [12] to the study of holomorphic mappings

$$f: A \longrightarrow V,$$

where A is an algebric, V a projective algebric variety. Given an algebraic subvariety $Z \subset V$; the two basic questions which are treated in this setting are in analogy to Nevanlinna theory: (A) can you find an upper bound on the size of $f^{-1}(Z)$ interms of Z and the "growth" of the mapping f; (B) can you find a lower bound on the size of $f^{-1}(Z)$, again interms of Z and the growth of the mapping. The most important special case of this problem is when $A=C^m$ and $V = P^n$, the complex projective space. Then f may be given by n meromorphic functions

$$f(z)=(f_1(z),\ldots,f_n(z)), \qquad z=(z_1\ldots,z_m) \in C^m.$$

The subvarieties Z will be the zero sets of collections of polynomials $P_\alpha(w_1,\ldots,w_n)$ and so the questions amount to globally studying solutions to the equations

$$P_\alpha(f_1(z),\ldots,f_n(z))=0.$$

We refer to L.Sario and K.Noshiro [20] for the extension of Nevanlinna theory for holomorphic mappings between Riemann surfaces and to the more Hermitian differential geometrical versions of S.S.Chern [7] and H.Wu [32]. There are several extensions of Nevanlinna theory to certain classes of non-helomorphic functions and the extension of E.F.Beckenbach and G.A.Hutchison [2] to triples of conjugate real harmonic functions.

Nevanlinna theory has also been used in case of exceptional values, asymptotic values for entire and meromorphic functions in different directions. For an entire function f(z) we say that a is an asymptotic value if there exists a curve starting from z=0 and extending up to infinity along which f(z) ---> a (finite). Valiron and Iverson have shown that infinity is always an asymptotic value for an entire function. Denjoy conjectured that an entire function of order ρ $(0 < \rho < \infty)$ has at most 2ρ asymptotic values. This conjecture was proved by L.V.Ahlfors. That this best possible result can be seen from the example $\frac{\sin\sqrt{z}}{\sqrt{z}}$. The result with 5ρ instead of 2ρ was proved by Carleman. But in case of meromorphic function of finite order the result of Denjoy-Ahlfors is not true. G.Valiron has constructed a meromorphic function of finite order having an infinity of asymptotic values which form uncountable set. Exceptional values and asymptotic values are related in some way.

With the above mentioned theory in the background, we now give the notations and the preliminary results that will be useful in our work in further developing the Nevanlinna theory.

Notations, Terminology and preliminary results.

Let $f(z)$ be a non-constant transcendental function meromorphic in the complex plane C. Let $\bar{C} = C \cup \{\infty\}$ the extended complex plane. For any complex number a, we denote by $n(r,a)=n(r,a,f)$ the number of roots of $f(z)=a$ in $|z| \leq r$, the multiple roots being counted with their multipticity, and let $\bar{n}(r,a)=\bar{n}(r,f,a)$ denote the number of distinct roots of $f(z)= a$ in $|z| \leq r$. For $a=\infty$, $n(r,a)=n(r,f)$ and $\bar{n}(r,a)=\bar{n}(r,f)$ respectively denote the number of poles and the number of distinct poles of $f(z)$ in $|z| \leq r$. We write

$$N(r,a) = \int_0^r \frac{n(t,a)-n(0,a)}{t}\,dt + n(0,a)\log r,$$

$$\bar{N}(r,a) = \int_0^r \frac{\bar{n}(t,a)-\bar{n}(0,a)}{t}\,dt + \bar{n}(0,a)\log r,$$

$$N(r,f)=N(r,\infty) = \int_0^r \frac{n(t,\infty)-n(0,\infty)}{t}\,dt + n(0,\infty)\log r.$$

$\bar{N}(r,f)$ being similarly defined.

Here the term $N(r,a)$ which refers to the number of roots of $f(z)=a$ in $|z| \leq r$, or to the number of poles of $f(z)$ if $a=\infty$, is called enumerating function. Also we write

$$m(r,a) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\phi})-a}\right| d\phi$$

$$m(r,\infty) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\phi})\right| d\phi.$$

The term $m(r,f)$ is a sort of averaged magnitude of $\log|f|$ on arcs of $|z| = r$ where $|f|$ is large, and the term $m(r,a)$ refers to the average smallness in a certain sense of $f-a$, on the circle $|z| = r$.

As usual for any complex number a, including ∞, we set -

$$\delta(a) = \delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)},$$

$$\Theta(a) = \Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r,a)}{T(r,f)},$$

$$\lambda(a) = \lambda(a,f) = 1 - \liminf_{r\to\infty} \frac{N(r,a)}{T(r,f)}$$

$$\theta(a) = \theta(a,f) = \liminf_{r\to\infty} \frac{N(r,a) - \bar{N}(r,a)}{T(r,f)} \text{ etc.}$$

The quantity $\delta(a)$ is called the deficiency of the value a. Finally the term $S(r,f)$ will denote any quantity satisfying $S(r,f) = o(T(r,f))$ as $r\to\infty$ through all values if f is of finite order and as $r \to \infty$ possibly outside a set of finite linear measure if f is of infinite order.

One of the fundamental theorem of the Nevanlinna's theory is

$$T(r,f) = N(r,a) + m(r,a) + O(1) \tag{1.1}$$

It provides an upper bound for $N(r,a)$ and so to the number of roots of the equation $f(z) = a$ valid for all r and a. This theorem in general form is

$$T(r,\frac{1}{f-a}) = T(r,f) + \log|f(0)-a| + \epsilon(a,R)$$

where $|\epsilon(a,R)| \leq \log^+|a| + \log 2$

and is normally known as the first fundamental theorem of

Nevanlinna (1964) see W.K.Hayman [13] . The more difficult question of lower bounds is found out with the help of Nevanlinn's second fundamental theorem which states that for $q \geqslant 3$ and distinct complex quantities $a_1, a_2, \ldots\ldots a_q$;

$$\sum_{i=1}^{q} m(r,a_i) \leqslant 2T(r,f) - N_1(r) + S(r,f) \qquad (1.2)$$

where $N_1(r)$ is a positive term related with multiple roots of the equation $f(z)=a$ and $S(r,f)$ is a small error term. This theorem tells us that the term $m(r,a)$ is small compared to $T(r,f)$ and so $N(r,a)$ comes near to $T(r,f)$, the maximum possible growth allowed by the first fundamental theorem. Thus the term

$$1 - \limsup_{r \to \infty} \frac{N(r,a)}{T(r,f)}$$

is defined and denoted by $\delta(a,f)$ and is called deficient value of a.

There are various directions in which the deficient values can be studied. In the present dissertation, we plan to work on deficient values of meromorphic functions, deficient values of their derivatives and deficient values of homogeneous differential polynomials of degree n, where by homogenous differential polynomials of degree n, we mean a finite sum of the terms of the form

$$a(z)\ (f(z))^{\lambda_0} (f'(z))^{\lambda_1} \ldots (f^{(k)}(z))^{\lambda_k}$$

where $\lambda_0 + \lambda_1 + \ldots \lambda_k = n$ and $a(z)$ is any meromorphic function satisfying $T(r, a(z)) = S(r.f)$.

Let $f(z)$ be entire or a meromorphic function we say that a is e.v.p. (exceptional value in the sense of Picard) if $n(r,a) = O(1)$ $(0 \leq |a| \leq \infty)$. a is called e.v.B. (exceptional value in the sense of Borel) if $\rho_1(a) < \rho$. a is called e.v.N. (exceptional value in the sense of Nevanlinna) if $\delta(a) > 0$. a is called e.v.E if

$$\liminf_{r \to \infty} \frac{T(r,f)}{n(r,a)\,\phi(r)} > 0$$

where $\phi(x)$ is any positive non-decreasing function such that

$$\int_A^{\infty} \frac{dx}{x\,\phi(x)} < \infty.$$

For an entire function every e.v.p. is an asymptotic value, and every e.v.B is also an asymptotic value in case if a function is of finite order. Nevanlinna put the question whether every e.v.N is also an asymptotic value. This was disproved by Arakclian, a Russian mathematician, who constructed an entire function of finite order having infinity of e.v.N which obiviously cannot all be asymptotic values because by Ahlfor's theorem, an entire function cannot have at most 2ρ asymptotic values. See Arakclian, "Doklady Akademy Nayuk, U.S.S.R., 1966." S.M.Shah in 1952 has proved that if f(z) is an entire function of finite order ρ having a as e.v.E then the number of asymptotic values of f(z) is precisely ρ and each asymptotic value is a. Nevanlinna conjectured that if α is e.v.N for an entire function or meromorphic function then α must be an asymptotic value. But this was proved to be false in 1941 by Madame Laurent Schwartz. She constructed a meromorphic function f(z) for which $\delta(0) = \delta(\infty) > 0$, and thus 0 and ∞ are e.v.N., but they were not asymptotic values. See [16]. For an entire function of infinite order it was proved to be false by W.K.Hayman and for finite order it was proved to be false by A.A.Goldberg. See [10]. But with some additional hypothesis the conjecture of Nevanlinna is true. A.Edrei and Fuchs have proved that if f(z) is an entire function of finite order and if $\sum_i \delta(a_i) = 2$, that is the total deficiency is attained, then each deficient value of f(z) is also an asymptotic value. See [9]. Later on by replacing some other smoother condition in the place of $\sum_i \delta(a_i) = 2$, Edrei proved that the restriction that f(z) must be of finite order can be removed and each deficient value will be asymptotic value. See A.Edrei [8].

The deficient values corresponding to zeros and poles being counted only once have also been studied extensively. Nevanlinna's theorem on deficient values states that if f(z) is meromorphic function then the set of values of a, for which $\delta(a) > 0$ or $\Theta(a) > 0$ is countable and $\sum_a \Theta(a) \leq 2$. This clearly implies that $\sum_a \delta(a) \leq 2$. If $\sum \delta(a) = 2$ then we say that the total deficiency is attained. S.K.Singh and H.S.Gopalkrishna [27] have shown by an example that a meromorphic function may be such that $\sum_{a \in C} \delta(a) = 1$ wheras $\sum_a \Theta(a) = 2$. Relative deficiencies i.e. deficiencies formed by considering the function and its derivative have also been studied in comparing them with the usual deficiencies [34]. For instance Xiong Qing-Lai in the above mentioned paper has proved that if f is a non-constant meromorphic function with ∞ and a finite value a with maximum defect them the relative defect and the usual defect of any value α with respect to $f^{(k)}$ are equal. The relative defects corresponding to distinct zeros and poles have been defined by A.P.Singh [24], where he has proved various relations between the relative defects and the relative defects corresponding to the distinct zeros and poles of f. Later in [25], these definitions have been carried over to form defects corresponding to two functions simultaneously, viz forming the defects with respect to distinct zeros $\Theta_{1,2}^{(k)}(a)$, common zeros $\Theta_{0}^{(k)}(a)$, of two meromorphic functions. Using these he has proved in [25] that if f_1 and f_2 are two meromorphic functions which have o and ∞ as exceptional values of defect 1, and if a_i are finite distinct non-zero complex numbers then $\sum_i \Theta_{1,2}^{(k)}(a_i) \leq 2$.

Deficient values of entire and meromorphic functions have been studied in the context of orders of the functions also. S.M.Shah and S.K.Singh have shown that if $\delta(a_1)=1$ and $\sum_{i=2}^{\infty} \delta(a_i)=1$, then the order ρ must be a positive integer. See [23]. But if the hypothesis only states that the total deficiency is attained, then ρ need not be an integer because of Nevanlinna who constructed an example of a meromorphic function for which total deficiency is attained and its order is $\frac{2n+1}{2}$. There is a longstanding conjecture of Nevanlinna that if $\sum_i \delta(a_i) = 2$ and function is of finite order, then the order of the meromorphic function must be either an integer or an integer divided by two,

Pflugar [18] proved that if f(z) is an entire function of finite order such that

$$\sum_a \delta(a) = 2,$$

then ρ must be an integer. However Singh and Gopalkrishna have shown that if the total deficiency for the l^{th} derivative of meromorphic function is attained ($l \geq 1$), then ρ must be a positive integer see [27] .

S.M.Shah and S.K.Singh have studied exceptional values in another context also. They have compared the growth of T(r,f') with respect to T(r,f) under different hypothesis. Most of our work is developed in this context only. For an entire function of finite order one has

$$\log M(r,f) \sim \log M(r,f')$$

Hence it is reasonable to conjecture that for an entire function of finite order,

$$T(r,f) \sim T(r,f').$$

Nevanlinna actually conjectured that for an entire function

$$T(r,f) \sim T(r,f'),$$

and for a meromorphic function either

$$T(r,f') \sim T(r,f),$$

or $$T(r,f') \sim 2T(r,f).$$

These conjectures have been tried by many persons. But S.M. Shah and S.K.Singh have proved that for a meromorphic function of finite order if

$$\delta(\alpha) = \delta(\infty) = 1, \ \alpha \neq \infty,$$

then $T(r,f') \sim T(r,f)$,

and if $\delta(\alpha_1) = \delta(\alpha_2) = 1, \ \alpha_1 \neq \alpha_2$

$$|\alpha_1| < \infty, \ |\alpha_2| < \infty,$$

then $T(r,f') \sim 2T(r,f)$.

See S.M.Shah and S.K.Singh [21] . Further these results were improved by the same authors. They proved that if

$$|\alpha_i| < \infty, \ \sum_1^{\infty} \delta(\alpha_i) = 2,$$

then $T(r,f') \sim 2T(r,f)$

where f(z) is a meromorphic function of finite order. See [22] .

These results were extended by P.K.Kamathan [14] and proved that if f(z) is a meromorphic function of finite order such that

$$\sum_{\alpha \neq \infty} \delta(\alpha) = 1 , \quad \delta(\infty) = 1 \quad \alpha \neq \infty, \qquad (1.3)$$

then $T(r,f^{(\lambda)}) \sim T(r,f)$

and if $\delta(\alpha_1) = \delta(\alpha_2) = 1, \ \alpha_1 \neq \alpha_2$, $|\alpha_1| < \infty, \ |\alpha_2| < \infty$, (1.4)

then $T(r,f^{(\lambda)}) \sim (\lambda+1)\, T(r,f)$.

In our work we have extended these results further in third chapter and proved with the same hypothesis as in (1.3) that for a homogenous polynomial P of degree n in a meromorphic function of finite order and P does not contain f

$$T(r,p) \sim n\, T(r,f)$$

and if (1.4) is true for a monomial of degree n, consisting just one term

$$a(z)\ (f(z))^{\lambda_0} (f'(z))^{\lambda_1} \ldots\ (f^{(k)}(z))^{\lambda_k}$$

Then with additional hypothesis

$$N(r,p) \leq (n+k\lambda_k)\, T(r,f)$$

we have shown that

$$T(r,p) \sim (n+k\lambda_k)\, T(r,f).$$

By weaking the hypothesis (1.3) P.K.Kamthan [15] has compared the growth of $T(r,f^{(\lambda)})$ with respect to $T(r,f)$ and proved that for a meromorphic function of finite order such that -

$$\delta(0,f) = \Theta(\infty,f) = 1,$$

Then

$$T(r,f^{(\lambda)}) \sim T(r,f) \sim \bar{N}\left(f, \frac{1}{f^{(\lambda)}-a}\right)$$

for all a, except possibly o and ∞. We have extended this result in chapter III for a homogeneous polynomial P of degree n in meromorphic function f of finite order (P does not involve f). In the same chapter we have considered the estimation of $\sum_{\alpha \neq \infty} \delta(\alpha,f)$ interms of deficient values $\delta(0,p)$, $\delta(\infty,p)$ which extends the work of P.K.Kamthan [15]. We have also found out certain other types of estimations of $\delta(0,p)$, $\sum_{\alpha \neq \infty} \delta(\alpha,p)$ and $\lambda(o,p)$ which are also extensions of work of same author.

The second chapter of our work is devoted to find the bounds for $\liminf\limits_{r \to \infty} \frac{T(r,f^{(k)})}{T(r,f)}$ and $\limsup\limits_{r \to \infty} \frac{T(r,f^{(k)})}{T(r,f)}$ interms of deficient values $\Theta(a_i, f)$ (where f is a meromorphic function and a_i's are complex constants) and compared the growth of $T(r, f^{(k)})$ with respect to $T(r,f)$ $(k \geq 1)$ under various hypothesis on deficient values. Several other results are also derived with the help of these bounds, and we have also used the above mentioned bounds and its corollaries to give some direct applications to the deficient values of two meromorphic functions using the theorem proved by A.P.Singh in [25] . In this chapter we have also defined the relative defects and the absolute defects of meromorphic functions and have obtained a relation between those two defects.

The method of the proof that we have followed in both these chapters is the classical theory of entire functions dealing with the order of entire functions and the classical Nevanlinna theory of meromorphic functions which heavily depends on the first and second fundamental theorems. We have made an extensive use of the Milloux's theorem which gives a bound for the Nevanlinna characteristic of a derivative with respect to the Nevanlinna characteristic function $T(r,f)$. We have also used some properties of meromorphic functions of non-integral order, and have also used the rate of growth of zeros and poles of a meromorphic function and compared it with $T(r,f)$. We have also used the principle of counting the number of zeros and poles of a meromorphic function and used it to compare with the number of zeros and poles of its derivatives. And finally the Nevanilinna's theory of deficient values has been used almost everywhere in our work.