CHAPTER II

CHAPTER II

DEFICIENT VALUES AND RELATIVE DEFECTS OF MEROMORPHIC FUNCTIONS.

We mentioned in the previous chapter that the Nevanlinna theory is heavily dependent on its two fundamental theorems, the first and second. There have been minor variations given by different authors for the second fundamental theorem, though basically they give the same result. One of these is given by Hayman [13, 31] in the following form. "Suppose that $f(z)$ is a non-constant meromorphic function in $|z| \leq r$. Let $a_1, a_2, \ldots, a_q$ where $q > 2$, be distinct finite complex numbers, $\delta > 0$, and suppose that $|a_j - a_i| \geq \delta$ for $1 \leq j < i \leq q$. Then

$$m(r,\infty) + \sum_{i=1}^{q} m(r,a_i) \leq 2\,T(r,f) - N_1(r) + S(r),$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N(r,\frac{1}{f'}) + 2\,N(r,f) - N(r,f')$$

and $$S(r) = m(r,\frac{f'}{f}) + m(r, \sum_{i=1}^{q}\frac{f'}{f-a_i}) + q\log^{+}\frac{3q}{\delta} + \log 2 +$$

$$+ \log\frac{1}{|f'(0)|},$$ with modifications if $f(0) = 0$ or ∞, or $f'(0) = 0$." In the above theorem it is not necessary to just consider $\sum_{i=1}^{q} m(r,a_i)$. Infact we can replace this

term by $n \sum_{i=1}^{q} m(r,a_i)$ for any positive integer n and still inequality remains valid. More precisely we have the following

<u>Theorem 1</u>. Let $f(z)$ be a non-constant meromorphic function in $|z| \leqslant r$. If $a_1, a_2, \ldots, a_q$ $(q \geqslant 2)$, be distinct finite complex numbers such that $|a_j - a_i| \geqslant \delta$ $(\delta > 0)$ for $1 \leqslant j < i \leqslant q$, then for all positive integers n, we have

$$m(r,\infty) + n \sum_{i=1}^{q} m(r,a_i) \leqslant 2\, T(r,f) - N_1(r) + S(r) \qquad (2.1)$$

where $N_1(r) = 2\, N(r,f) + N(r,\frac{1}{f'}) - N(r,f')$

and $S(r) = m(r,\frac{f'}{f}) + m(r,\sum \frac{f'}{(f-a_i)^n}) + nq \log^+ \frac{3q}{\delta} + n \log 2 +$

$+ \log \frac{1}{|f'(0)|}$.

<u>Proof</u>. Set $F(z) = \sum_{i=1}^{q} \frac{1}{(f(z)-a_i)^n}$,

we first suppose that for some i,

$|f(z) - a_i| < \frac{\delta}{3q}$.

Then for $j \neq i$,

$$|f(z)-a_j| \geqslant |a_j - a_i| - |f(z) - a_i|$$
$$\geqslant \delta - \frac{\delta}{3q}$$
$$\geqslant \frac{2\delta}{3} \quad (\text{since } q \geqslant 1)$$

Hence

$$\frac{1}{|f(z)-a_j|} \leqslant \frac{3}{2\delta}$$

$$\leqslant \frac{3}{2} \frac{1}{3q|f(z)-a_i|} \text{ since } |f(z)-a_i| < \frac{\delta}{3q}$$

Thus

$$\frac{1}{|f(z)-a_j|} \leqslant \frac{1}{2q|f(z)-a_i|} \qquad (2.2)$$

Consider

$$|F(z)| \geqslant \left|\frac{1}{(f(z)-a_i)^n}\right| - \sum_{j\neq i}\left|\frac{1}{(f(z)-a_j)^n}\right|$$

$$\geqslant \frac{1}{|f(z)-a_i|^n} - \sum_{j\neq i} \frac{1}{2^n q^n |f(z)-a_i|^n} \quad \text{(using 2.2)}$$

$$= \frac{1}{|f(z)-a_i|^n}\left\{1 - \frac{q-1}{2^n q^n}\right\}$$

$$\geqslant \frac{1}{|f(z)-a_i|^n} \cdot \frac{1}{2^n},$$

since $1 \geqslant \frac{1}{2^n} + \frac{1}{2^n}$ for $n \geqslant 1$ and

$$1 - \frac{q-1}{2^n q^n} \geqslant 1 - \frac{q^n}{2^n q^n} = 1 - \frac{1}{2^n}$$

which gives $1 - \frac{q-1}{2^n q^n} \geqslant \frac{1}{2^n}$.

Hence

$$\log^+|F(z)| \geqslant \log^+ \frac{1}{|f(z)-a_i|^n} - n \log 2$$

$$= \sum_{j=1}^{q} \log^+ \frac{1}{|f(z)-a_j|^n} - \sum_{j\neq i} \log^+ \frac{1}{|f(z)-a_j|^n} - n \log 2 \qquad (2.3)$$

But since for $j \neq i$, $|f-a_j| \geqslant |a_j-a_i| - |f-a_i|$

$$> \delta - \frac{\delta}{3q}$$

$$= \frac{(3q-1)\,\delta}{3q}$$

$$\geqslant \frac{\delta}{3q} \; .$$

We have

$$\log^+ \frac{1}{|f-a_j|^n} \leqslant \log^+ \left(\frac{3q}{\delta} \right)^n$$

or

$$\sum_{j \neq i} \log^+ \frac{1}{|f-a_j|^n} \leqslant (q-1) \log^+ \left(\frac{3q}{\delta} \right)^n$$

$$\leqslant nq \log^+ \left(\frac{3q}{\delta} \right) .$$

Hence from (2.3), we have

$$\log^+ |F(z)| \geqslant \sum_{j=1}^{q} \log^+ \frac{1}{|f(z)-a_j|^n} - nq \log^+ \frac{3q}{\delta} - \qquad (2.4)$$
$$- n\log 2.$$

Next we consider the case when

$$|f(z)-a_i| \geqslant \frac{\delta}{3q} \quad \text{for all } i \, .$$

Then we have

$$\log^+ \frac{1}{|f(z)-a_i|^n} \leqslant \log^+ \left(\frac{3q}{\delta} \right)^n$$

and so

$$\sum_{i=1}^{q} \log^+ \frac{1}{|f(z)-a_j|^n} \leqslant nq \log^+ \frac{3q}{\delta} \; .$$

This shows that right hand side of (2.4) is negative.

But left hand side of (2.4) is non-negative and therefore (2.4) is trivially true in this case and it is true in all cases. Multiplying (2.4) both sides by $\frac{1}{2\pi}$ and integrating over $[0,2\pi]$ we get

$$m(r,F) \geqslant \sum_{i=1}^{q} m(r, \frac{1}{(f-a_i)^n}) - nq \log^+ \frac{3q}{\delta} - n \log 2.$$

$$= n \sum_{i=1}^{q} m(r,a_i) - nq \log^+ \frac{3q}{\delta} - n \log 2. \qquad (2.5)$$

Now, to get required inequality we consider

$$m(r,F) = m(r, \frac{1}{f} \frac{f}{f'} f'F)$$

$$= m(r,\frac{1}{f}) + m(r,\frac{f}{f'}) + m(r,f'F).$$

But from (1.10) of Hayman [13, 4] we have

$$T(r,f) = T(r,\frac{1}{f}) + \log|f(0)|,$$

This gives

$$m(r,\frac{f}{f'}) = m(r,\frac{f'}{f}) + N(r,\frac{f'}{f}) - N(r,\frac{f}{f'}) + \log\left|\frac{f(0)}{f'(0)}\right|$$

and

$$m(r,\frac{1}{f}) = T(r,f) - N(r,\frac{1}{f}) + \log\frac{1}{|f(0)|}.$$

So we get finally

$$m(r,F) \leqslant T(r,f) - N(r,\frac{1}{f}) + \log\frac{1}{|f(0)|} + m(r,\frac{f'}{f}) + N(r,\frac{f'}{f}) -$$
$$- N(r,\frac{f}{f'}) + m(r,f'F) + \log\left|\frac{f(0)}{f'(0)}\right|.$$

This inequality combined with (2.5) gives

$$n\sum_{i=1}^{q} m(r,a_i)+m(r,\infty)\leqslant m(r,F)+m(r,f)+nq\log^+\frac{3q}{\delta}+n\log 2$$

$$\leqslant T(r,f)-N(r,\tfrac{1}{f})+N(r,\tfrac{f'}{f})-N(r,\tfrac{f}{f'})+$$

$$+\ m(r,\tfrac{f'}{f})+m(r,f'F)+\log\frac{1}{|f'(0)|}+$$

$$+\ T(r,f)-N(r,f)+nq\log^+(\tfrac{3q}{\delta})+n\log 2.$$

Now, by Jensen's formula

$$N(r,\tfrac{f'}{f})-N(r,\tfrac{f}{f'})=\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{f(re^{i\theta})}{f'(re^{i\theta})}\right|d\theta-\log\left|\frac{f(0)}{f'(0)}\right|$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|d\theta-\log|f(0)|-$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\log|f'(re^{i\theta})|d\theta+\log|f'(0)|$$

$$=N(r,\tfrac{1}{f})-N(r,f)-N(r,\tfrac{1}{f'})+N(r,f').$$

Hence we obtain lastly

$$n\sum_{i=1}^{q} m(r,a_i)+m(r,\infty)\leqslant 2\,T(r,f)-\Big\{2\,N(r,f)-N(r,f')+$$

$$+\ N(r,\tfrac{1}{f'})\Big\}+S(r)$$

where $S(r)$ is defined as in Theorem 1 and this completes proof.

We now find bounds for $\frac{T(r,f^{(k)})}{T(r,f)}$ in terms of the deficient values. We start with the following lemma.

<u>Lemma 1.</u> If f is a meromorphic function and if $a_1, a_2, \ldots a_q$ are distinct elements of $\mathbb{C}$ then

$$\sum_{i=1}^{q} m(r,a_i,f) + N(r,\frac{1}{f^{(k)}}) \leqslant T(r,f^{(k)}) + S(r,f) \qquad (2.6)$$

where q,k are any positive integers and $S(r,f)=o(T(r,f))$ as $r \to \infty$ through all the values if f is of finite order and $S(r,f)= o(T(r,f))$ as $r \to \infty$ except possibly for a set of finite linear measure if f is of infinite order.

For the proof we shall require the following well known lemma of Milloux [13, 55]

<u>Lemma 2</u>. Let l be a positive integer and

$$\psi(z) = \sum_{i=0}^{l} a_i(z) f^{(i)}(z).$$

Then

$$m(r,\frac{\psi(z)}{f(z)}) = S(r,f),$$

and

$$T(r,\psi) \leqslant (l+1)T(r,f)+S(r,f) .$$

Proof. If $q = 1$ then writing

$$m(r,\frac{1}{f-a_1})+N(r,\frac{1}{f^{(k)}}) = m(r,\frac{f^{(k)}}{f-a_1}\frac{1}{f^{(k)}}) + N(r,\frac{1}{f^{(k)}})$$

$$\leq m(r,\frac{f^{(k)}}{f-a_1}) + m(r,\frac{1}{f^{(k)}}) + N(r,\frac{1}{f^{(k)}})$$

$$\leq m(r,\frac{f^{(k)}}{f-a_1}) + T(r,\frac{1}{f^{(k)}})$$

and using Lemma 2 and Nevanlinna's first fundamental theorem, the result follows.

So, let $q \geq 2$.

Set

$F(z) = \sum_{i=1}^{q} \frac{1}{f(z)-a_i}$, then by inequality (2.1) of Hayman [13, 33]

$$\sum_{i=1}^{q} m(r,a_i,f) \leq m(r,F) + O(1)$$

$$= m(r,\frac{Ff^{(k)}}{f^{(k)}}) + O(1)$$

$$\leq m(r,Ff^{(k)}) + m(r,\frac{1}{f^{(k)}}) + O(1).$$

$$\leq \sum_{i=1}^{q} m(r,\frac{f^{(k)}}{f-a_i}) + m(r,\frac{1}{f^{(k)}}) + O(1).$$

The result now follows by adding $N(r,\frac{1}{f^{(k)}})$ to both the sides and using Lemma 2 and the first fundamental Theorem of Nevanlinna.

We now prove

<u>Theorem 2.</u> If f is a meromorphic function of order ρ and $a_1, a_2, \ldots, a_q (q \geq 1)$ are distinct elements of $\mathbb{C}$, then for any positive integer k,

$$\liminf_{r \to \infty} \frac{T(r,f^{(k)})}{T(r,f)} \geq k \sum_{i=1}^{q} \Theta(a_i,f) - q(k-1) \quad (2.7)$$

where $r \to \infty$ without restriction if f is finite and $r \to \infty$ outside an exceptional set of finite measure if $\rho = +\infty$.

<u>Proof.</u> By (2.6), we have

$$\sum_{i=1}^{q} m(r,a_i,f) \leq T(r,f^{(k)}) - N\left(r,\frac{1}{f^{(k)}}\right) + S(r,f).$$

Adding $\sum_{i=1}^{q} N(r,a_i,f)$ to both sides,

$$\sum_{i=1}^{q} T(r,a_i,f) \leq T(r,f^{(k)}) + \sum_{i=1}^{q} N(r,a_i,f) - N\left(r,\frac{1}{f^{(k)}}\right) + S(r,f)$$

$$= T(r,f^{(k)}) + k\sum_{i=1}^{q} \bar{N}(r,a_i,f) - N_0\left(r,\frac{1}{f^{(k)}}\right) + S(r,f)$$

where $N_0\left(r, \frac{1}{f^{(k)}}\right)$ is formed with the zeros of $f^{(k)}$ which are not zeros of any of the $f-a_i$ $(i = 1,2,\ldots,q)$.

Since $N_0\left(r, \frac{1}{f^{(k)}}\right) \geq 0$ and

$T(r,a_i,f) = T(r,f) + O(\log r)$, it follows that

$$qT(r,f) \leqslant T(r,f^{(k)}) + k\sum_{i=1}^{q}\bar{N}(r,a_i,f) + S(r,f)$$

So,

$$q \leqslant \liminf_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)} + k\sum_{i=1}^{q}\limsup_{r\to\infty}\frac{\bar{N}(r,a_i,f)}{T(r,f)} + \limsup_{r\to\infty}\frac{S(r,f)}{T(r,f)}$$

$$= \liminf_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)} + k\sum_{i=1}^{q}[1-\Theta(a_i,f)].$$

Thus

$$k\sum_{i=1}^{q}\Theta(a_i,f) - q(k-1) \leqslant \liminf_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)}.$$

<u>Remark</u>. (i) In particular if $k = 1$, then (2.7) reduces to

$$\liminf_{r\to\infty}\frac{T(r,f')}{T(r,f)} \geqslant \sum_{i=1}^{q}\Theta(a_i,f)$$

Now making $q \to \infty$, we obtain

$$\liminf_{r\to\infty}\frac{T(r,f')}{T(r,f)} \geqslant \sum_{i=1}^{\infty}\Theta(a_i,f) = \sum_{a\in C}\Theta(a,f)$$

which yields Theorem 2 of [27].

(ii) In the above theorem we have found a lower bound for $\liminf_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)}$ for functions of any order. If now f is of finite order, then we can also find an upper bound for $\limsup_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)}$. More precisely we have the following

Theoram 3. If f is a meromorphic function of finite order then for positive integers k,q

$$k \sum_{i=1}^{q} \Theta(a_i,f) - q(k-1) \leq \liminf_{r \to \infty} \frac{T(r,f^{(k)})}{T(r,f)} \leq \limsup_{r \to \infty} \frac{T(r,f^{(k)})}{T(r,f)}$$

$$\leq k + 1 - k\Theta(\infty,f) \qquad (2.8)$$

Proof. In view of Theorem 1, it is sufficient to prove the right hand side of inequality (2.8). We have,

$$T(r,f^{(k)}) = m(r,f^{(k)}) + N(r,f^{(k)})$$

$$\leq m(r,\frac{f^{(k)}}{f}) + m(r,f) + N(r,f) + k\bar{N}(r,f)$$

$$= T(r,f) + k\bar{N}(r,f) + S(r,f)$$

Thus

$$\limsup_{r \to \infty} \frac{T(r,f^{(k)})}{T(r,f)} \leq 1 + k \limsup_{r \to \infty} \frac{\bar{N}(r,f)}{T(r,f)}$$

$$= 1 + k\left[1 - \Theta(\infty,f)\right]$$

So,

$$\limsup_{r \to \infty} \frac{T(r,f^{(k)})}{T(r,f)} \leq k + 1 - k\Theta(\infty,f) \qquad (2.9)$$

This completes the proof.

Remark. In particular if k = 1 and $\sum_{a \in C} \Theta(a,f) = 2$, then from (2.8) on making $q \to \infty$ we obtain

$$2 - \Theta(\infty,f) \leq \liminf_{r \to \infty} \frac{T(r,f')}{T(r,f)} \leq \limsup_{r \to \infty} \frac{T(r,f')}{T(r,f)} \leq 2 - \Theta(\infty,f)$$

Thus

$$\lim_{r \to \infty} \frac{T(r,f')}{T(r,f)} = 2 - \Theta(\infty,f),$$

which gives corollary 2.1 of [27].

We now give various applications of Theorem 3.

Corollary 1. If f is a meromorphic function of finite order such that $\Theta(\infty,f) = 1$ and $\Theta(a,f) = 1$ for some $a \neq \infty$, then

$$T(r,f^{(k)}) \sim T(r,f) \qquad (2.10)$$

Proof. Since $\sum_{a \in C} \Theta(a,f) \leq 2$, it follows that $q = 1$ and hence from (2.8), we have

$$k\Theta(a,f) - (k-1) \leq \liminf_{r \to \infty} \frac{T(r,f^{(k)})}{T(r,f)} \leq \limsup_{r \to \infty} \frac{T(r,f^{(k)})}{T(r,f)}$$

$$\leq k + 1 - k$$

which gives, $\lim_{r \to \infty} \frac{T(r,f^{(k)})}{T(r,f)} = 1.$

This proves (2.10).

Corollary 2. Let f be a meromorphic function of finite order.

(i) If $\Theta(a_i,f) = \frac{1}{2}$ for $i = 1,2,3$, $(a_i \neq \infty)$ and $\Theta(\infty,f)=\frac{1}{2}$, then $T(r,f') \sim \frac{3}{2} T(r,f)$. (2.11)

(ii) And if $\Theta(a_i,f) = \frac{1}{2}$ for $i =1,2,3,4$ where a_i are finite and distinct, then

$$T(r,f') \sim 2\, T(r,f) \qquad (2.12)$$

Proof. (i) Putting $k = 1$ and $q = 3$ we obtain from (2.8) that

$$\frac{3}{2} \leq \liminf_{r \to \infty} \frac{T(r,f')}{T(r,f)} \leq \limsup_{r \to \infty} \frac{T(r,f')}{T(r,f)} \leq \frac{3}{2}$$

Which gives the desired result

$$\lim_{r \to \infty} \frac{T(r,f')}{T(r,f)} = \frac{3}{2}$$

(ii) Since $\Theta(a_i,f)=\frac{1}{2}$ for finite a_i; $i= 1,2,3,4$ we have $\Theta(\infty,f) = 0$ and so from (2.8) for $k = 1$ we obtain as above

$$\lim_{r \to \infty} \frac{T(r,f')}{T(r,f)} = 2$$

and hence we get (2.12).

<u>Remark</u>. Let us note that there do exist meromorphic functions satisfying the hypothesis of corollary 2(i). For example the Weierstrass's elliptic function p(z) is one such example. Also if f satisfies (ii) of the above corollary then by Corollary 3 of [28] it follows that ∞ which is clearly not e.v.N cannot be e.v.V also.

<u>Corollary 3</u>. If f is entire function of a finite order such that $\Theta(a_i,f) = \frac{1}{2}$ for finite a_i, $i = 1,2$ then

$$T(r,f') \sim T(r,f)$$

<u>Proof</u>. Since f is an entire function, we have $\Theta(\infty,f)=1$ and so as earlier by putting $k = 1$ and $q = 2$ we get

$$\lim_{r \to \infty} \frac{T(r,f')}{T(r,f)} = 1$$

and hence

$$T(r,f') \sim T(r,f).$$

Remark. Once again we observe that there do exist entire functions satisfying the hypothesis of the above corollary. For example for $f(z) = \sin z$, it is known that $\Theta(1) = \Theta(-1) = \frac{1}{2}$, see [13.45].

We end this chapter by proving some relations dealing with the usual defects and relative defects of meromorphic functions. Milloux introduced the concept of absolute defect viz. $\delta(\alpha, f')$. This definition was later taken up by Xiong-Lai [34], who defined the term

$$\delta_r^{(k)}(\alpha, f) = 1 - \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{f^{(k)} - \alpha}\right)}{T(r, f)}$$

and called it the relative defect of α with respect to f, and in contrast the usuall defect $\delta(\alpha, f^{(k)})$ was denoted by $\delta_a^{(k)}(\alpha, f)$, and he found various relations between $\delta_r^{(k)}(\alpha, f)$ and $\delta_a^{(k)}(\alpha, f)$. Later A.P.Singh [24] defined the relative defect corresponding to the distinct zeros and distinct poles viz.

$$\Theta_r^{(k)}(\alpha, f) = 1 - \limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)} - \alpha}\right)}{T(r, f)},$$

and he found various relations between $\Theta_r^{(k)}(\alpha, f)$ and $\delta(\infty, f)$, $\Theta(\alpha, f)$ etc.

Here we shall find a relation between $\Theta_r^{(k)}(\alpha,f)$ and $\Theta_a^{(k)}(\alpha,f)$ where

$$\Theta_a^{(k)}(\alpha,f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-\alpha}\right)}{T(r,f^{(k)})}.$$

Thus we shall prove the following

<u>Theorem</u> 4. Let f(z) be a meromorphic function. Then for each positive integer k

$$(k+1)\,\Theta_a^{(k)}(\alpha,f) \leqslant k + \Theta_r^{(k)}(\alpha,f).$$

<u>Proof</u>. Using Lemma 2, we have

$$T(r,f^{(k)}) \leqslant (k+1)\,T(r,f) + S(r,f).$$

And so,

$$\limsup_{r\to\infty} \frac{T(r,f^{(k)})}{T(r,f)} \leqslant k+1.$$

Our conclusion now follows from

$$\Theta_r^{(k)}(\alpha,f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-\alpha}\right)}{T(r,f)}$$

$$= 1 - \limsup_{r\to\infty}\left[\frac{\bar{N}\left(r, \frac{1}{f^{(k)}-\alpha}\right)}{T(r,f^{(k)})} \cdot \frac{T(r,f^{(k)})}{T(r,f)}\right]$$

$$\geqslant 1 - \limsup_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-\alpha}\right)}{T(r,f^{(k)})}\ \limsup_{r\to\infty} \frac{T(r,f^{(k)})}{T(r,f)}$$

$$\geqslant 1 - \limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)} - \alpha}\right)}{T(r, f^{(k)})} (k + 1)$$

$$= (k + 1) \left\{ 1 - \limsup_{r \to \infty} \bar{N}\left(r, \frac{1}{f^{(k)} - \alpha}\right) \right\} - k$$

$$= (k + 1) \, \Theta_a^{(k)} (\alpha, f) - k.$$

The above concept of relative defects corresponding to distinct poles was also taken up by A.P. Singh [25] for two meromorphic functions f_1 and f_2, and he defined

$$\Theta_{1,2}(\infty) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_{1,2}(r, \infty)}{T(r, f_1) + T(r, f_2)},$$

$$\Theta_0(\infty) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_0(r, \infty)}{T(r, f_1) + T(r, f_2)}$$

where

$$\bar{N}_0(r, \infty) = \int_0^r \frac{\bar{n}_0(t, \infty) - \bar{n}_0(0, \infty)}{t} \, dt,$$

where $\bar{n}_0(r, \infty)$ denotes the number of common poles of f_1 and f_2 in $|z| \leqslant r$, the poles being counted without their multipilicity and $\bar{N}_{1,2}(r, \infty) = \bar{N}(r, \infty, f_1) + \bar{N}(r, \infty, f_2) - 2 \bar{N}_0(r, \infty)$.

He proved

Therem.5 Let f_1 and f_2 be two meromorphic functions of finite order, and let

$$T(r, f_i') \sim a \, T(r, f_i)$$

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where $a \geq 1$ and $i = 1,2$. Then

$$\Theta_{1,2}(\infty) + 2\,\Theta_0(\infty) \leq 4 - a.$$

As an immediate consequences of the above theorem and using corollaries 1 to 3 of Theorem 3 we have the following corollaries.

Corollary 1'.

If f_1 and f_2 are two meromorphic functions of finite order such that

$$\Theta(\alpha, f_j) = 1 \text{ and } \Theta(\infty, f_j) = 1,$$

for $\alpha \neq \infty$ a nd $j = 1,2$, then

$$\Theta_{1,2}(\infty) + 2\,\Theta_0(\infty) \leq 3.$$

Corollary 2':

Let f_1, f_2 be two meromorphic functions of finite order,

(i) If $\Theta(a_i, f_j) = \frac{1}{2}$,

$$\Theta(\infty, f_j) = \frac{1}{2}$$

for $i = 1,2,3$ ($a_i \neq \infty$) and $j = 1,2$, then

$$\Theta_{1,2}(\infty) + 2\,\Theta_0(\infty) \leq \frac{5}{2}$$

(ii) And if f_1, f_2 be two meromorphic functions of finite order and if

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$$\Theta(a_i, f_j) = \frac{1}{2}$$

for i = 1,2,3,4, j = 1,2, and

where a_i are finite, distinct then

$$\Theta_{1,2}(\infty) + 2\Theta_0(\infty) \leqslant 2.$$

Corollary 3':

If f_1, f_2 are entire functions of finite order such that

$$\Theta(a_i, f_j) = \frac{1}{2} \text{ for } i = 1,2,\ j = 1,2$$

then

$$\Theta_{1,2}(\infty) + 2\Theta_0(\infty) \leqslant 3.$$