CHAPTER III

CHAPTER III

NEVANLINNA CHARACTERISTIC OF HOMOGENEOUS DIFFERENTIAL POLYNOMIALS AND THEIR DEFICIENT VALUES

Let f(z) be a non-constant meromorphic function in the complex plane, and let $m(r,\alpha,f)$, $N(r,\alpha,f)=N(r,\frac{1}{f-\alpha})$ $\bar{N}(r,\alpha,f)$, $N(r,f)$, $T(r,f)$ etc., have the usual meaning as explained in Chapter I. Similiarly for deficient values $\delta(\alpha,f)$, $\Theta(\infty,f)$, $\lambda(\alpha,f)$ see page-10 in Chapter I.

By a homogenous differential polynomial of degree n we shall mean a finite sum of the form

$$a(z)(f(z))^{l_0}(f'(z))^{l_1}\ldots(f^{(k)}(z))^{l_k} \tag{3.1}$$

where $l_0+l_1+\ldots+l_k= n$ and a(z) is any meromorphic function satisfying $T(r,a(z)) = S(r,f)$, where $S(r,f) = o(T(r,f))$ as $r \to \infty$.

A monomial of degree n is a homogeneous differential polynomial having just one term. Throughout this chapter it is assumed that f(z) is a transcendental function of finite order and by P(f) we shall mean a homogeneous differential polynomial of degree n which does not contain f. Thus P(f) in this chapter will be a finite sum of the form

$$a(z)(f'(z))^{l_1} \ldots (f^{(k)}(z))^{l_k} \tag{3.2}$$

where $l_1 + \ldots + l_k = n$.

We shall first prove some relations between Nevanlinna characteristic of f and p, and later we shall use these to find various relations between deficient values of homogenous differential polynomials.

Theorem 1. Let f be a meromorphic function and p(f) be a homogeneous differential polynomial in f of degree n as explained in (3.2). If

$$\sum_{\alpha \neq \infty} \delta(\alpha, f) \geq 1-\nu \; ; \; \delta(\infty, f) \geq 1-\nu \quad (0 \leq \nu \leq 1)$$

then

$$n(1-\nu m) \leq \liminf_{r \to \infty} \frac{T(r,p)}{T(r,f)} \leq \limsup_{r \to \infty} \frac{T(r,p)}{T(r,f)} \leq$$

$$\leq n\,(1+\nu m) \tag{3.3}$$

where $f^{(m)}$ is the highest derivative occurring in p. For the proof we shall need the following lemma :

Lemma 1. If p(f) is as in (3.2) and $\alpha_1, \ldots, \alpha_q$ $(q > 2)$ are distinct finite complex numbers, then

$$n \sum_{i=1}^{q} m(r, \alpha_i, f) + N(r, \tfrac{1}{p}) + S(r,f) \leq T(r,p)$$

$$\leq nT(r,f) + mn\bar{N}(r,f) + S(r,f) \tag{3.4}$$

where $f^{(m)}$ is the highest derivative occurring in p.

The first inequality of (3.4) is Lemma 2 of [11] and also the second inequality follows easily using (10) and (11) of the same paper. However, we give its proof as we shall be using this inequality frequently.

Since the poles of p occur only at the poles of f or at the poles of the coefficients a(z) of p, and since $(T(r,a(z)) = S(r,f)$, we ignore the poles of a(z). At a pole of f of order k it is easily seen that p has a pole of order at most nk + mn.

So,

$$N(r,p) \leqslant n\, N(r,f) + mn\bar{N}(r,f) + S(r,f) \tag{3.5}$$

Also clearly (see Lemma 1 of [11])

$$\begin{aligned} m(r,p) &\leqslant m(r,f^n) + m(r, \frac{p}{f^n}) \\ &\leqslant nm(r,f) + S(r,f). \end{aligned}$$

Adding with (3.5) we have

$$T(r,p) \leqslant nT(r,f) + mn\bar{N}(r,f) + S(r,f) \tag{3.6}$$

as desired.

Let us note that the above lemma includes a result of Kamthan [14,6] .

Proof of Theorem 1. From the second inequality in (3.4), we get

$$\limsup_{r\to\infty} \frac{T(r,p)}{T(r,f)} \leq n + mn\left((1-\Theta(\infty,f)\right)$$
$$\leq n + mn\left((1-\delta(\infty,f)\right)$$
$$\leq n + mn\nu$$
$$= n(1+m\nu) \qquad (3.7)$$

And considering the first part of inequality in (3.4), we have for any α_i

$$T(r,p) \geq n\sum_{i=1}^{q} m(r,\alpha_i,f) + N(r,\frac{1}{p}) + S(r,f)$$

which easily yields

$$\liminf_{r\to\infty} \frac{T(r,p)}{T(r,f)} \geq \sum_{i=1}^{q} \liminf_{r\to\infty} \frac{m(r,\alpha_i,f)}{T(r,f)}$$

Now making $q \to \infty$ and using the hypothesis it follows that

$$\liminf_{r\to\infty} \frac{T(r,p)}{T(r,f)} \geq n(1-\nu)$$
$$\geq n(1-\nu m) \text{ since } m \geq 1.$$

This along with (3.7) completes the proof of the theorem.

An immediate consequence of the above theorem is the following

<u>Corollary</u> If f is a meromorphic function of finite order such that $\delta(\infty, f) = 1$ and $\sum_{\alpha \neq \infty} \delta(\alpha, f) = 1$ and if p is a homogeneous differential polynomial of degree n satisfying (3.2), then

$$T(r,p) \sim nT(r,f)$$

The proof follows easily as our hypothesis imply that $\nu = 0$.

<u>Theorem 2</u>. If p is a homogeneous differential polynomial of degree n in a meromorphic function f of finite order and satisfying

$$N(r,f) + N(r,\frac{1}{f}) = S(r,f) \qquad (3.8)$$

Then

$$nT(r,f) \sim T(r,p) \sim \bar{N}(r,\frac{1}{p-b}) \quad \text{for all b except}$$

possibly 0 and ∞.

<u>Proof</u>. With the same hypothesis (3.8) it is already proved in Theorem 2 of [11] that

$$T(r,p) \sim \bar{N}(r,\frac{1}{p-b})$$

Our aim will be to prove the remaining part of the asymptotic relation. However for sake of completness we shall outline some of the steps of Theorem 2 of [11].

Now

$$nm(r,\frac{1}{f}) = m(r,\frac{1}{f^n})$$
$$\leq m(r,\frac{1}{p}) + m(r,\frac{p}{f^n})$$
$$= m(r,\frac{1}{p}) + S(r,f) \text{ by Lemma 1 of [11].}$$

Adding $nN(r,\frac{1}{f})$ to both sides and using (3.8) one has

$$nT(r,\frac{1}{f}) \leq m(r,\frac{1}{p}) + S(r,f)$$
$$\leq T(r,\frac{1}{p}) + S(r,f).$$

So,

$$nT(r,f) \leq T(r,p) + S(r,f) \qquad (3.9)$$

from which it follows that

$$S(r,f) = S(r,p). \qquad (3.10)$$

since the poles of p can occur only at the poles of f or at the poles of the coefficients a(z) of p and $T(r, a(z)) = S(r,f)$ we have

$$\bar{N}(r,p) \leq \bar{N}(r,f) + S(r,f)$$

so, by (3.8) and (3.9),

$$\bar{N}(r,p) = S(r,p) \qquad (3.11)$$

By (3.5),

$$N(r,p) \leq n(m+1)N(r,f) + S(r,f)$$

where n is the degree of p and $f^{(m)}$ is the highest derivative of f occurring in p.

Hence by (3.8) and (3.10), we get

$$N(r,p) = S(r,p) \qquad (3.12)$$

Again,

$$\bar{N}(r,\frac{1}{p}) \leq \bar{N}(r,\frac{1}{f^n}) + \bar{N}(r,\frac{f^n}{p})$$

$$\leq \bar{N}(r,\frac{1}{f}) + T(r,\frac{f^n}{p})$$

$$= \bar{N}(r,\frac{1}{f}) + T(r,\frac{p}{f^n}) + O(1)$$

$$= \bar{N}(r,\frac{1}{f}) + N(r,\frac{p}{f^n}) + S(r,f)$$

$$\leq \bar{N}(r,\frac{1}{f}) + N(r,p) + N(r,\frac{1}{f^n}) + S(r,f)$$

$$\leq (n+1)\, N(r,\frac{1}{f}) + S(r,p), \text{ by } (3.10) \text{ and } (3.12)$$

$$= S(r,p) \quad \text{by } (3.8) \text{ and } (3.10).$$

Thus $\bar{N}(r,\frac{1}{p}) = S(r,p)$. (3.13)

If $b \in C$ and $b \neq o$ then by Nevanlinna's second fundamental theorem [13, Theorem 2.5],

we have

$$T(r,p) \leq \bar{N}(r,p) + \bar{N}(r,\frac{1}{p}) + \bar{N}(r,\frac{1}{p-b}) + S(r,f)$$

$$= \bar{N}(r,\frac{1}{p-b}) + S(r,p), \text{ by } (3.11) \text{ and } (3.13).$$

Therefore from (3.9) we get

$$nT(r,f) \leq \bar{N}(r,\frac{1}{p-b}) + S(r,f) \tag{3.14}$$

(since $S(r,p) = S(r,f)$ always).

Now by first fundamental theorem of Nevanlinna, we have

$$T(r,p) \geq \bar{N}(r,\frac{1}{p-b}) + O(1) \tag{3.15}$$

So, from (3.14) and (3.15) we have

$$\liminf_{r \to \infty} \frac{T(r,p)}{T(r,f)} \geq \liminf_{r \to \infty} \frac{\bar{N}(r,\frac{1}{p-b})}{T(r,f)} \geq n \qquad (3.16)$$

Again, from (3.6) we have

$$T(r,p) \leq nT(r,f) + mn\bar{N}(r,f) + S(r,f)$$
$$\leq [n + O(1)]\, T(r,f) \text{ as } r \to \infty \qquad (3.17)$$

Hence from (3.15) and (3.17) one has

$$\liminf_{r \to \infty} \frac{\bar{N}(r,\frac{1}{p-b})}{T(r,f)} \leq \liminf_{r \to \infty} \frac{T(r,p)}{T(r,f)} \leq n \qquad (3.18)$$

and hence the theorem follows from (3.16) and (3.18).

We now prove that if $p(z)$ is a monomial of degree n containing f with the exponent of highest derivative as l_k then under certain conditions

$$T(r,p) \sim (n + kl_k)\, T(r,f)$$

More precisely we have the following theorem :

<u>Theorem 3.</u> Let $f(z)$ be a meromorphic function of finite order and let $\alpha_1 \neq \alpha_2$ be two finite complex numbers. Let

$$\limsup_{r \to \infty} \frac{\bar{N}(r,\alpha_i)}{T(r,f)} = 0, \qquad (i = 1,2)$$

Then

$$T(r,p) \sim (m + kl_k)\, T(r,f)$$

where $p(z) = (f)^{l_0}(f')^{l_1}\ldots(f^{(k)})^{l_k}$ is a monomial of degree m satisfying

$$N(r,p) \leqslant (m + kl_k)\, N(r,f)$$

Proof. The poles of p occur at the ples of f, and if f has a pole of order n then

$$\text{number of poles of } p(z) = l_0 n + l_1(n + 1) + \ldots l_k(n + k)$$
$$= (l_0 + \ldots + l_k)\, n + (l_1 + 2l_2 + \ldots kl_k)$$
$$\geqslant mn + kl_k$$
$$\geqslant m + kl_k$$

Hence $N(r,p) \geqslant (m + kl_k)\, \bar{N}(r,f)$.

Now from Nevanlinna's second fundamental theorem we have

$$T(r,f) \leqslant \bar{N}(r,\frac{1}{f - \alpha_1}) + \bar{N}(r,\frac{1}{f - \alpha_2}) + \bar{N}(r,f) + S(r,f).$$

Since $\bar{N}(r,f) \leqslant T(r,f)$, it follows on using the hypothesis that

$$\bar{N}(r,f) \sim T(r,f)$$

So,

$$T(r,p) \geqslant N(r,p) \geqslant (m + kl_k)\, T(r,f)$$

which gives

$$\liminf_{r \to \infty} \frac{T(r,p)}{T(r,f)} = \liminf_{r \to \infty} \left[\frac{T(r,p)}{\bar{N}(r,f)} \cdot \frac{\bar{N}(r,f)}{T(r,f)}\right]$$
$$\geqslant m + kl_k$$

Thus

$$\liminf_{r \to \infty} \frac{T(r,p)}{T(r,f)} \geqslant m + kl_k \tag{3.19}$$

Next

$$T(r,p) = m(r,p) + N(r,p)$$

$$\leqslant m(r,f^m) + m(r,\frac{p}{f^m}) + N(r,p)$$

$$= m.m(r,f) + N(r,p) + S(r,f)$$

And so using the hypothesis we have

$$T(r,p) \leqslant m.m(r,f) + (m + kl_k)\, N(r,f) + S(r,f)$$

$$= mT(r,f) + kl_k N(r,f) + S(r,f)$$

$$\leqslant (m + kl_k)\, T(r,f) + S(r,f).$$

So,

$$\limsup_{r \to \infty} \frac{T(r,p)}{T(r,f)} \leqslant (m + kl_k) \tag{3.20}$$

From (3.19) and (3.20) we have

$$\lim_{r \to \infty} \frac{T(r,p)}{T(r,f)} = m + kl_k$$

and hence the desired result.

<u>Note</u>. There do exist monomials satisfying the condition

$N(r,p) \leqslant (m + kl_k)\, N(r,f)$. For example $p(z)=f^{(k)}$ is one of them. In this case the above theorem yields

$$T(r,f^{(k)}) \sim (k+1)\, T(r,f) \qquad \ldots$$

which is the result of kamthan [14,8] .

We now prove

<u>Theorem 4.</u> For $p(f)$ as defined in Theorem 1,

(i) $$\limsup_{r \to \infty} \frac{N(r,p)}{T(r,p)} \leq \frac{(m+1)\nu}{1-\nu m}$$

(ii) $$\limsup_{r \to \infty} \frac{N(r,\frac{1}{p})}{T(r,p)} \leq \frac{2\nu m}{1+\nu m}$$

<u>Proof.</u> From (3.5) we obtain

$$\frac{N(r,p)}{T(r,p)} \leq \frac{T(r,f)}{T(r,p)}\left[n\frac{N(r,f)}{T(r,f)} + mn\,\frac{\bar{N}(r,f)}{T(r,f)} + \frac{S(r,f)}{T(r,f)}\right].$$

Using the first part of inequality (3.3) we obtain

$$\limsup_{r \to \infty} \frac{N(r,p)}{T(r,p)} \leq \frac{1}{n(1-\nu m)}\left\{n(1-\delta(\infty,f)) + mn(1-\Theta(\infty,f))\right\}$$

$$\leq \frac{1}{n(1-\nu m)}\left[(n\nu + mn\nu)\right]$$

and hence (i) follows.

To prove (ii) we consider the first part of inequality (3.4). Dividing by $T(r,p)$ and taking superior limit as $r \to \infty$ we obtain on simplifying

$$\limsup_{r \to \infty} \frac{N(r,\frac{1}{p})}{T(r,p)} \leq 1 - \left(\liminf_{r \to \infty} \frac{T(r,f)}{T(r,p)}\right) \times$$

$$\times \quad \left(n \sum_{i=1}^{q} \liminf_{r \to \infty} \frac{m(r,\alpha_i,f)}{T(r,f)}\right)$$

$$\leq 1 - \frac{n}{n(1+m\nu)} \sum_{i=1}^{q} \delta(\alpha_i, f)$$

$$\leq 1 - \frac{(1-\nu m)}{1+\nu m)}$$

$$= \frac{2\nu m}{1+\nu m} \text{ as desired.}$$

<u>Theorem 5</u>. Let $|\alpha_i| < \infty$ $(i = 1,2,\ldots q)$. If p is a homogeneous differential polynomial of order n satisfying (3.2), then

$$\left[1-\delta(0,p)+\lambda(0,p)\right] \sum_{i=1}^{q} \delta(\alpha_i,f) \leq \left[1+m\left(1-\Theta(\infty,f)\right)\right]\lambda(0,p) \quad \ldots \quad (3.21)$$

where $f^{(m)}$ is the highest derivatice of f occurring in p $(m \geq 0)$

<u>Proof</u>. Let $\lim\limits_{r \to \infty} \begin{array}{c}\sup\\ \inf\end{array} \dfrac{T(r,p)}{T(r,f)} = \begin{array}{c}A\\ B\end{array}$

Adding $n \sum_{i=1}^{q} N(r,\alpha_i,f)$ to both sides of first inequality of (3.4), we obtain

$$nqT(r,f) + N(r,\frac{1}{p}) + S(r,f) \leq T(r,p) + n \sum_{i=1}^{q} N(r,\alpha_i,f)$$

So,

$$nq + \frac{N(r,\frac{1}{p})}{T(r,p)} \frac{T(r,p)}{T(r,f)} + \frac{S(r,f)}{T(r,f)} \leqslant \frac{T(r,p)}{T(r,f)} + n\sum_{i=1}^{q} \frac{N(r,\alpha_i,f)}{T(r,f)} \qquad \ldots\ldots(3.22)$$

Hence

$$nq + \liminf_{r\to\infty} \frac{N(r,\frac{1}{p})}{T(r,p)} \liminf_{r\to\infty} \frac{T(r,p)}{T(r,f)} \leqslant \liminf_{r\to\infty} \frac{T(r,p)}{T(r,f)} +$$

$$+ n\sum_{i=1}^{q} \limsup_{r\to\infty} \frac{N(r,\alpha_i,f)}{T(r,f)}$$

or we have

$$nq + (1 - \lambda(0,p))B \leqslant B + n\sum_{i=1}^{q} (1 - \delta(\alpha_i,f))$$

which reduces on simplifying to

$$n\sum_{i=1}^{q} \delta(\alpha_i,f) \leqslant B\,\lambda(0,p) \qquad (3.23)$$

Again from (3.22) we have

$$nq + \limsup_{r\to\infty} \frac{N(r,\frac{1}{p})}{T(r,p)} \liminf_{r\to\infty} \frac{T(r,p)}{T(r,f)} \leqslant \limsup_{r\to\infty} \frac{T(r,p)}{T(r,f)} +$$

$$+ n\sum_{i=1}^{q} \limsup_{r\to\infty} \frac{N(r,\alpha_i,f)}{T(r,f)},$$

and so,

$$nq + (1 - \delta(0,p))B \leqslant A + n\sum_{i=1}^{q} (1 - \delta(\alpha_i,f)),$$

therefore on rearranging,

$$(1 - \delta(0,p))\, B \leq A - n \sum_{i=1}^{q} (1 - \Theta(\alpha_i, f)),$$

and as $1 - \delta(0,p) \geq 0$, we find on multiplying this with the corresponding inequalities of (3.23)

$$n(1 - \delta(0,p)) \sum_{i=1}^{q} \delta(\alpha_i, f) \leq (A - n \sum_{i=1}^{q} \delta(\alpha_i, f))\lambda(o,p) \quad (3.24)$$

But from Lemma 1, we have

$$\frac{T(r,p)}{T(r,f)} \leq n + mn \frac{\bar{N}(r,f)}{T(r,f)} + \frac{S(r,f)}{T(r,f)}$$

and so

$$A = \limsup_{r \to \infty} \frac{T(r,p)}{T(r,f)} \leq n + mn\,(1 - \Theta(\infty, f)) \ \ldots \quad (3.25)$$

Therefore from (3.24) and (3.25) we get

$$(1 - \delta(o,p)) \sum_{i=1}^{q} \delta(\alpha_i, f) \leq \Big[1 + m\,(1 - \Theta(\infty, f)) - \sum_{i=1}^{q} \delta(\alpha_i, f)\Big] \lambda(0,p)$$

which on rearranging its terms gives the inquality (3.21), and this completes the proof.

Theorem 6. Let f be a meromorphic function of finite order and let p (z) be a homogeneous differential polynomial of degree n as defined in (3.2). Further let

$S(r,f) = S(r,p)$, then

then

$$\delta(0,p) \geq \frac{1}{m+1} \sum_{\alpha \neq \infty} \delta(\alpha,f) \tag{3.26}$$

where $f^{(m)}$ is the highest derivative occupying in p.

Further if f is entire then

$$\delta(0,p) \geq \sum_{\alpha \neq \infty} \delta(\alpha,f) \tag{3.27}$$

Proof. Let $\alpha_1, \alpha_2, \ldots \alpha_q$ be distinct finite complex numbers and let

$$F(z) = \sum_{i=1}^{q} \frac{1}{(f(z) - \alpha_i)^n}$$

Then it follows from lemma 1 that

$$n \sum_{i=1}^{q} m(r,\alpha_i,f) \leq m(r,\tfrac{1}{p}) + S(r,f).$$

So, dividing by $T(r,p)$ on both sides and using

$S(r,f) = S(r,p)$

we deduce that

$$n \sum_{i=1}^{q} \liminf_{r \to \infty} \frac{m(r,\alpha_i,f)}{T(r,p)} \leq \delta(0,p) \tag{3.28}$$

But from (3.6) we obtain

$$T(r,p) \leq (m+1)\, nT(r,f) + S(r,f)$$

and using this in (3.28), one has

$$\delta(0,p) \geq \sum_{i=1}^{q} \liminf_{r\to\infty} \frac{m(r,\alpha_i,f)}{(m+1)\,T(r,f)}$$

which yields

$$\delta(0,p) \geq \frac{1}{m+1} \sum_{i=1}^{q} \delta(\alpha_i,f)$$

On making $q \to \infty$, we obtain (3.26).

Next if f is entire function of finite order, then

$$\begin{aligned} T(r,p) &= m(r,p) \\ &= m(r,\frac{p}{f^n}) + m(r,f^n) \\ &= nm(r,f) + S(r,f) \\ &\leq nT(r,f) + S(r,f) \end{aligned}$$

and hence from (3.28) we have

$$\sum_{i=1}^{q} \liminf_{r\to\infty} \frac{m(r,\alpha_i,f)}{T(r,f)} \leq \delta(0,p).$$

On making $q \to \infty$, we obtain (3.27).

Let us note that if $p(f)$ is a monomial then the condition $S(r,f)= S(r,p)$ is automatically satisfied, since if

$$p(f)=(f')^{l_1}(f'')^{l_2}\ldots(f^{(k)})^{l_k} \text{ when } l_1+l_2+\ldots+l_k= n$$

then clearly $T(r,p) \leq AT(r,f) + S(r,f)$ for some constant A. Also

$$p(f) = f^n\left(\frac{f'}{f}\right)^{l_1}\left(\frac{f''}{f}\right)^{l_2}\cdots\left(\frac{f^{(k)}}{f}\right)^{l_k}$$

and so

$$nT(r,f) \leq T(r,p) + l_1T(r,\frac{f}{f'}) + \ldots + l_kT(r,\frac{f}{f^{(k)}})$$

$$= T(r,p) + l_1T(r,\frac{f'}{f}) + \ldots + l_kT(r,\frac{f^{(k)}}{f}) +$$

$$+ S(r,f).$$

Thus using Milloux's theorem (Lemma 2 of chapter II)

$$nT(r,f) \leq T(r,p) + l_1N(r,\frac{f'}{f}) + \ldots + l_kN(r,\frac{f^{(k)}}{f})$$

But $N(r,\frac{f^{(i)}}{f}) = i\left[\bar{N}(r,\frac{1}{f}) + \bar{N}(r,f)\right]$ for $i = 1, \ldots k$

$$\leq i\left[N(r,\frac{1}{p}) + N(r,p)\right]$$

$$\leq i\left[T(r,p) + T(r,p)\right] + S(r,p)$$

$$\leq 2iT(r,p) + S(r,f)$$

Thus $nT(r,f) \leq BT(r,p)\ S(r,f)$, for some constant B. Combining this above it follows that $S(r,f) = S(r,p)$.

Also since $\lambda(0,f^{(1)}) \geq \delta(0,f^{(1)})$, Theorem 3 of Kamthan [15] become a particular case of our theorem. We now use the above theorem to find an upper bound for $\delta(\infty, p)$.

<u>Theorem 7</u>. Let f be a meromorphic function of order $\rho_f < 1$. If p is a homogeneous differential polynomial in f satisfying (3.2)

and if $S(r,f) = S(r,p)$ then

$$\delta(\infty, p) \leq 2 - K(\rho_p) - \frac{1}{m+1} \sum_{\alpha \neq \infty} \delta(\alpha, f) \ldots \qquad (3.29)$$

where ρ_p denote order of p and $K(\rho_p) \geq 1 - \rho_p$.

<u>Proof</u>. Since $\rho_{fg} \leq \max(\rho_f, \rho_g), \rho_{f+g} \leq \max(\rho_f, \rho_g)$

and $\rho_f = \rho_{f'}$, it follows that $\rho_p \leq \rho_f \leq 1$.

Therefore, using a result of Hayman [13,101], we have

$$\limsup_{r \to \infty} \frac{N(r,P) + N(r,\frac{1}{p})}{T(r,p)} \geq K(\rho_p) \text{ where } K(\rho_p) \geq 1 - \rho_p$$

This easily yields

$$2 - \delta(\infty, p) - \delta(0,p) \geq K(\rho_p)$$

Thus

$$\delta(0,p) \leq 2 - K(\rho_p) - \delta(\infty, p) \qquad (3.30)$$

The proof now follows using the previous theorem.

<u>Remark (i)</u> If f is entire function and the coefficients $a(z)$ in p are entire functions then clearly p is entire and hence $\delta(\infty, p) = 1$. Hence from (3.30) we have

$$\delta(0,p) \leq 1 - K(\rho_p)$$

which with (3.27) gives

$$1 - K(\rho_p) \geq \sum_{\alpha \neq \infty} \delta(\alpha, f)$$

(ii) Let us also note that if $f(z)$ is a meromorphic function of non-integral order and p is a non-zero homogeneous differential polynomial such that each of the terms of p contain f, then A.P.Singh in [26] proved that inequality (3.30) holds. Our theorem proves the case when none of the terms of p contain f. However, in these cases we have been able to prove the result only for the case when order of f is less than one. It looks like the result may be true for any meromorphic function of non-integral order. However we have been unable to prove it.

Our next theorem also deals with finding certain different types of estimation for $\delta(0,p)$ and $\lambda(0,p)$.

<u>Theorem 8.</u> Let f and p be defined as in (3.2). Let $\{a_i\}$ $(i = 1, \ldots, s)$, $s > 2$ $|a_i| < \infty$ and $\{b_j\}$ $(j = 1, \ldots, t)$, $t > 2$, $0 < |b_j| < \infty$ be two sets of complex numbers. Let

$$\lim_{r \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{T(r,p)}{T(r,f)} = \begin{matrix}A\\ B\end{matrix} \text{, then}$$

$$B(t-1)\,\delta(0,p) - A\sum_{j=1}^{t}\delta(b_j,p) \geq nt\sum_{i=1}^{s}\delta(a_i,f) +$$

$$+ \Theta(\infty,f) + B(t-1) - At - 1 \quad (3.31)$$

and

$$B(t-1)\,\lambda(0,p) - B\sum_{j=1}^{t}\Theta(b_j,p) \geq nt\sum_{i=1}^{s}\delta(a_i,f) +$$

$$+\Theta(\infty, f) - (B + 1) \tag{3.32}$$

<u>Corollary.</u> For above f and p,

$$B\,\delta(0,p) \geq n \sum_{i=1}^{s} \delta(a_i, f) + B - A$$

$$B\,\lambda(0,p) \geq n \sum_{i=1}^{s} \delta(a_i, f)$$

The proof of the corollary follows by dividing with t and then making $t \to \infty$, in (3.31) and (3.32) and using the fact that $\sum_{j=1}^{\infty} \delta(b_j, p)$ is a bounded quantity.

For the proof of the theorem we shall need the following:

<u>Lemma 2.</u> For the hypothesis of Theorem 8,

$$nt\,T(r,f) \leq nt \sum_{i=1}^{s} N(r,a_i,f) + \sum_{j=1}^{t} \bar{N}(r,b_j,p) +$$

$$+ \bar{N}(r,f) - (t-1)\,N(r,\tfrac{1}{p}) + S(r,f).$$

<u>Proof of Lemma 2.</u> From the first part of inequality (3.4) we have

$$n \sum_{i=1}^{s} m(r,a_i,f) \leq T(r,p) - N(r,\tfrac{1}{p}) + S(r,f).$$

Adding $n \sum_{i=1}^{s} N(r,a_i,f)$ and then multiplying by t on both sides one obtains

$$nstT(r,f) \leq tT(r,p) + nt \sum_{i=1}^{s} N(r,a_i,f) - tN(r,\frac{1}{p}) + $$
$$+ S(r,f) \quad \ldots \quad (3.33)$$

But by second fundamental theorem of Nevanlinna when applied to p, and since $S(r,p) = S(r,f)$, one gets

$$tT(r,p) \leq \sum_{j=0}^{t} N(r,\frac{1}{p-b_j}) - N(r,\frac{1}{p'}) + \bar{N}(r,p) + S(r,f)$$

where b_j's are finite distinct $(t+1)$ numbers and where b_o is chosen to be 0.

Also since the coefficients $a(z)$ of p satisfy $T(r,a(z)) = S(r,f)$, we get

$$\bar{N}(r,p) \leq \bar{N}(r,f) + S(r,f)$$

Thus

$$tT(r,p) \leq \sum_{j=0}^{t} N(r,\frac{1}{p-b_j}) - N(r,\frac{1}{p'}) + \bar{N}(r,f) + S(r,f).$$

Thus we have

$$tT(r,p) \leq \sum_{j=1}^{t} \bar{N}(r,\frac{1}{p-b_j}) - N_0(r,\frac{1}{p'}) + \bar{N}(r,f) +$$
$$+ N(r,\frac{1}{p}) + S(r,f) \qquad (3.34)$$

where $N_o(r,\frac{1}{p'})$ is formed with the zeros of p' which are not the zeros of $p-b_j$ $(j= 1,\ldots,t)$.

The lemma now follows from (3.33) and (3.34).

Proof of Theorem 8. By Lemma 2, we have

$$nst \leq nt \sum_{i=1}^{s} \frac{N(r,a_i,f)}{T(r,f)} + \frac{T(r,p)}{T(r,f)} \sum_{j=1}^{t} \frac{\bar{N}(r,b_j,p)}{T(r,p)} +$$

$$+ \frac{\bar{N}(r,f)}{T(r,f)} - (t-1) \frac{T(r,p)}{T(r,f)} \frac{N(r,\frac{1}{p})}{T(r,p)} + \frac{S(r,f)}{T(r,f)}$$

ans so

$$nst \leq nt \sum_{i=1}^{s} \limsup_{r \to \infty} \frac{N(r,a_i,f)}{T(r,f)} + \left(\limsup_{r \to \infty} \frac{T(r,p)}{T(r,f)} \right) \times$$

$$\left(\sum_{j=1}^{t} \limsup_{r \to \infty} \frac{\bar{N}(r,b_j,p)}{T(r,p)} \right) + \limsup_{r \to \infty} \frac{\bar{N}(r,f)}{T(r,f)} +$$

$$+ \liminf_{r \to \infty} \left[- (t-1) \frac{T(r,p)}{T(r,f)} \frac{N(r,\frac{1}{p})}{T(r,p)} \right]$$

and hence we get

$$nst \leq nt \left[\sum_{i=1}^{s} \left\{ 1 - \delta(a_i,f) \right\} \right] + A \sum_{j=1}^{t} \left\{ 1 - \Theta(b_j,p) \right\} +$$

$$+ 1 - \Theta(\infty,f) - (t-1) B \left(1 - \delta(0,p) \right)$$

which on rearrangement gives (3.31).

Also from Lemma 2 we get

$$\liminf_{r\to\infty} (nst) \leq \liminf_{r\to\infty} \left\{ nt \sum_{i=1}^{s} \frac{N(r,a_i,f)}{T(r,f)} + \left(\frac{T(r,p)}{T(r,f)}\right) \times \right.$$

$$\left(\sum_{j=1}^{t} \frac{\bar{N}(r,b_j,p)}{T(r,p)}\right) + \frac{\bar{N}(r,f)}{T(r,f)} - (t-1)\frac{T(r,P)}{T(r,f)}\frac{N(r,\frac{1}{p})}{T(r,p)} +$$

$$\left. + \frac{S(r,f)}{T(r,f)} \right\}$$

and so

$$nst \leq nt \sum_{i=1}^{s} \limsup_{r\to\infty} \frac{N(r,a_i,f)}{T(r,f)} + \left(\liminf_{r\to\infty} \frac{T(r,p)}{T(r,f)}\right) \times$$

$$\left(\sum_{j=1}^{t} \limsup_{r\to\infty} \frac{\bar{N}(r,b_j,p)}{T(r,p)}\right) + \limsup_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} -$$

$$- (t-1) \liminf_{r\to\infty} \frac{T(r,p)}{T(r,f)} \liminf_{r\to\infty} \frac{N(r,\frac{1}{p})}{T(r,p)} .$$

which gives

$$nst \leq nt \sum_{i=1}^{s} (1 - \delta(a_i,f)) + B \sum_{j=1}^{t} (1 - \Theta(b_j,p)) +$$

$$+ 1 - \Theta(\infty,f) - (t-1)\, B\, (1 - \lambda(0,p))$$

and this results in (3.32).