CHAPTER-I

I N T R O D U C T I O N

1. BRIEF SURVEY OF NEVANLINNA'S WORK

From 1919 until the early 50's Rolf Nevanlinna's mathematical papers fall under the heading "Complex Analysis". After that, it is difficult to give one single title. It could perhaps be "Differential Geometry and Applications". More accurately, we say the titles may be "Linear spaces, Absolute Analysis and topics of Mathematical Physics". The first function- theoretic papers dealt with the interpolation of bounded analytic functions, schlict functions and the moment problem. In 1922, Nevanlinna focussed his research on the value distribution properties of meromorphic functions. The following decade, during which his subject dominated his research, was undoubtedly his most important mathematical period. Very significant was his work also in the thirties, still connected with the value distribution theory but perhaps more appropriately entitled "Harmonic measure and Applications". The war, coupled with the fact that in 1941-45 Nevanlinna was Rector of the University of Helsinki, caused a break in his research. But immediately after the war, he resumed his studies concentrating on the general theory of Riemann surfaces. From the last twenties on, Nevanlinna had been dealing with Riemann Surfaces in connection with his studies on the deficient values of meromorphic functions. He seems to have shifted his interest to the general theory just before the outbreak of the war.

Nevanlinna surveyed his function-theoretic work in three monographs. "Le theoreme de Picard - Borel et la theorie des fonctions meromorphes". Gauthier-Villars 1929, describes the new Nevanlinna theory for meromorphic functions. The monumental "Eindeutige analytische funktionen", Springer-Verlag 1936, deals with the harmonic measure and its applications and presents the value distribution theory of meromorphic functions, with regard to the topological features of the theory introduced in the early thirties, "The monograph Uniformisierung", Springer-Verlag 1953, is on Riemann Surfaces.

We shall now give an idea of his main work on the value distribution theory of meromorphic functions, which cultimates in Nevanlinna's First and Second Main Theorems.

The basic problem of the theory is to study the roots of the equation $f(z) = a$, where f is meromorphic function in the complex plane and a given complex number or ∞ . If f is a polynomial of degree n, the theory is very simple and symmetric. For every complex a, the equation has precisely n - roots, with due regard to multiplicity. Also, the growth of f near infinity is determined by n; as $Z \to \infty$ we have the asymptotic equation

$$\lim \frac{f(z)}{z^n} = C \qquad C \neq 0, \infty$$

By the turn of the century, the results on polynomials had been largely generalised to entire functions. The starting points were the Weierstrass product formula from 1876 for an entire function with prescribed zeros and Picard's theorem from 1880 that a non-constant entire function takes all complex values upto one possible exception.

2. NEVANLINNA'S THEORY OF MEROMORPHIC FUNCTIONS :

We know that, if $P(z)$ is a polynomial of degree n, then the equation $P(z) = a$ has n-roots for all values of 'a'. Keeping in mind this analogy consider a transcendental entire function as a polynomial of degree infinite. Then for every transcendental entire function $f(z)$, $f(z) = a$ should have infinity of solutions for all values of 'a'. But in reality this is not true; for instance, the equation $e^z = 0$ has no solution. As Picard proved that for the transcendental entire function $f(z)$, if we leave the possibility of one value of 'a' then the equation $f(z) = a$ has infinity of solutions. This theorem known as Picard's theorem is an improvement of the theorem of Weierstrass which states that if $f(z)$ is analytic having an isolated essential singularity at 'a' then the image by f for every deleted neighbourhood of 'a' is dense in the finite complex plane.

We shall define as usual the order ρ of an entire function by

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r,f)}{\log r};$$

where

$$M(r,f) = \max_{|z| = r} |f(z)| .$$

Also, by $\rho_1(a)$, we mean the exponent of convergence of the a - points of f(z) and is defined as

$$\rho_1(a) = \limsup_{r \to \infty} \frac{\log^+ n(r,a)}{\log r};$$

where n(r, a) denotes the zeros of f(z) - a in $|z| \leqslant r$, and where $\log^+ x$ denotes log x if $x \geqslant 1$ and is zero if $x \leqslant 1$.

It is well known that

$\rho_1(a) \leqslant \rho$ for all a.

See for e.g. [1, 1].

If n(r,a) = o, 'a' is said to be exceptional value in the sense of Picard (e.V.P.). If $\rho_1(a) < \rho$, 'a' is said to be exceptional value in the sense of Borel (e.V.B.). Borel's theorem for entire functions of finite order states that there can be at most one e.V.B. If ρ is infinite the classical theorem of Borel gives no information. We shall presently develop

The Nevanlinna theory of meromorphic functions which extends the theorem of Picard and Borel.

Let f(z) be a function meromorphic (i.e. regular except for poles) and not constant in the complex plane. For any 'a' in the extended complex plane $\bar{C}$ we denote by

$$n(r,a) = n(r,a,f) = n\left(r, \frac{1}{f-a}\right),$$

the number of roots of f(z) = a with due count of multiplicity in $|z| \leq r$, For $a = \infty$,

$$n(r,a) = n(r, \infty) = n(r,f),$$

stands as usual for the number of poles of f(z) in $|z| \leq r$. We set

$$N(r,a) = N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n(t,a) - n(0,a)}{t}\, dt + n(0,a)\log r;$$

$$N(r,\infty) = N(r,f) = \int_0^r \frac{n(t,\infty) - n(0,\infty)}{t}\, dt + n(0,\infty)\log r;$$

$$m(r,a) = m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| d\theta;$$

$$m(r,\infty) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta;$$

where

$$\log^{+t} |x| = \text{Max} \left\{ \log |x| , 0 \right\} .$$

The functions

$$T(r,a) = T(r, \frac{1}{f-a}) = m(r, \frac{1}{f-a}) + N(r, \frac{1}{r-a}),$$

and

$$T(r,\infty) = T(r,f) = m(r,f) + N(r,f) . \qquad \ldots (1.1)$$

The terms of (1.1) are derived from the famous Poisson-Jensen formula [12, 1], which states that, if f(z) is meromorphic in $|z| \leq R$, $(0 \leq R < \infty)$ and if a_μ $(\mu = 1, 2, \ldots M)$ are the zeros and b_ν (ν = 1 to N) are poles of f(z) in $|z| < R$, then if $z = re^{i\theta}$ $(0 < r < R)$ and if $f(z) \neq 0, \infty$ we have

$$\log |f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log |f(Re^{i\phi})| \frac{R^2 - r^2}{R^2+r^2-2Rr\cos(\theta-\phi)} d\phi$$

$$+ \sum_{\mu=1}^{M} \log \left| \frac{R(z - a_\mu)}{R^2 - \bar{a}_\mu z} \right| -$$

$$- \sum_{\nu=1}^{N} \log \left| \frac{R(z - b_\nu)}{R^2 - \bar{b}_\nu z} \right| .$$

The term T(r,f) is called the Nevanlinna characteristic functions of f(z) and play a fundamental role in the theory of meromorphic functions. This shares many properties of $\log M(r,f)$ with which we measure the growth of an entire function where

$$M(r,f) = \max_{|z| = r} |f(z)| .$$

Therefore it is natural that Nevanlinna characteristic function is used for measuring the growth of a meromorphic function.

For $m(r,a)$ we shall use the term proximity function. Equation (1.1) shows that $T(r,f)$ is the sum of two terms, the proximity function $m(r,f)$ which measures the proximity of $f(z)$ to ∞ on the circle $|z| = r$ and the enumerative function $N(r,f)$ which gives a weighted average of the number of infinitudes in the disk $|z| \leqslant r$.

If $f(z)$ is an entire function, $N(r,f)$ vanishes and we have

$$T(r,f) = m(r,f) \leqslant \log M(r,f).$$

On the same lines the proximity of $f(z)$ to the value $w = a$ on $|z| = r$ is measured by $m(r,a)$ and the weighted average of the number of a-values in the disk $|z| \leqslant r$ is given by $N(r,f)$.

In order to estimate the proximity function, we need two properties of the function

$$\log^+ p = \max(\log p, 0),\ p > 0.$$

They are stated as follows :

$$\left.\begin{aligned} &\log^+ (p_1 p_2 \ldots p_m) \leq \sum_{k=1}^{m} \log^+ p_k \\ &\log^+ \left(\sum_{k=1}^{m} p_k\right) \leq \sum_{k=1}^{m} \log^+ p_k + \log m \end{aligned}\right\} \quad \ldots (1.2)$$

The second of these relations gives

$$m(r, f-a) \leq m(r,f) + \log^+ |a| + \log 2.$$

It is a surprising fact that for an entire function f(z), the Nevanlinna characteristic function T(r,f) is connected to log M(r,f) by the following inequality :

$$T(r,f) \leq \log M(r,f) \leq \frac{R+r}{R-r}\, T(R,f)$$

where

$$0 \leq r < R.$$

With the use of above inequality it is very easy to show that for an entire function f,

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log \log M(r,f)}{\log r}.$$

This motivates the following :

Definition :

The order ρ of a meromorphic function f(z) is defined by,

$$\rho = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} .$$

Then the following properties on order are well known.

i) $\rho(f) = \rho(f')$.

ii) $\rho(f+g) \leq \text{Max} \{\rho(f), \rho(g)\}$

iii) $\rho(f.g) \leq \text{Max} \{\rho(f), \rho(g)\}$.

See for example [30] , [7, 42] .

The Nevanlinna's theory of meromorphic function is based on two fundamental theorems, known as the first and second fundamental Theorems of Nevanlinna respectively. We now state his first Fundamental Theorem.

I <u>Nevanlinna's First Fundamental Theorem</u> :

For every complex number 'a',

$$m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) = T(r,f) - \log |f(o)-a| + \epsilon(a,r)$$

where

$$|\epsilon(a, r)| \leq \log^{+}|a| + \log 2.$$

If we allow r to vary, then the above theorem can be simply written as

$$m(r,a) + N(r,a) = T(r,f) + O(1). \qquad \ldots (1.3)$$

for every 'a' finite or infinite.

Thus the enumerative function

$m(r,a) + N(r,a)$

is independent of 'a'.

It is easy to see that if f(z) is meromorphic in the finite plane and if its characteristic function is bounded then f(z) is necessarily a constant. More generally, if $T(r,f) = o(\log r)$ then f(z) is a rational function. See for e.g. [15, 213].

From the definitions and the relations (1.2) we note the following inequalities :

$$m(r, f_1 f_2 \dots f_m) \leqslant \sum_{k=1}^{m} m(r, f_k);$$

$$N(r, f_1 f_2 \dots f_m) \leqslant \sum_{k=1}^{m} N(r, f_k);$$

and as a consequence we have

$$T(r, f_1 f_2 \dots f_m) \leqslant \sum_{k=1}^{m} T(r, f_k),$$

and

$$T(r, f_1 + f_2 + \dots + f_m) \leqslant \sum_{k=1}^{m} T(r, f_k) + \log m.$$

Further, if k is any constant $k \neq 0$, then

$$\left| T(r, kf) - T(r,f) \right| \leqslant \left| \log |k| \right|.$$

Similarly as we saw above,

$$\left| T(r,f-a) - T(r,f) \right| \leq \log^{+}|a| + \log 2.$$

If we combine the last two relations with one more relation, viz.

$$T\left(r, \frac{1}{f}\right) = T(r, f) - \log |f(o)| ;$$

We see that replacing f(z) by a linear fractional transform

$$\frac{\alpha f(z) + \beta}{\gamma f(z) + \delta}, \quad \alpha\delta - \beta\gamma \neq 0;$$

changes the characteristic function by a bounded function of r. Also, it can be shown as in [15, 213] that $N(r, \frac{1}{f-a})$ is a monotone increasing function of r and a convex function of log r. The function m(r, f-a) is rather irregular. So it is somewhat surprising that like log M(r,f), T(r,f) which is m (r,f) + N (r,f) is also convex function of log r and an increasing function of r. Also, T (r,f) is differentiable and $rT'(r, f)$ is non-decreasing. See [26, 2] .

The base of above facts about T(r,f) is due to Henri Cartan [3] who proved :

If f(z) is meromorphic in $|z| < R \leq \infty$ and if $f(o) \neq \infty$ then

$$T(r,f) = \log^{+} |f(o)| + \frac{1}{2\pi} \int_{0}^{2\pi} N(r, e^{i\theta}) \, d\theta .$$

The first fundamental theorem fails to tell us which of the two terms m(r,a) or N(r,a) of (1.3) is normally the more important one. But it becomes clear from the second fundamental theorem, which is stated below, that in general, it is N(r,a) which dominates.

II Nevanlinna's Second Fundamental Theorem :

Let f(z) be a transcendental meromorphic function of order ρ. Let $a_1, a_2, \ldots a_q$ $(q \geq 3)$ be distinct numbers (finite or infine), then

$$(q-2)\ T(r,f) < \sum_{i=1}^{q} N(r,a_i) - N_1(r) + S\ (r,f),$$

where

$$N_1(r) = N(r, \frac{1}{f'}) + 2N\ (r,f) - N(r,f'),$$

and

$$S(r,f) = O\ (\log r) \text{ if } \rho < \infty$$

$$S(r,f) = O\left[\log r + \log T\ (r,f)\right]$$ for all r; except possibly for a set of r of linear measure finite if $\rho = \infty$.

Using the first Fundamental Theorem and second Fundamental Theorem of Nevanlinna it is easy to prove the following classical theorems :

Picard's Theorem :

If f(z) is a transcendental meromorphic function, then f(z) - a has infinity of zeros for all 'a $\in \bar{C}$ except possibly two values of 'a'. In case a = ∞ , we as usual understand by a zero of $f(z)-a$, a pole of $f(z)$.

The above theorem is best possible in the sense that two exceptions may exist, for instance, e^z omits 0, and tan z omits $\pm i$.

Borel's Theorem :

If $f(z)$ is a transcendental meromorphic function of order ρ $(0 \leq \rho \leq \infty)$, then $\rho_1(a) = \rho$ for all values of 'a' except possibly two values of 'a'.

If we consider the only distinct zeros of $f(z)-a_i$, then the Nevanlinna's second Fundamental Theorem can be put in the form :

Let $f(z)$ be a meromorphic function of order ρ and let $a_1, a_2, \ldots a_q \in \overline{C}$ be distinct and let $q \geq 3$. Then,

$$(q-2)\ T(r,f) < \sum_{i=1}^{q} \overline{N}(r, a_i) + S(r,f),$$

where

$$\overline{N}(r,a) = \int_0^r \frac{\overline{n}(t,a) - \overline{n}(0,a)}{t}\, dt + \overline{n}(0,a) \log r;$$

where $n(r,a)$ denotes the number of distinct roots of $f(z)=a$ in $|z| \leq r$ and where $S(r,f)$ has the same meaning as earlier.

Another interesting theorem due to Nevanlinna is his uniqueness theorem [22] which states that if the meromorphic functions f and g share five values ignoring multiplicity, then either $f = g$ or f and g are both constants and where f and g sharing the value c means $f(z)-c$ and $g(z)-c$ have the same zeros.

For quite some time, no work was done on shared values, until recently when Rubel and Yang [25] proved the following result :

If a non-constant entire function f and its derivative f' share two finite values counting multiplicities (CM), then $f = f'$.

The same theorem has been proved by Gray G.Gundersen [10] , for non-constant meromorphic function. An immediate consequence of it is the result which states that if a and b are two distinct complex constants and ω is a non-constant entire function then the algebraic differential equation

$$f' = \frac{(a-be^{\omega})f + ab\,(e^{\omega}-1)}{(1-e^{\omega}) + ae^{\omega} - b}$$

does not possess a non-constant meromorphic solution f. Another interesting consequence is obtained by combining the above theorem and the theorem [8] stating that the meromorphic functions f and f' share the value $a \neq 0$, ignoring multiplicities (IM) if and only if there is a non-constant entire function h such that

$$f = a\left(1 + \frac{h}{h'}\right);$$

where h has only simple zeros and $h''(z) = 0$ implies either $h'(z) = 0$ or $h(z) = 0$; the consequence being if h and g are non-constant entire functions such that

i) h', g' share 0 CM,

ii) h'', g'' share 0 CM,

and iii) $a(1 + h/h') = b(1 + g/g')$,

for distinct non-zero numbers a and b, then

$h'(z) = C\, e^{-z}$;

and

$g'(z) = K\, e^{-z}$;

for non-zero constants C and K.

R. Nevanlinna also found results on the two meromorphic functions f, g that share four values CM and all pairs f, g that share three values CM. He proved the following theorems:

I. If two distinct non-constant meromorphic functions f and g share four values $a_i \Big|_{i=1}^{4}$ CM, then f is a Mobius transformation of g, two of the shared values, say a_1 and a_2 must be Picard values, and the cross ratio

$(a_1, a_2, a_3, a_4) = -1$.

For example, if h is a non-constant entire function then e^h, e^{-h} share $0, \infty, \pm 1$ CM.

Recently Gray G. Gundersen [11] has shown that the hypothesis of theorem I can be relaxed somewhat by proving the following theorem :

II. If two non-constant meromorphic functions f and g share three values CM and share a fourth value IM, then f and g share all four values CM (hence if $f \not\equiv g$, the conclusion of above theorem hold).

On the other hand, the following example shows that we cannot simply replace "CM" by "IM" in I, for, let h be a non-constant entire function and b be a non-zero constant then

$$f = \frac{e^h + b}{(e^h - b)^2} \quad ,$$

and

$$g = \frac{(e^h + b)^2}{8b^2 \ (e^h - b)}$$

whare 0, ∞, 1/b and $-1/8b$ by different multiplicities (DM) at every point. In contrast to theorem I, f is not a Mobius transformation of g, none of the shared values are Picard values, and the cross ratio of any permutation of the standard values does not equal -1. In the same paper [11] Gundersen has tried to "close the gap" between above example and theorem II by proving the following theorem:

III. If two non-constant meromorphic functions f and g share four values $a_i \big|_{i=1}^{4}$ a_1 and a_2 both CM and a_3 and a_4 both IM, then f and g share all four values CM.

In the above we have considered meromorphic functions f and g that share 3, 4 or 5 values. Work on meromorphic functions, that share two values only, has also been considered. However, in this case, the sharing of the values by the corresponding derivatives of the functions has also to be considered. In fact C.C.Yang in his paper [37] has classified all possible types of meromorphic functions f and g that are possible if f and g and f', g' share the value 0 CM, and further if the zeros of f and g are simple. In fact, he proved that :

Suppose two transcendental entire functions f and g satisfy the following three conditions;

a) f and g share 0 CM, and all the zeros are simple,

b) f' and g' share 0 CM,

c) $$\rho = \max\left\{\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r},\ \limsup_{r\to\infty}\frac{\log\log M(r,g)}{\log r}\right\} < 1 .$$

Then f and g satisfy exactly one of the following two relations :

i) $f(z) = C(g(z))^{K}$

where C and K are constants;

or

ii) $f(z) = C_1 \; e^{\nu(z)} + C_2,$

$g(z) = C_3 \; (C_2 \; e^{-\nu(z)} + C_1 \;),$

where C_1, C_2 and C_3 are constants and ν is entire of order less than one.

Later G. Gundersen in his paper [9] gave all possible classifications if f and g and f' and g' share 0 CM (i.e. ignoring whether the zeros are simple or not simple). Then he showed

IV. f and g are entire functions of finite order such that f, g share 0 CM and f', g' share 0 CM if and only if we have exactly one of the following four cases :

i) $f(z) = C \; g(z)$ where $C \neq 0$ is a constant and f is entire with order $(f) < \infty$;

ii) $f(z) = e^{p(z)}$, $g(z) = a \; e^{bp(z)}$;

where $a \neq 0$, and $b \neq 0, 1$ are constants and p is a non-constant polynomial;

iii) $f(z) = a \; (e^{p(z)} - 1)^n$, $g(z) = b(1 - e^{-p(z)})^n$;

where a, b are non-zero constants, n is a positive integer, and p is a non-constant polynomial;

iv) $$f(z) = \exp\left[\sum_{n=0}^{2N} a_n (2\pi i)^{-n} \int \frac{p'(z)\ (p(z))^n}{1 - e^{-p(z)}}\, dz \right],$$

$$g(z) = \exp\left[\sum_{n=0}^{2N} a_n\ (2\pi i)^{-n} \int \frac{p'(z)\ (p(z))^n}{e^{p(z)} - 1}\, dz \right].$$

where N is a positive integer, p is a non-constant polynomial and $a_n \Big|_{n=0}^{N}$ are rational numbers ($a_{2N} > 0$) such that $\sum_{n=0}^{2N} a_n k^n$ is a non-negative integer where k is an integer.

V. Let f and g be entire functions such that f, g share 0 CM and f', g' share 0 CM then we have one of the following four cases :

i) $f(z) = C\ g(z)$,

where C is a non-zero constant.

ii) $f(z) = a\ (e^{h(z)} - 1)^n$, $g(z) = b(1 - e^{-h(z)})^n$,

where a, b are non-zero constants, n is a positive integer, and h is a non-constant entire function.

iii) the multiplicities of the zeros of f, g are bounded and as $r \to \infty$,

$$N(r,0,f) = N(r,0,g) = O(\log T(r,f)) + O(\log T(r,g))$$

n.e. and

$$N(r,0,f') = N(r,0,g') = O(\log T(r,f')) + O\ (\log T(r,g'))$$ n.e.

or

iv) the multiplicities of the zeros of f, g are unbounded,

and $$\lim_{r \to \infty} \frac{\bar{N}(r,o,f)}{T(r,f) + T(r,g)} = o \text{ n.e.,}$$

and $$\lim_{r \to \infty} \frac{\bar{N}(r,o,f')}{T(r,f') + T(r,g')} = o \text{ n.e.,}$$

where n.e. stands for nearly everywhere, and is to mean the interval $o \leqslant r < \infty$ minus a set of finite linear measure.

For any $a \in \bar{C}$, the quantity

$$\delta(a,f) = \liminf_{r \to \infty} \frac{m(r,a)}{T(r,f)}$$

is called the Nevanlinna deficiency of the value 'a' with respect to the function f(z) where f is meromorphic. It is clear, by the first fundamental theorem of Nevanlinna, that

$$\delta(a,f) = \liminf_{r \to \infty} \frac{m(r,a)}{T(r,f)} = 1 - \limsup_{r \to \infty} \frac{N(r,a)}{T(r,f)},$$

If we change N to $\bar{N}$ in the above relation, the quantity on R.H.S. becomes

$$1 - \limsup_{r \to \infty} \frac{\bar{N}(r,a)}{T(r,f)},$$

which is denoted by Ⓗ(a, f).

If all the roots of the equation

$$f(z) = a \ \left(\frac{1}{f(z)} = o \text{ if } a = \infty \right)$$

are multiple roots, then we call to the value 'a' as the completely ramified value with respect to f(z).

Also as usual we set

$$\lambda(a) = \lambda(a,f) = 1 - \liminf_{r \to \infty} \frac{N(r,a)}{T(r,f)}$$

$$\Theta(a) = \Theta(a,f) = \liminf \frac{N(r,a) - \bar{N}(r,a)}{T(r,f)} \qquad \text{etc.}$$

The deficient values corresponding to zeros and poles being counted only once have also been studied extensively. One interesting theorem known as Nevanlinna's theorem on deficient values states that if f(z) is meromorphic function then the set of values a, for which $\Theta(a) > 0$ is countable and $\sum_a \Theta(a) \geq 2$.

From this since $\delta(a) < \Theta(a)$, it is easily seen that

$$\sum_a \delta(a) \geq 2.$$

The total deficiency is said to be attained if $\sum \delta(a) = 2$. S.K.Singh and H.S.Gopalkrishna, in [32] have shown by an example, that a meromorphic function may be such that $\sum_{a \in \mathbb{C}} \delta(a) = 1$ whereas $\sum \Theta(a) = 2$. Since $\sum_{a \in \mathbb{C}} \delta(a,f) \leq 2$, a question that arises is, what would be the value of $\sum_a \delta(a,f)^{\alpha}$, where α is a real number.

A. Weierstrass [34] showed that a meromorphic function f(z) of finite lower order satisfies

$$\sum \delta\ (a,f)^{\alpha} < \infty \quad \text{as long as} \quad \alpha \geq \tfrac{1}{3}\,.$$

See for example, [23] .

In the above assertion we can not replace α by any positive number less than 1/3. See for example [12, 98] .

These facts naturally pose the problem to determine the upper bound $\Delta\ (\alpha,\mu)$ of $\sum \delta\ (a,f)^{\alpha}$.

Overall meromorphic functions f(z) of order $\leq \mu$ $(1/3 \leq \alpha < 1)$. T. Murai has thus show that for $1/3 < \alpha < 1$,

$$\mu^{1-\alpha} \leq \Delta\ (\alpha,\ \mu) \leq C_{\alpha}\mu^{1-\alpha} \quad (\mu \geq 1),$$

where C_{α} is a constant depending only on α. An immediate consequence of the above result are

$$\lim_{\mu \to \infty} \frac{\log \Delta(\alpha,\mu)}{\log \mu} = (1-\alpha) \quad (\ 1/3 < \alpha < 1\ );$$

and if we let $\lambda = \dfrac{C\ (\alpha,\ S)}{\mu}$, then

$$\sum_{\delta(\cdot,f)<\lambda} \delta\ (a,f) < \sum_{\delta(\cdot,f)<\lambda} \delta\ (a,f)^{\alpha} \quad \lambda^{1-\alpha} \leq C_{\alpha}\ \mu^{1-\alpha}\ \lambda^{1-\alpha} \leq \frac{\ }{2}$$

Another interesting result proved by Nevanlinna [22, 53] is

Let $f(z)$ be a meromorphic function in $|z| < \infty$ and let

$$k(f) = \limsup \frac{N(t,0) + N(t,\infty)}{T(t,f)},$$

Then there is a constant $C(\rho)$ such that for a non-integral order ρ of f,

$$k(f) \geq C(\rho) > 0.$$

At the same time he made the following conjecture :

$$K(\rho) = \inf K(f) = \begin{cases} \dfrac{|\sin \pi\rho|}{q + |\sin \pi\rho|} & (q \leq \rho \leq q + \tfrac{1}{2}) \\ \dfrac{|\sin \pi\rho|}{q+1} & (q + \tfrac{1}{2} \leq \rho \leq q+1) \end{cases}$$

where infimum is taken over all meromorphic functions of order ρ.

To date, the above conjecture has not been solved. However, for functions of order $\leq \rho$, Edrei and Fuchs (see e.g. [6]) have been able to prove that

$$K(\rho) = \begin{cases} 1 & 0 \leq \rho < 1/2 \\ \sin \pi\rho & \tfrac{1}{2} \leq \rho \leq 1 \end{cases}.$$

Hellerstein and Williamson [18] have been able to prove th the conjecture is true for entire functions of order ρ w. only negative zeros. Recently, M. Ozawa [23] proved under restrictive conditions, viz. let $f(z) = \dfrac{\prod E\left(\frac{z}{a_n}, q\right)}{\prod E\left(\frac{z}{b_n}, q\right)}$

be a meromorphic function of order ρ $(q < \rho < q + 1)$ and

let $\int_a^{\infty} T(t)^{-\alpha-1}\, dt \to \infty$ as $\alpha \to \infty$ decreasingly, let

$$S(t,E) = \frac{1}{2\pi} \int_E \log\left| f(t e^{i\theta})\right| d\theta + N(t, \infty),$$

where E is a measurable subset of $[-\pi, \pi]$ and let $L(\rho)$

be the constant defined by

$$L(\rho) = \begin{cases} \dfrac{|\sin \pi\rho|}{q + |\sin \pi\rho|} & q < \rho < q + 1/2 \\ \\ \dfrac{|\sin \pi\rho|}{q + 1} & q + 1/2 \leq \rho < q + 1 . \end{cases}$$

then,

$$L(\rho) \liminf_{t \to \infty} \frac{S(t, E)}{T(t,f)} \leq K(f),$$

for any measurable subset E of $[-\pi, \pi]$. And if for any positive ϵ, there is a sequence $\{r_n(\epsilon)\}$ such that any t in $[r_n(\epsilon), R_n(\epsilon)]$ with

$$R_n(\epsilon) = r_n(\epsilon) \log 1/\epsilon ,$$

$$T(t)t^{-\rho} \leq kT(r_n(\epsilon))\, r_n(\epsilon)^{-\rho} \quad (K : \text{bounded})$$

$$T(r_n(\epsilon))\, r_n(\epsilon)^{-\rho+\epsilon} \leq T(t)\, t^{-\rho+\epsilon},$$

and

$$r_n(\epsilon) \to \infty \text{ as } n \to \infty, \text{ we have}$$

$$L(\rho) \liminf_{t \to \infty} \frac{S(t, E)}{T(t, f)} \leq K(f),$$

for any measurable subset E of $[-\pi, \pi]$. In a subsequent paper [24] M. Ozawa also proved,

Let f(z) be a meromorphic function of regular growth of order ρ. Then

$$K(f) \geq L(\rho) \liminf_{t \to \infty} \frac{S(t,E)}{T(t,f)},$$

and

Let f(z) be a meromorphic function defined by a quotient of two canonical products of genus q

$$f(z) = \frac{\Pi E(z/a_n, q)}{\Pi E(z/b_n, q)}.$$

Suppose that the order λ and the order μ of f(z) satisfies

$$q \leq \mu < \lambda < q + 1.$$

Let β be a number satisfying $\mu < \beta < \lambda$. Then for any E,

$$\sup_{\mu < \beta < \lambda} L(\beta) \liminf_{t \to \infty} \frac{S(t,E)}{T(t,f)} \leq K(f).$$

The term,

$$1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{f-\alpha})}{T(r,f)},$$

as mentioned earlier is called a deficient value and is denoted by $\delta\ (\alpha, f)$. If instead of considering f, the derivative f' is considered, then the properties of the term

$$\delta_r\ (\alpha,\ f') = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{f'-\alpha})}{T(r,f)}$$

were considered by H. Milloux [19] . Later K.L.Hiong [17] defined the relative defects of the value α with respect to $f^{(k)}$ viz.

$$\delta_r^{(k)}\ (\alpha, f) = 1 - \limsup_{r \to \infty} \frac{N\ (r, \frac{1}{f^{(k)} - \alpha})}{T(r,f)} \quad \ldots (1.4)$$

and the usual defect viz.

$$1 - \limsup_{r \to \infty} \frac{N\ (r, \frac{1}{f^{(k)} - \alpha})}{T(r, f^{(k)})},$$

he denoted by $\delta_a^{(k)}$ $(\alpha,\ f)$. In his paper, he found various relations between the two defects. An interesting result regarding the relative defect was that, unlike the absolute defect, the relative defect of $f^{(k)}$ could have negative values with - K as its lower bound. In the case of K = 1, Milloux [17, 163] gave a example to show that the lower bound could be reached.

If in (1.4) only distinct zeros are considered, then the corresponding term

$$\text{Ⓗ}_r^{(k)}(\alpha, f) = 1 - \limsup_{r \to \infty} \frac{\bar{\bar{N}}\left(r, \frac{1}{f^{(k)}-\alpha}\right)}{T(r,f)}$$

was defined by A.P.Singh [28] in which he obtained various relations between $\text{Ⓗ}_r^{(k)}(\alpha, f)$ and $\delta(\infty, f)$, $\delta(o, f)$ etc., for instance, he showed that if f(z) is a meromorphic function, then for every non-negative integer k,

$$\text{Ⓗ}_r^{(k)}(\alpha, f) \leqslant 2 - \left\{\delta(o, f) + \text{Ⓗ}(\infty, f)\right\}$$

where $\alpha \neq 0, \infty$.

And,

if $f_q(z)$ is an entire function and $a_i \Big|_{i=1}^{p}$ and

$b_j \Big|_{j=1}^{q}$ are finite complex numbers, distinct within each set and such that $bj \neq 0$ for any j, and if $\sum_{i=1}^{\infty} \delta(a_i, f) = 1$,

then
$$\sum_{bj \neq \infty} \text{Ⓗ}_r^{(k)}(bj, f) \leqslant 1.$$

In a subsequent paper [29] A.P.Singh extended his results for two meromorphic functions having common roots. In order to do this new notations, dealing with common roots and disjoint roots would be essential. Thus we shall first define these new notations :

Let $f_1(z)$ and $f_2(z)$ be two functions, meromorphic and non-constant. Let $n_o(r,a)$ denote the number of common roots in the disk $|z| \leq r$ of the two equations $f_1(z) = a$ and $f_2(z) = a$. Also, let $\bar{n}_o(r,a)$ denote the number of common roots in the disk $|z| \leq r$ of the two equations $f_1(z) = a$ and $f_2(z) = a$ where the multiplicity is disregarded (i.e. each root being counted only once). Set

$$\bar{N}_o(r,a) = \int_o^r \frac{\bar{n}_o(t,a) - \bar{n}_o(o,a)}{t} dt + \bar{n}_o(o,a) \log r;$$

$$\bar{N}_{1,2}(r,a) = \bar{N}\left(r, \frac{1}{f_1-a}\right) + \bar{N}\left(r, \frac{1}{f_2-a}\right) - 2\bar{N}_o(r,a).$$

Let $\bar{n}_o^{(k)}(r,a)$, $\bar{N}_{1,2}^{(k)}(r,a)$ etc. denote the corresponding quantities with respect to $f_1^{(k)}$ and $f_2^{(k)}$. Set

$$Ⓗ_{1,2}(a) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_{1,2}(r,a)}{T(r,f_1) + T(r,f_2)},$$

$$Ⓗ_{1,2}^{k}(a) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_{1,2}^{(k)}(r,a)}{T(r,f_1) + T(r,f_2)}$$

$$\delta_{1,2}(a) = 1 - \limsup_{r \to \infty} \frac{N_{1,2}(r,a)}{T(r,f_1) + T(r,f_2)},$$

$Ⓗ_o(a)$, $Ⓗ_o^{(k)}(a)$ being similarly defined.

With these notations, A.P.Singh proved several theorems on deficient values. We shall just list a few of these :

i) Let $f_1(z)$, $f_2(z)$ be two meromorphic functions such that

$$N(r, 1/f_1) = S(r, f_1) \text{ and}$$

$$N(r, 1/f_2) = S(r, f_2) .$$

Then, for any $a \neq 0, \infty$,

$$\Theta^{(k)}_{1,2}(a) + 2\Theta^{(k)}_{o}(a) \leq 5 - (\Theta_{1,2}(\infty) + 2\Theta_{o}(\infty)).$$

ii) Let $f_1(z)$ and $f_2(z)$ be two meromorphic functions, which have 0 and ∞ as exceptional value of defect 1. Let a_i be distinct non-zero complex numbers then,

$$\sum_i \Theta^{(k)}_{1,2}(a_i) \leq 2.$$

iii) Let f_1 and f_2 be two meromorphic functions of finite order and let $T(r, f_i') \sim aT(r, f_i)$ where $a \geq 1$ and $i = 1, 2$. Then

$$\Theta_{1,2}(\infty) + 2\Theta_o(\infty) \leq 4 - a .$$

Let us now mention some results on sum and products dealing with meromorphic functions and its derivative, viz. the monomials, differential polynomials and homogenous differential polynomials. Here, by a differential polynomial in a meromorphic function f, we mean a finite sum of the form

$$a_o(z)\ (f)^{l_o}\ (f')^{l_1}\ \ldots\ (f^{(\nu)})^{l_\nu}$$

where $l_o, l_1, \ldots, l_\nu$ are integers ≥ 0 and

$$T(r, a_o(z)) = S(r,f)$$

A differential polynomial having just one term is called a monomial. Also, if $l_o + l_1 + \ldots + l_\nu = n$ for all the terms of the differential polynomial then that differential polynomial is called homogenous differential polynomial of degree n.

Results on differential polynomials have also been extensively studied, for example see [36], [2], [37], and [5]. Also recently A.P.Singh [30], found a relation connecting the order of a meromorphic function and its homogenous differential polynomials; that, "if f(z) is transcendental meromorphic function and $\phi(z)$ is a non-zero homogenous differential polynomial of degree n satisfying that each of the exponents of f are integers ≥ 1, then $\rho(f) = \rho(\phi)$."

As a consequence of this is that if f(z) is a meromorphic function of finite order ρ and if ρ is not an integer, and ϕ is a non-zero homogenous differential polynomial as defined in the above theorem then

$$\delta(o, f) + \delta(\infty, f) \leq 2 - K(\rho)$$

where

$$K(\rho) \geq 1 - \rho \quad \text{if } 0 < \rho < 1$$

and

$$K(\rho) \geq \frac{(q + 1 - \rho)(\rho - q)}{2\rho(q + 1)(2 + \log(q+1))}$$

if $\rho > 1$ and $q = [\rho]$

Another result on order of homogenous differential polynomial has been recently proved by H.S.Gopalkrishna and S.S.Bhoosnurmath [14].

Let f be a meromorphic function satisfying

$$\bar{N}(r, f) + \bar{N}(r, 1/f) = S(r,f)$$

If P is a homogenous differential polynomial in f which does not reduce to a constant, then order of P equals order of f and further

$$\bar{N}(r,p) + \bar{N}(r, 1/p) = S(r,P)$$

The proof of this theorem follows from another interesting result of the same authors [13] which states that if f is a meromorphic function of finite order and P is a homogenous differential polynomial in f of degree n which does not reduce to a constant, then

$$n(1 - m\alpha) \leq \liminf_{r \to \infty} \frac{T(r,P)}{T(r,f)}$$

$$\leq \limsup_{r \to \infty} \frac{T(r,p)}{T(r,f)}$$

$$\leq n(1 + m\alpha)$$

In the present dissertation, we have extended the definition of relative defect of meromorphic functions to include monomials and homogenous differential polynomials. Thus, for instance, we have defined

$$\delta_r(a, P_n(f)) = 1 - \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{P_n(f) - a}\right)}{T(r,f)},$$

so that $\delta_r^{(k)}(a,f)$ becomes a particular case which is obtained by taking n = 1. With the help of this definition we have obtained several results on relative defects.

The second chapter deals with homogenous differential polynomials, where we have extended several results of A.P. Singh [27]. Thus, for instance, we have shown that (see theorem 2.3) under certain conditions on f, the relative defect for homogenous differential polynomial, viz. $\delta_r(a, D_n(f))$ equals

$$1 - n + \liminf_{r \to \infty} \frac{m\left(r, \frac{1}{D_n(f) - a}\right)}{T(r,f)}.$$

Towards the end of the second chapter, we have found bounds for $\delta_r(\infty, D_n(f))$, where $D_n(f)$ denotes a homogenous differential polynomial in the derivatives of f, and which does not contain f as its factor. Thus, for instance, we have shown

$$\delta_r(\infty, D_n(f)) \geq 1 - nt - n(K_1 + K_2 + \dots + K_t) + nt\,\delta(\infty, f) + n(K_1 + K_2 + \dots + K_t)\,\Theta(\infty, f).$$

Our third and the last chapter deals with the relative defects of monomials and homogeneous differential polynomials, where the zeros and poles are counted only once. Thus we have defined

$$\Theta_r(\alpha, P_n(f)) = 1 - \limsup \frac{\bar{N}\left(r, \frac{1}{P_n(f)-\alpha}\right)}{T(r,f)}$$

and have thus shown, for instance that if P_n is a monomial of degree n, then

$$\Theta_r(\alpha, P_n) \leq 2 - \Theta(\infty, f) - \delta(0, f),$$

from which we see that $\Theta_r(\alpha, P_n)$ is bounded by a quantity which is independent of the degree of the monomial. Also several other interesting results have been obtained. In this chapter we have also found bounds for $\Theta_r(a_i, P_n)$ in terms of $\delta_r(0, P_n)$. Thus for instance, we have shown that under certain hypothesis (See theorem 3.7), we obtain

$$\sum_{i=1}^{p} \textcircled{H}_r (a_i, P_n) + \textcircled{H}_r (o, P_n) + \textcircled{H} (\omega, f) \leq 2+p \left\{ \delta_r(o, P_n) - n\delta(o, f) \right\} .$$

The method of proof of our work is the classical Nevanlinna Theory, a good account of which can easily be found in the book of W.K.Hayman entitled "Meromorphic Functions", and also in Nevanlinna's books [22] and [23] .

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