CHAPTER II

RELATIVE DEFECTS OF HOMOGENEOUS DIFFERENTIAL POLYNOMIALS

Introduction :

Let f(z) be a meromorphic function in the complex plane. As stated in our introduction chapter the term

$$\delta(\alpha, f) = 1 - \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{r-\alpha}\right)}{T(r,f)}$$

is called the deficiency of the value 'α' with respect to f. Milloux [19] introduced the concept of absolute defect of α with respect to the derivative f'. This definition was later extended by Xiong Qing Lai [17] who introduced the term

$$\delta_r^{(k)}(\alpha, f) = 1 - \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{f^{(k)} - \alpha}\right)}{T(r,f)}$$

and called it the relative defect of α with respect to $f^{(k)}$. In contrast the usual defect of α with respect to $f^{(k)}$ denoted by

$$\delta_a^{(k)}(\alpha, f) = 1 - \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{f^{(k)} - \alpha}\right)}{T(r, f^{(k)})}$$

was called the absolute defect of α with respect to $f^{(k)}$. Several results regarding these relative defects, absolute defects and relative defects corresponding to the roots being counted once were found by Xiong-Qing Lai [17] and A.P. Singh [27] .

In the present dissertation, we define the relative defects corresponding to the homogeneous differential polynomials and find several relations for these in terms of the usual Nevanlinna deficient values.

Notations, Terminology :

Let f(z) be a non-constant meromorphic function. As earlier, we define m(r,f), $N(r, \frac{1}{f-\alpha})$, N(r,f), T(r,f) etc. by

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta;$$

$$N\left(r, \frac{1}{f-\alpha}\right) = \int_0^r \frac{n(t,\alpha)}{t} dt$$

where $n(t,\alpha)$ denotes the number of zeros of $f(z) - \alpha$ in $|z| \leq t$;

$$N(r,f) = \int_0^r \frac{n(t,\infty)}{t} dt$$

where $n(t,\infty)$ denotes the number of poles of f(z) in $|z| \leq t$;

$$T(r,f) = m(r,f) + N(r,f)$$

Let S(r,f) denote any quantity satisfying S(r,f) = $= o(T(r,f))$ if f is of finite order and $S(r,f) = O(T(r,f))$

possibly outside an exceptional set of finite linear measure if f is of infinite order. Also by a homogeneous differential polynomial $D_n(f)$ of degree n, we shall mean a finite sum of the form

$$a(z)\,(f(z))^{l_o}\,(f'(z))^{l_1}\,\ldots\,(f^{(k)}(z))^{l_k}$$

where $l_o + l_1 + \ldots + l_k = n$ and $a(z)$ is a meromorphic function :

$$T(r,\ a(z)) = S(r,f)$$

A homogeneous differential polynomial having one term will be called a monomial of degree n. The term

$$\delta_r\,(\alpha,\ D_n(f)\,) = 1 - \limsup_{r \to \infty} \frac{N\left(r,\ \dfrac{1}{D_n(f)-\alpha}\right)}{T(r,f)}$$

will be called the relative defect of α with respect to the homogeneous differential polynomial $D_n(f)$. With these notations we shall now prove the following theorems :

<u>Theorem 2.1</u>

Let f(z) be a meromorphic function. Let $D_n(f)$ be a homogeneous differential polynomial of degree n not containing f and which does not reduce to zero. Furthermore, let

$$\delta\,(\infty, f) = 1 \text{ and } \sum_{\alpha_i \neq \infty} \delta\,(\alpha_i,\ f) = 1\ .$$

Then

$$\frac{T(r, D_n(f))}{T(r,f)} \to n \text{ as } r \to \infty$$

For the proof we shall need the following lemma.

<u>Lemma 2.1</u> -

Let f(z) be a meromorphic function and P(f) be a monomial of degree n not containing f. Also, let

$$\delta(\infty, f) = 1 \text{ and } \sum_{\alpha_i \neq \infty} \delta(\alpha_i, f) = 1.$$

Then,

$$\frac{T(r, P(f))}{T(r,f)} \to n \text{ as } r \to \infty .$$

Before we start with the proof of lemma 2.1 we shall mention Milloux's theorem which has been used in the proof of our lemma and also quite frequently throughout the Dissertation.

<u>Milloux's Theorem</u> [12, 55] :

Let l be a positive integer and

$$\phi(z) = \sum_{\nu=0}^{l} a_\nu(z) f^{(\nu)}(z). \text{ Then}$$

$$m\left(r, \frac{\phi(z)}{f(z)}\right) = S(r,f) \text{ and}$$

$$T(r, \phi) \leq (l + 1) T(r,f) + S(r,f).$$

Proof of Lemma 2.1 :

By hypothesis, P(f) is of the form

$$P(f) = a_o (f')^{l_1} (f'')^{l_2} \ldots (f^{(k)})^{l_k}$$

where $l_1 + l_2 + \ldots + l_k = n$ and $T(r, a_o) = S(r,f)$.

Consider,

$$T(r,p(f)) = m\,(r,\ p(f)) + N(r,\ P(f)\ )$$

$$= m(r,\ \frac{p(f)}{f^n}\ .\ f^n\ ) + N\ (r,\ p(f))$$

$$\leqq m\ (r,\ \frac{p(f)}{f^n}) + m\ (r,\ f^n) + N\ (r,\ p(f))$$

Now

$$m(r,\ \frac{p(f)}{f^n}) = m\ (r\ ,\ \frac{a_o(f')^{l_1}(f'')^{l_2} \ldots (f^{(k)})^{l_k}}{f^n}\ )$$

$$\leqslant m\ (r,\ a_o) + m\ (r,\ (\frac{f'}{f})^{l_1}) + m(r,\ (\frac{f''}{f})^{l_2}) +$$

$$+ \ldots\ldots + m\ (r,\ (\frac{f^{(k)}}{f})^{l_k})$$

$$\leqslant T\ (r,\ a_o\ ) + l_1\ m\ (r,\ f'/f) + l_2\ m(r,\ \frac{f''}{f}) +$$

$$+ \ldots + l_k\ m(\ r,\ \frac{f^{(k)}}{f}\ )\ .$$

Hence, by Milloux's theorem it follows that

$$m\ (r,\ \frac{p(f)}{f^n}\ ) = S\ (r,f)\ . \qquad \ldots (2.1)$$

Hence,

$$T(r, p(f)) \leq m(r,f^n) + N(r, p(f)) + S(r,f) .$$

And so

$$T(r,p(f)) \leq n\, m(r,f) + N(r,p(f)) + S(r,f) . \quad \ldots (2.2)$$

Now, we know that

$$N(r,f.g) \leq N(r,f) + N(r,g).$$

Therefore,

$$N(r,p(f)) = N(r, a_o (f')^{l_1} (f'')^{l_2} \ldots (f^{(k)})^{l_k})$$

$$\leq N(r, a_o) + N(r, (f')^{l_1}) + \ldots + N(r, (f^{(k)})^{l_k})$$

$$\leq T(r,a_o) + l_1 N(r,f') + l_2 N(r,f'') + \ldots +$$

$$+ l_k N(r, f^{(k)})$$

Since $N(r, f^{(k)}) \leq (k+1) N(r,f)$ it follows that

$$N(r, p(f)) \leq T(r, a_o) + 2l_1 N(r,f) + 3l_2 N(r,f) + \ldots +$$
$$+ (k+1) N(r,f)$$

and so

$$N(r, p(f)) \leq T(r, a_o) + AN(r,f)$$

where A is some constant.

Since $\delta(\infty, f) = 1$ it follows that $N(r,f) = S(r,f)$ and so one gets from above

$$N(r, p(f)) = S(r,f) . \quad \ldots (2.3)$$

Thus (2.2) and (2.3) imply

$$T(r, p(f)) \leqslant n\, m(r,f) + S(r,f)$$
$$\leqslant nT(r,f) + S(r,f)$$

Therefore,

$$\limsup_{r \to \infty} \frac{T(r, p(f))}{T(r,f)} \leqslant n \qquad \ldots (2.4)$$

Next we have from [12, 23]

$$n \sum_{i=1}^{q} m(r,a_i) \leqslant m\left(r, \sum_{i=1}^{q} \frac{1}{(f-a_i)^n}\right) + O(1)$$

$$= m\left(r, \sum_{i=1}^{q} \frac{p(f)}{(f-a_i)^n} \cdot \frac{1}{p(f)}\right) + O(1).$$

And so

$$n \sum_{i=1}^{q} m(r,a_i) \leqslant m\left(r, \frac{1}{(p(f)}\right) + m\left(r, \sum_{i=1}^{q} \frac{p(f)}{(f-a_i)^n}\right) + O(1). \qquad \ldots (2.5)$$

But

$$m\left(r, \sum_{i=1}^{q} \frac{p(f)}{(f-a_i)^n}\right) \leqslant \sum_{i=1}^{q} m\left(r, \frac{p(f)}{(f-a_i)^n}\right)$$

$$= \sum_{i=1}^{q} m\left(r, \frac{a_o(f')^{l_1}(f'')^{l_2} \ldots (f^{(k)})^{l_k}}{(f-a_i)^n}\right)$$

and hence

$$m\left(r, \sum_{i=1}^{q} \frac{p(f)}{(f-a_i)^n}\right) \leqslant \sum_{i=1}^{q} \left\{ m\left(r, a_o\left(\frac{f'}{f-a_i}\right)^{l_1}\left(\frac{f''}{f-a_i}\right)^{l_2} \ldots \left(\frac{f^{(k)}}{f-a_i}\right)^{l_k}\right)\right\}$$

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$$\leq \sum_{i=1}^{q} \left\{ m(r, a_o) + m\left(r, \left(\frac{f'}{f-a_i}\right)^{l_1}\right) + m\left(r\left(\frac{f''}{f-a_i}\right)^{l_2}\right) + \right.$$

$$\left. + \cdots + m\left(r, \left(\frac{f^{(k)}}{f-a_i}\right)^{l_k}\right) \right\}$$

$$\leq \quad qT(r, a_o) + l_1 \sum_{i=1}^{q} m\left(r, \frac{f'}{f-a_i}\right) +$$

$$+ l_2 \sum_{i=1}^{q} m\left(r, \frac{f''}{f-a_i}\right) + \cdots + l_k \sum_{i=1}^{q} m\left(r, \frac{f^{(k)}}{f-a_i}\right).$$

Thus, using Milloux's theorem, we get

$$m\left(r, \sum_{i=1}^{q} \frac{p(f)}{(f-a_i)^n}\right) = S(r,f). \qquad \ldots (2.6)$$

Thus (2.5) reduces to

$$n \sum_{i=1}^{q} m(r, a_i) \leq m\left(r, \frac{1}{p(f)}\right) + S(r,f)$$

$$\leq T\left(r, \frac{1}{p(f)}\right) + S(r,f).$$

Thus by Nevanlinna's first fundamental theorem it follows that

$$n \sum_{i=1}^{q} m(r, a_i) \leq T(r, p(f)) + S(r,f). \qquad \ldots (2.7)$$

Dividing throughout by $T(r,f)$ and then taking limit inferior as $r \to \infty$ of both the sides, it easily follows that

$$n \sum_{i=1}^{q} \liminf_{r\to\infty} \frac{m(r, a_i)}{T(r,f)} \leq \liminf_{r\to\infty} \left[\frac{T(r,p(f))}{T(r,f)} + \frac{S(r,f)}{T(r,f)} \right]$$

$$\leq \liminf_{r\to\infty} \frac{T(r,p(f))}{T(r,f)} + \limsup_{r\to\infty} \frac{S(r,f)}{T(r,f)}$$

Thus

$$n \sum_{i=1}^{q} \delta(a_i, f) \leq \liminf_{r\to\infty} \frac{T(r, P(f))}{T(r,f)}$$

Making $q \to \infty$ and using $\sum_{\alpha_i \neq \infty} \delta(\alpha_i, f) = 1$,

we get

$$n \leq \liminf_{r\to\infty} \frac{T(r, p(f))}{T(r,f)} . \qquad \dots (2.8)$$

From (2.4) and (2.8) we get

$$n \leq \liminf_{r\to\infty} \frac{T(r, P(f))}{T(r,f)} \leq \limsup_{r\to\infty} \frac{T(r, P(f))}{T(r,f)} \leq n.$$

This immediately gives

$$\lim_{r\to\infty} \frac{T(r, p(f))}{T(r,f)} = n$$

which completes the proof.

<u>Proof of Theorem 2.1.</u>

Consider

$$T(r, D_n(f)) = m(r, D_n(f)) + N(r, D_n(f))$$

$$= m\left(r, \frac{D_n(f)}{f^n} . f^n\right) + N(r, D_n(f))$$

$$= m\left(r, \frac{D_n(f)}{f^n}\right) + m(r, f^n) + N(r, D_n(f)).$$

Thus

$$T(r, D_n(f)) = n\, m(r,f) + m\left(r, \frac{D_n(f)}{f^n}\right) + N(r, D_n(f)). \quad \ldots (2.9)$$

Now, $D_n(f)$ is a finite sum of the form $\sum P_n(f)$ where $P_n(f)$ are monomials of degree n.

Thus,

$$m\left(r, \frac{D_n(f)}{f^n}\right) = m\left(r, \frac{\sum P_n(f)}{f^n}\right) \leqslant \sum m\left(r, \frac{P_n(f)}{f^n}\right).$$

But by result (2.1) of lemma 2.1,

$$m\left(r, \frac{P_n(f)}{f^n}\right) = S(r,f).$$

Therefore,

$$m\left(r, \frac{D_n(f)}{f^n}\right) = S(r,f).$$

Next,

$$N(r, D_n(f)) = N(r, \sum P_n(f))$$

$$\leqslant \sum N(r, P_n(f))$$

But again as in (2.3), $N(r, P_n(f)) = S(r,f)$ and therefore,

$$N(r, D_n(f)) = S(r,f). \quad \ldots (2.10)$$

With (2.10) inequality (2.9) gets converted into

$$T(r, D_n(f)) \leqslant n\, m(r,f) + S(r,f)$$

$$\leqslant n\, T(r,f) + S(r,f)$$

Dividing throughout by $T(r,f)$ and then taking limit superior as $r \to \infty$ of both the sides, we get

$$\limsup_{r \to \infty} \frac{T(r, D_n(f))}{T(r,f)} \leqslant n . \qquad \ldots (2.11)$$

Next, we have

$$n \sum_{i=1}^{q} m(r, a_i) \leqslant m\left(r, \sum_{i=1}^{q} \frac{1}{(f-a_i)^n}\right) + O(1)$$

$$= m\left(r, \sum_{i=1}^{q} \frac{D_n(f)}{(f-a_i)^n} \cdot \frac{1}{D_n(f)}\right) + O(1).$$

Thus,

$$n \sum_{i=1}^{q} m(r,a_i) \leqslant m\left(r, \frac{1}{D_n(f)}\right) + m\left(r, \sum_{i=1}^{q} \frac{D_n(f)}{(f-a_i)^n}\right) + O(1). \qquad \ldots (2.12)$$

But

$$m\left(r, \sum_{i=1}^{q} \frac{D_n(f)}{(f-a_i)^n}\right) = m\left(r, \sum_{i=1}^{q} \frac{\left(\sum P_n(f)\right)}{(f-a_i)^n}\right).$$

But once again by (2.6),

$$m\left(r, \sum_{i=1}^{q} \frac{P_n(f)}{(f-a_i)^n}\right) = S(r,f).$$

And so

$$m\left(r, \sum_{i=1}^{q} \frac{D_n(f)}{(f-a_i)^n}\right) = S(r,f).$$

Making use of this, inequality (2.12) get converted into

$$n \sum_{i=1}^{q} m(r,a_i) \leqslant m\left(r, \frac{1}{D_n(f)}\right) + S(r,f)$$

$$\leqslant T(r, \frac{1}{D_n(f)}) + S(r,f)$$

Using the first fundamental theorem it now follows that

$$n \sum_{i=1}^{q} m(r, a_i) \leqslant T(r, D_n(f)) + S(r,f). \quad \ldots (2.13)$$

Dividing throughout by $T(r,f)$ and then taking limit inferior as $r \to \infty$ of both the sides we get

$$n \sum_{i=1}^{q} \liminf_{r\to\infty} \frac{m(r,a_i)}{T(r,f)} \leqslant n \liminf_{r\to\infty} \sum_{i=1}^{q} \frac{m(r,a_i)}{T(r,f)}$$

$$\leqslant \liminf_{r\to\infty} \left(\frac{T(r,D_n(f)) + S(r,f)}{T(r,f)} \right)$$

$$\leqslant \liminf_{r\to\infty} \frac{T(r,D_n(f))}{T(r,f)} + \limsup_{r\to\infty} \frac{S(r,f)}{T(r,f)}$$

Thus

$$n \sum_{i=1}^{q} \delta(a_i, f) \leqslant \liminf_{r\to\infty} \frac{T(r, D_n(f))}{T(r,f)}$$

Making $q \to \infty$ and using $\sum_{\alpha_i \neq \infty} \delta(\alpha_i, f) = 1$, we get

$$n \leqslant \liminf_{r\to\infty} \frac{T(r,D_n(f))}{T(r,f)}. \quad \ldots (2.14)$$

Combining (2.11) and (2.14) we obtain

$$n \leqslant \liminf_{r\to\infty} \frac{T(r,D_n(f))}{T(r,f)} \leqslant \limsup_{r\to\infty} \frac{T(r,D_n(f))}{T(r,f)} \leqslant n.$$

Thus,

$$\lim_{r \to \infty} \frac{T(r, D_n(f))}{T(r,f)} = n .$$

Equivalently, $\lim_{r \to \infty} \frac{T(r,D_n(f))}{nT(r,f)} = 1.$

This completes the proof of theorem 2.1.

<u>Theorem 2.2</u> :

Let f(z) be a meromorphic function and $D_n(f)$ be a homogeneous differential polynomial of degree n. Also let

$$\lim_{r \to \infty} \frac{T(r,D_n(f))}{T(r,f)} = C.$$

Then

$$\delta_r(a, D_n(f)) = 1 - C + \liminf_{r \to \infty} \frac{m\left(r, \frac{1}{D_n(f)-a}\right)}{T(r,f)}$$

The proof of this theorem follows as in theorem 2.1 on using the following lemma.

<u>Lemma 2.2</u> :

Let f(z) be a meromorphic function and $P_n(f)$ be a monomial of degree n. Also let

$$\lim_{r \to \infty} \frac{T(r, P_n(f))}{T(r,f)} = C.$$

Then,

$$\delta_r\,(a, P_n(f)) = 1 - C + \liminf_{r\to\infty} \frac{m\left(r, \dfrac{1}{P_n(f)-a}\right)}{T(r,f)} .$$

Proof of lemma 2.2 :

By Nevanlinna's first fundamental theorem we have

$$m\left(r, \frac{1}{P_n(f)-a}\right) + N\left(r, \frac{1}{P_n(f)-a}\right) = T(r, P_n(f)) + O(1).$$

This gives,

$$\liminf_{r\to\infty} \frac{T(r,P_n(f))}{T(r,f)} = \liminf_{r\to\infty} \left(\frac{m\left(r, \dfrac{1}{P_n(f)-a}\right) + N\left(r, \dfrac{1}{P_n(f)-a}\right)}{T(r,f)} \right)$$

$$\leq \liminf_{r\to\infty} \frac{m\left(r, \dfrac{1}{P_n(f)-a}\right)}{T(r,f)} + \limsup_{r\to\infty} \frac{N\left(r, \dfrac{1}{P_n(f)-a}\right)}{T(r,f)}$$

Similarly,

$$\limsup_{r\to\infty} \frac{T(r, P_n(f))}{T(r,f)} \geq \liminf_{r\to\infty} \frac{m\left(r, \dfrac{1}{P_n(f)-a}\right)}{T(r,f)} +$$

$$+ \limsup_{r\to\infty} \frac{N\left(r, \dfrac{1}{P_n(f)-a}\right)}{T(r,f)}$$

But by hypothesis $\lim\limits_{r\to\infty} \dfrac{T(r, P_n(f))}{T(r,f)}$ exists

and hence we get from above

$$\lim_{r\to\infty} \frac{T(r,P_n(f))}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r, \frac{1}{P_n(f)-a})}{T(r,f)} + \limsup_{r\to\infty} \frac{N(r, \frac{1}{P_n(f)-a})}{T(r,f)}$$

Thus

$$-\limsup_{r\to\infty} \frac{N(r, \frac{1}{P_n(f)-a})}{T(r,f)} = -\lim_{r\to\infty} \frac{T(r, P_n(f))}{T(r,f)} + \liminf_{r\to\infty} \frac{m(r, \frac{1}{P_n(f)-a})}{T(r,f)}$$

From this, it now follows that

$$\delta_r (a, P_n(f)) = 1 - C + \liminf_{r\to\infty} \frac{m(r, \frac{1}{P_n(f)-a})}{T(r,f)}$$

which is nothing but our lemma 2.2.

<u>Theorem 2.3</u> :

Let f(z) be a meromorphic function with

$\sum_{\alpha_i \neq \infty} \delta(\alpha_i, f) = 1$ and $\delta(\infty, f) = 1$. Let $D_n(f)$ be a homogeneous differential polynomial of degree n. Then for any 'a',

$$\delta_r (a, D_n(f)) = 1 - n + \liminf_{r\to\infty} \frac{m(r, \frac{1}{D_n(f)-a})}{T(r,f)} .$$

The proof can easily be filled in by the reader after considering the following lemma.

<u>Lemma 2.3</u> :

Let f(z) be a meromorphic function with $\sum_{\alpha_i \neq \infty} \delta(\alpha_i, f) = 1$ and $\delta(\infty, f) = 1$.

Let $P_n(f)$ be a monomial of degree n not containing f. Then for any 'a' we have

$$\delta_r(a, P_n(f)) = 1 - n + \liminf_{r \to \infty} \frac{m\left(r, \dfrac{1}{P_n(f) - a}\right)}{T(r,f)}$$

<u>Proof of lemma 2.3</u> :

By lemma 2.1, we have

$$\lim_{r \to \infty} \frac{T(r, P_n(f))}{T(r,f)} = n.$$

Also, lemma 2.2 gives

$$\delta_r(a, P_n(f)) = 1 - C + \liminf_{r \to \infty} \frac{m\left(r, \dfrac{1}{P_n(f)-a}\right)}{T(r,f)} ; \qquad \ldots (2.15)$$

where $C = \lim_{r \to \infty} \dfrac{T(r, P_n(f))}{T(r,f)}$

Thus, we see that

$C = n.$

Hence with C = n, equation (2.15) becomes

$$\delta_r(a,P_n(f)) = 1 - n + \liminf_{r\to\infty} \frac{m\left(r, \dfrac{1}{P_n(f) - a}\right)}{T(r,f)}$$

which we wanted to show

Remark :

Putting n = 1 and considering a homogeneous differential polynomial consisting of one term, viz. monomial, we get Theorem [1, 27] proved by A.P.Singh.

Theorem 2.4 :

Let f(z) be a transcendental meromorphic function and let

$$\lim_{r\to\infty} \frac{T(r, P_n(f))}{T(r,f)} = C,$$

where C is a constant and $P_n(f)$ is a monomial of degree n not containing f. Then

$$n \sum_{i=1}^{q} \delta(a_i, f) \leq 1 - C + \delta_r(0, P_r(f))$$

Proof -

Let $a_1, a_2, \ldots, a_q$ be distinct complex numbers and let

$$F(Z) = \sum_{i=1}^{q} \frac{1}{(f(z)-a_i)^n}.$$

Then by inequality (2.5) of [21, 23], we have

$$n \sum_{i=1}^{q} m(r, \frac{1}{f-a_i}) \leq m(r, F) + O(1)$$

$$= m\left(r, \sum_{i=1}^{q} \frac{1}{P_n(f)} \cdot \frac{P_n(f)}{(f-a_i)^n}\right) + o(1)$$

$$\leq m\left(r, \frac{1}{P_n(f)}\right) + m\left(r \sum_{i=1}^{q} \frac{P_n(f)}{(f-a_i)^n}\right) + o(1)$$

But if we use the result (2.6), we get

$$m\left(r, \sum_{i=1}^{q} \frac{P_n(f)}{(f-a_i)^n}\right) = S(r,f).$$

Therefore, we have

$$n \sum_{i=1}^{q} m\left(r, \frac{1}{f-a_i}\right) \leq m\left(r, \frac{1}{P_n(f)}\right) + S(r,f) .$$

Dividing throughout by $T(r,f)$ and then taking limit inferior as $r \to \infty$, we get

$$n \sum_{i=1}^{q} \liminf_{r \to \infty} \frac{m(r, \frac{1}{f-a_i})}{T(r,f)} \leq \liminf_{r \to \infty} \left[\frac{m(r, \frac{1}{P_n(f)}) + S(r,f)}{T(r,f)} \right]$$

and so

$$n \sum_{i=1}^{q} \liminf_{r \to \infty} \frac{m(r, \frac{1}{f-a_i})}{T(r,f)} \leq \liminf_{r \to \infty} \frac{m(r, \frac{1}{P_n(f)})}{T(r,f)} +$$

$$+ \limsup_{r \to \infty} \frac{S(r,f)}{T(r,f)} =$$

$$= \liminf_{r \to \infty} \frac{m\left(r, \dfrac{1}{P_n(f)}\right)}{T(r,f)} .$$

This is nothing but

$$n \sum_{i=1}^{q} \delta(a_i, f) \leq \liminf_{r \to \infty} \frac{m\left(r, \dfrac{1}{P_n(f)}\right)}{T(r,f)} .$$

By lemma 2.2, it now follows that

$$n \sum_{i=1}^{q} \delta(a_i, f) \leq 1 - C + \delta_r(0, P_n(f)) .$$

Thus we get the required theorem.

In our next two theorems we shall find lower bounds for the relative defects of homogeneous differential polynomials in terms of distinct poles of the function f. More precisely we shall prove,

Theorem 2.5 :

Let f(z) be a transcendental meromorphic function and let $P_n(f)$ be a monomial of degree n not containing f. Then for any integer $m \geq 1$,

(i) $\delta_r(a, P_n(f)) \geq 1 - n - mn + mn\,Ⓗ(\infty, f)$.

(ii) And if a = 0 then

$\delta_r(0, P_n(f)) \geq 1 - n - mn + n\delta(0,f) + mn\,Ⓗ(\infty, f)$.

Theorem 2.6 :

Let f(z) be a transcendental meromorphic function.Let

$$D_n(f) = P_1(f) + P_2(f) + \dots + P_t(f).$$

be a homogeneous differential polynomial of degree n and not containing the term f. Further, let $D_n(f)$ not reduces to zero and K_i be the highest derivative occurring in $P_i(f)$ $(i = 1, 2, \dots, t)$. Then

i) $\delta_r(a, D_n(f)) \geq 1 - n - n(k_1 + k_2 + \dots + k_t) \times [1 - \Theta(\infty, f)]$,

ii) $\delta_r(0, D_n(f)) \geq 1 - 2n + n\delta(0,f) - n(k_1 + k_2 + \dots + k_t)[1 - \Theta(\infty, f)]$.

<u>Proof of Theorem 2.5</u> :

We first prove (i)

By Nevanlinna's first fundamental theorem we have

$$m\left(r, \frac{1}{f^n}\right) + N\left(r, \frac{1}{f^n}\right) = m(r, f^n) + N(r, f^n) + S(r,f).$$

And hence

$$N\left(r, \frac{1}{f^n}\right) - N(r, f^n)$$

$$= m(r, f^n) - m(r, 1/f^n) + S(r,f)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f^n|\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f^n}\right| d\theta + S(r,f).$$

Since $\log x = \log^+ x - \log^+ \frac{1}{x}$, it follows that

$$N\left(r, \frac{1}{f^n}\right) - N(r, f^n) = \frac{1}{2\pi}\int_0^{2\pi} \log |f^n|\, d\theta + S(r,f) \quad \dots (2.16)$$

Next

$$N(r, P_n(f) - a) - N(r, \frac{1}{P_n(f)-a})$$

$$= m(r, \frac{1}{P_n(f) - a}) - m(r, P_n(f)-a) + S(r,f).$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{P_n(f)-a} \right| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log^+ |P_n(f) - a| \, d\theta + S(r,f).$$

As above we get

$$N(r, P_n(f) - a) - N(r, \frac{1}{P_n(f)-a})$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{1}{P_n(f)-a} \right| d\theta + S(r,f). \qquad \ldots (2.17)$$

Similarly, it is easily seen that

$$N(r, \frac{P_n(f)-a}{f^n}) - N(r, \frac{f^n}{P_n(f)-a})$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f^n}{P_n(f)-a} \right| d\theta + S(r,f). \qquad \ldots (2.18)$$

From (2.16), (2.17) and (2.18) we get

$$N(r, \frac{1}{P_n(f)-a}) = N(r, P_n(f)-a) - \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{1}{P_n(f)-a} \right| d\theta + S(r,f)$$

$$= N(r, P_n(f)-a) - \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f^n}{P_n(f)-a} \cdot \frac{1}{f^n}\right| d\theta + S(r,f)$$

$$= N(r, P_n(f)-a) - \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f^n}{P_n(f)-a}\right| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{1}{f^n}\right| d\theta + S(r,f).$$

and so

$$N(r, \frac{1}{P_n(f)-a}) = N(r,P_n(f)-a) - \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f^n}{P_n(f)-a}\right| d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|f^n|\, d\theta + S(r,f)$$

$$= N(r,P_n(f)-a) - \left[N(r, \frac{P_n(f)-a}{f^n}) - N(r, \frac{f^n}{P_n(f)-a})\right] + \left[N(r, \frac{1}{f^n}) - N(r,f^n)\right] + S(r,f).$$

And hence

$$N(r, \frac{1}{P_n(f)-a}) = N(r, \frac{1}{f^n}) - N(r, f^n) + N(r, P_n(f)-a) - \left[T(r, \frac{P_n(f)-a}{f^n}) - m(r, \frac{P_n(f)-a}{f^n})\right] +$$

$$+ \left[T\left(r, \frac{f^n}{P_n(f)-a}\right) - m\left(r, \frac{f^n}{P_n(f)-a}\right) \right] + S(r,f). \quad \ldots (2.19)$$

But by first fundamental theorem we have

$$T\left(r, \frac{f^n}{P_n(f)-a}\right) = T\left(r, \frac{P_n(f)-a}{f^n}\right) + S(r,f). \quad \ldots (2.20)$$

Also, we have

$$P_n(f) = a_o(z)\ (f')^{l_1}\ (f'')^{l_2} \ldots\ (f^{(m)})^{l_m}$$

where $l_1 + l_2 + \ldots + l_m = n.$

Now, if z_o is a pole of order k for f then z_o is a pole of order (k+1) for f'. Therefore, z_o is a pole of order $l_1(k+1)$ for $(f')^{l_1}$ and so on.

Thus z_o is a pole of multiplicity for $(k+1)\ l_1 + (k+2)\ l_2 + \ldots + (k+m)\ l_m$. But

$$(k+1)\ l_1 + (k+2)\ l_2 + \ldots + (k+m) l_m$$

$$= k(l_1 + l_2 + \ldots + l_m) + (l_1 + 2l_2 + \ldots + ml_m)$$

$$\leqslant kn + m\ (l_1 + l_2 + \ldots + l_m)$$

$$\leqslant kn + mn$$

$$= n\ (k+m);$$

and therefore

$$N(r, P_n(f)) \leqslant n\ (N(r,f) + m\bar{N}(r,f)).$$

But we know that the poles of $P_n(f)$ are the same as the poles of $P_n(f)-a$. And so

$$N(r, P_n(f)) = N(r, P_n(f) - a)$$

Thus,

$$N(r,P_n(f) -a) \leqslant n (N (r,f) + m\bar{N} (r,f)). \quad \ldots (2.21)$$

With the help of (2.20) and (2.21) equation (2.19) gets transformed into

$$N(r, \frac{1}{P_n(f)-a}) \leqslant N(r, \frac{1}{f^n}) - N(r, f^n) +n(N(r,f)+m\bar{N} (r,f))+$$
$$+ m (r, \frac{P_n(f)-a}{f^n}) - m (r, \frac{f^n}{P_n(f)-a}) +S(r,f).$$
$$\leqslant N(r, \frac{1}{f^n}) - nN(r,f) + nN (r,f) +$$
$$+ mn \bar{N}(r,f) + m(r, \frac{P_n(f)-a}{f^n}) +S(r,f). \quad \ldots (2.22)$$

But

$$m(r, \frac{P_n(f)-a}{f^n}) \leqslant m (r, \frac{P_n(f)}{f^n}) + m(r, \frac{a}{f^n}),$$

And so using (2.1) and the fact that $T(r,a) = S(r,f)$, it easily follows that

$$m(r, \frac{P_n(f)-a}{f^n}) \leqslant m (r, \frac{1}{f^n}) + S (r,f). \quad \ldots (2.23)$$

Therefore,

$$N\left(r, \frac{1}{P_n(f)-a}\right) \leqslant N\left(r, \frac{1}{f^n}\right) + mn\,\bar{N}(r,f) + m\left(r, \frac{1}{f^n}\right) + S(r,f)$$

$$= T(r, 1/f^n) + mn\,\bar{N}(r,f) + S(r,f).$$

Thus by the first fundamental theorem

$$N\left(r, \frac{1}{P_n(f)-a}\right) \leqslant nT(r,f) + mn\,\bar{N}(r,f) + S(r,f).$$

Dividing throughout by $T(r,f)$ and taking limit superior as $r \to \infty$ of both the sides of above inequality we get

$$\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{P_n(f)-a}\right)}{T(r,f)} \leqslant n + mn \limsup_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} + \limsup_{r\to\infty} \frac{S(r,f)}{T(r,f)}.$$

Thus

$$1 - \delta_r(a, P_n(f)) \leqslant n + mn\,(1 - \Theta(\infty, f));$$

and hence

$$\delta_r(a, P_n(f)) \geqslant 1 - n - mn + mn\,\Theta(\infty, f).$$

which proves (i)

For the proof of (ii) we consider the inequality (2.22)

$$N\left(r, \frac{1}{P_n(f)-a}\right) \leqslant N\left(r, \frac{1}{f^n}\right) + mn\,\bar{N}(r,f) + m\left(r, \frac{P_n(f)-a}{f^n}\right) + S(r,f).$$

If we consider $a = 0$ then from (2.23) (or (2.1)) immediately we will have

$$m\left(r, \frac{P_n(f)-0}{f^n}\right) = m\left(r, \frac{P_n(f)}{f^n}\right) = S(r,f).$$

Using this in the above inequality, we get

$$N\left(r, \frac{1}{P_n(f)}\right) \leqslant N\left(r, \frac{1}{f^n}\right) + mn\,\bar{N}(r, f) + S(r,f).$$

Dividing throughout by T(r,f) and then taking limit superior as $r \to \infty$, we get

$$\limsup_{r\to\infty} \frac{N\left(r, \frac{1}{P_n(f)}\right)}{T(r,f)} \leqslant n \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f}\right)}{T(r,f)} +$$

$$+ mn \limsup_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} +$$

$$+ \limsup_{r\to\infty} \frac{S(r,f)}{T(r,f)} .$$

Thus

$$1- \delta_r(0, P_n(f)) \leqslant n\left[1 - \delta(0,f)\right] + mn\left[1 - \Theta(\infty,f)\right]$$

which reduces to

$$\delta_r(0, P_n(f)) \geqslant 1 - n - mn + n\delta(0,f) + mn\,\Theta(\infty,f).$$

<u>Remark</u> :

Putting n = 1 in (i) and (ii) above we get Theorem 3 of [27].

Proof of Theorem 2.6 :

We first prove (i).

As in the proof of Theorem 2.5, using Nevanlinna's first fundamental theorem we have

$$m\left(r, \frac{1}{f^n}\right) + N\left(r, \frac{1}{f^n}\right) = m(r,f^n) + N(r,f^n) + S(r,f),$$

and hence

$$N\left(r, \frac{1}{f^n}\right) - N(r,f^n)$$

$$= m(r,f^n) - m(r, 1/f^n) + S(r,f)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f^n|\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f^n}\right| d\theta + S(r,f).$$

And so

$$N(r, 1/f^n) - N(r,f^n) = \frac{1}{2\pi}\int_0^{2\pi} \log |f^n|\, d\theta + S(r,f) \ldots (2.24)$$

Similarly,

$$N(r, D_n(f) - a) - N\left(r, \frac{1}{D_n(f)-a}\right)$$

$$= m\left(r, \frac{1}{D_n(f)-a}\right) - m(r, D_n(f)-a) + S(r,f)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{D_n(f)-a}\right| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^+ |D_n(f)-a|\, d\theta +$$

$$+ S(r,f).$$

And so

$$N(r, D_n(f)-a) - N(r, \frac{1}{D_n(f)-a}) = \frac{1}{2\pi}\int_0^{2\pi} \log \left|\frac{1}{D_n(f)-a}\right| d\theta + $$

$$+ S(r,f). \qquad \ldots (2.25)$$

Similarly

$$N(r, \frac{D_n(f)-a}{f^n}) - N(r, \frac{f^n}{D_n(f)-a})$$

$$= m(r, \frac{f^n}{D_n(f)-a}) - m(r, \frac{D_n(f)-a}{f^n}) + S(r,f)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{f^n}{D_n(f)-a}\right| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{D_n(f)-a}{f^n}\right| d\theta +$$

$$+ S(r,f).$$

And so

$$N(r, \frac{D_n(f)-a}{f^n}) - N(r, \frac{f^n}{D_n(f)-a}) = \frac{1}{2\pi}\int_0^{2\pi} \log \left|\frac{f^n}{D_n(f)-a}\right| d\theta +$$

$$+ S(r,f). \qquad \ldots (2.26)$$

Combining (2.24), (2.25) and (2.26) we will get

$$N(r, \frac{1}{D_n(f)-a})$$

$$= N(r, D_n(f)-a) - \frac{1}{2\pi}\int_0^{2\pi} \log \left|\frac{1}{D_n(f)-a}\right| d\theta + S(r,f)$$

$$= N(r,D_n(f)-a) - \frac{1}{2\pi}\int_0^{2\pi} \log \left|\frac{f^n}{D_n(f)-a} \cdot \frac{1}{f^n}\right| d\theta + S(r,f)$$

$$= N(r, D_n(f)-a) - \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f^n}{D_n(f)-a}\right| d\theta +$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \log|f^n|\, d\theta + S(r,f)$$

$$= N(r,D_n(f)-a) - \left[N(r, \frac{D_n(f)-a}{f^n}) - N(r, \frac{f^n}{D_n(f)-a})\right] +$$

$$+ \left[N(r, \frac{1}{f^n}) - N(r, f^n)\right] + S(r,f).$$

Thus,

$$N(r, \frac{1}{D_n(f)-a}) = N(r, \frac{1}{f^n}) - N(r, f^n) + N(r,D_n(f)-a) -$$

$$-\left[T(r, \frac{D_n(f)-a}{f^n}) - m(r, \frac{D_n(f)-a}{f^n})\right] +$$

$$+ \left[T(r, \frac{f^n}{D_n(f)-a}) - m(r, \frac{f^n}{D_n(f)-a})\right] + S(r,f). \qquad \ldots (2.27)$$

Again by first fundamental theorem of Nevanlinna we have

$$T(r, \frac{f^n}{D_n(f)-a}) = T(r, \frac{D_n(f)-a}{f^n}) + S(r,f). \qquad \ldots (2.28)$$

But by counting the poles it easily follows that

$$N(r, D_n(f)) \le nt\, N(r,f) + n(k_1 + k_2 + \ldots + k_t)\, \bar{N}(r, f).$$

Also, we know that, the number of poles of $D_n(f)$ equals the number of poles of $D_n(f)-a$.

Thus,

$$N(r, D_n(f)) = N(r, D_n(f)-a);$$

and hence the above inequality becomes

$$N(r, D_n(f)-a) \leqslant nt\ N(r,f) + n(k_1 + k_2 + \dots + k_t)\ \bar{N}(r,f). \quad \dots (2.29)$$

With the use of (2.28), (2.29) equation (2.27) reduces to

$$N(r, \frac{1}{D_n(f)-a}) \leqslant N(r, \frac{1}{f^n}) - N(r,f^n) + nt\,N(r,f) + n(k_1 + k_2 + \dots + k_t)\ \bar{N}(r,f) + m(r, \frac{D_n(f)-a}{f^n}) - m(r, \frac{f^n}{D_n(f)-a}) + S(r,f).$$

And so

$$N(r, \frac{1}{D_n(f)-a}) \leqslant N(r, \frac{1}{f^n}) - n(1-t)\ N(r,f) + n(k_1+k_2+\dots+k_t)\ \bar{N}(r,f) + m(r, \frac{D_n(f)-a}{f^n}) + S(r,f). \quad \dots (2.30)$$

But

$$m(r, \frac{D_n(f)-a}{f^n}) = m(r, \sum_{i=1}^{t} \frac{P_i(f)-a}{f^n})$$

$$= m(r, \sum_{i=1}^{t} (\frac{P_i(f)}{f^n} - \frac{a}{f^n}))$$

$$\leqslant m(r, \sum_{i=1}^{t} \frac{P_i(f)}{f^n}) + m(r, \frac{a}{f^n}) ;$$

and as in the proof of theorem (2.5) we will get

$$m(r, \frac{D_n(f)-a}{f^n}) = m(r, \frac{1}{f^n}) + S(r,f). \qquad \ldots (2.31)$$

Therefore,

$$N(r, \frac{1}{D_n(f)-a}) \leqslant$$

$$\leqslant N(r, \frac{1}{f^n}) - n(1-t) N(r,f) + n(k_1 + k_2 + \ldots + k_t) \bar{N}(r,f) + m(r, \frac{1}{f^n}) + S(r,f)$$

$$= T(r, \frac{1}{f^n}) - n(1-t) N(r,f) + n(k_1 + k_2 + \ldots + k_t) \bar{N}(r,f) + S(r,f).$$

From first fundamental theorem it now follows that

$$N(r, \frac{1}{D_n(f)-a}) \leqslant nT(r,f) + n(k_1+k_2+ \ldots +k_t) \bar{N}(r,f)+S(r,f).$$

Dividing throughout by T(r,f) and then taking limit superior as $r \to \infty$ we get

$$\limsup_{r\to\infty} \frac{N(r, \frac{1}{D_n(f)-a})}{T(r,f)} \leqslant n + n(k_1+k_2+ \ldots + k_t) \times$$

$$\times \limsup_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} + \limsup_{r\to\infty} \frac{S(r,f)}{T(r,f)} ;$$

which immediately gives

$$1 - \delta_r (a, D_n(f)) \leq n + n (k_1 + k_2 + \dots + k_t)(1 - \Theta(\;, f))$$

which yields

$$\delta_r(a, D_n(f)) \geq 1 - n - n(k_1 + k_2 + \dots + k_t)(1 - \Theta(\infty, f)),$$

which proves (i) .

To prove (ii) consider the inequality (2.30). Then

$$N (r, \frac{1}{D_n(f)-a}) \leq N(r, \frac{1}{f^n}) - n (1-t) N (r, f) +$$

$$+ n(k_1 + k_2 + \dots + k_t) \bar{N} (r, f) +$$

$$+ m (r, \frac{D_n(f)-a}{f^n}) + S(r, f).$$

Let a = 0. Then from inequality (2.31) we have

$$m (r, \frac{D_n(f)-o}{f^n}) = m (r, \frac{1}{f^n}) + S(r, f).$$

Thus

$$m (r, \frac{D_n(f)}{f^n}) \leq T(r, \frac{1}{f^n}) + S(r, f)$$

$$= nT(r, f) + S(r, f).$$

And hence for a = 0, it follows that

$$N (r, \frac{1}{D_n(f)}) \leq n N(r, \frac{1}{f}) - n (1-t) N (r, f) +$$

$$+ n(k_1 + k_2 + \dots + k_t) \bar{N}(r,f) + nT(r,f) + S(r,f).$$

$$\leq nN(r,\frac{1}{f}) + n(k_1 + k_2 + \dots + k_t) \bar{N}(r,f) +$$

$$+ nT(r,f) + S(r,f)$$

Dividing throughout by $T(r,f)$ and then taking limit superior as $r \to \infty$, we will get

$$\limsup_{r \to \infty} \frac{N(r, \frac{1}{D_n(f)})}{T(r,f)} \leq$$

$$\leq n \limsup_{r \to \infty} \frac{N(r, \frac{1}{f})}{T(r,f)} + n(k_1 + k_2 \dots + k_t) \limsup_{r \to \infty} \frac{\bar{N}(r,f)}{T(r,f)} +$$

$$+ n + \limsup_{r \to \infty} \frac{S(r,f)}{T(r,f)}$$

which gives

$$1 - \delta_r(0, D_n(f)) \leq n(1 - \delta(0,f)) + n(k_1 + k_2 + \dots +$$

$$+ \dots k_t)(1 - \Theta(\infty, f)) + n$$

After adjusting the term finally we will get

$$\delta_r(0, D_n(f)) \geq 1 - n - n\big(1 - \delta(0,f)\big) - n(k_1 + k_2 + \dots +$$

$$+ \dots + k_t)(1 - \Theta(\infty, f))$$

$$= 1 - 2n + n\delta(0,f) - n(k_1 + k_2 + \dots + k_t)(1 - \Theta(\infty, f)).$$

which completes the proof of theorem 2.6.

In our next theorem we shall find a lower bound for the relative defects corresponding to the poles of the homogeneous differential polynomials in terms of the deficiencies corresponding to the poles of f. More precisely we shall prove :

Theorem 2.7 :

Let f(z) be a transcendental meromorphic function and let $D_n(f)$ be a homogeneous differential polynomial as stated in theorem 2.6. Then

$$\delta_r(\infty, D_n(f)) \geq 1 - nt - n(k_1+k_2 \ldots + k_t) + nt\,\delta(\infty, f) + $$
$$+ n(k_1 + k_2 + \ldots + k_t)\ \textcircled{H}(\infty, f).$$

We first prove the following lemma :

Lemma 2.4 :

Let f(z) be a transcendental meromorphic function. Then for any integer $k \geq 1$,

$$\delta_r(\infty, P_n(f)) \geq 1 - n(1+k) + n\delta(\infty, f) + k\ \textcircled{H}(\infty, f).$$

where $P_n(f)$ is a monomial of degree n and not containing f.

Proof of Lemma 2.4 :

Set

$$P_n(f) = a_o(f')^{l_1}(f'')^{l_2} \ldots (f^{(k)})^{l_k}$$

where $l_1 + l_2 + \dots + l_k = n$.

Then we have

$$N(r, P_n(f))$$

$$= N(r, a_o(f')^{l_1} (f'')^{l_2} \dots (f^{(k)})^{l_k})$$

$$\leqslant N(r, a_o) + N(r,(f')^{l_1}) + N(r,(f'')^{l_2}) + \dots +$$

$$+ N(r, (f^{(k)})^{l_k}) + S(r,f)$$

$$\leqslant T(r,a_o) + l_1 N(r, f') + l_2 N(r,f'') + \dots + l_k N(r,f^{(k)}) +$$

$$+ S(r,f).$$

$$\leqslant l_1 \left[N(r,f) + \bar{N}(r,f)\right] + l_2 \left[N(r,f) + 2\bar{N}(r,f)\right] +$$

$$+ \dots + l_k \left[N(r,f) + k\bar{N}(r,f)\right] + S(r,f)$$

$$= (l_1 + l_2 + \dots + l_k) N(r,f) + (l_1 + 2l_2 + \dots + kl_k)\bar{N}(r,f) +$$

$$+ S(r,f).$$

Thus,

$$N(r,P_n(f)) \leqslant (l_1 + l_2 + \dots + l_k)N(r,f) + k(l_1 + l_2 + \dots$$

$$\dots + l_k)\bar{N}(r,f) + S(r,f).$$

Since $l_1 + l_2 + \dots + l_k = n$, we obtain

$$N(r, P_n(f)) \leqslant n N(r,f) + nK \bar{N}(r,f) + S(r,f). \qquad \dots (2.32)$$

Dividing both the sides by $T(r,f)$ and then taking limit superior as $r \to \infty$, we get

$$\limsup_{r\to\infty} \frac{N(r,P_n(f))}{T(r,f)} \leqslant n \limsup_{r\to\infty} \frac{N(r,f)}{T(r,f)} + nk \limsup_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} +$$

$$+ \limsup_{r\to\infty} \frac{S(r,f)}{T(r,f)}$$

which gives

$$1 - \delta(\infty, P_n(f)) \leqslant n\,(1 - \delta(\infty, f)) + nk\,(1 - \Theta(\infty, f)).$$

which yields

$$\delta(\infty, P_n(f)) \geqslant 1 - n(1+k) + n\left[\delta(\infty, f) + k\,\Theta(\infty, f)\right].$$

This proves lemma 2.4.

We shall now prove theorem 2.7.

<u>Proof of theorem 2.7</u> :

We have

$$N(r, D_n(f)) \leqslant N(r,P_1) + N(r, P_2) + \dots + N(r,P_t).$$

Using (2.32) we therefore get

$$N(r, D_n(f)) \leqslant n\left[N(r,f) + k_1 \bar{N}(r,f)\right] + n\left[N(r,f) + \right.$$

$$\left. + k_2 \bar{N}(r,f)\right] + \dots + n\left[N(r,f) + k_t \bar{N}(r,f)\right] + S(r,f)$$

$$= ntN(r,f) + n(k_1+k_2+\dots+k_t)\,\bar{N}(r,f) + S(r,f).$$

Dividing both the sides by $T(r,f)$ and then taking limit superior as $r \rightarrow \infty$, as usual, we get on simplification

$$\delta_T(\infty, D_n(f)) \geqslant 1 - nt - n(k_1 + k_2 + \ldots + k_t) + + nt\,\delta(\infty, f) + n(k_1 + k_2 + \ldots \ldots + k_t)\ \Theta(\infty, f)$$

which completes the proof of the theorem 2.7.

<u>Corollary</u>

Putting $n = 1$, $t = 1$ we get Theorem 3(ii) of [27] .

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