CHAPTER-III

RELATIVE DEFECTS FOR DISTINCT ROOTS OF MONOMIALS

Introduction :

Let $f(z)$ be a meromorphic function in the complex plane. Let $n(t,\alpha) = n(t, \frac{1}{f-\alpha})$ denote the number of roots of $f(z) = \alpha$ in $|z| \leqslant t$, the multiple roots being counted with their multiplicity.

Also, let $\bar{n}(t, \alpha) = \bar{n}(t, \frac{1}{f-\alpha})$

denote the number of distinct roots of $f(z) = \alpha$ in $|z| \leqslant t$. For $\alpha = \infty$, $n(t, \alpha) = n(t,f)$ and $\bar{n}(t, \alpha) = \bar{n}(t,f)$ respectively denote the number of poles and the number of distinct poles of $f(z)$ in $|z| \leqslant t$. We get

$$N(r, \alpha) = \int_0^r \frac{n(t, \alpha) - n(o, \alpha)}{t} dt + n(o, \alpha) \log r,$$

$$\bar{N}(r, \alpha) = \int_0^r \frac{\bar{n}(t, \alpha) - \bar{n}(o, \alpha)}{t} dt + \bar{n}(o, \alpha) \log r,$$

$$N(r, \frac{1}{f-\alpha}) = N(r,\alpha),\ N(r,f) = N(r, \infty).$$

The other terms being similarly defined.

As usual let

$$\delta(\alpha, f) = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{f-\alpha})}{T(r,f)};$$

$$\text{Ⓗ}(\alpha, f) = 1 - \limsup_{r \to \infty} \frac{\bar{N}(r, \frac{1}{f-\alpha})}{T(r,f)};$$

and if $P_n(f)$ denotes a differential polynomial of degree n, we set

$$\delta_r(\alpha, P_n(f)) = 1 - \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{P_n(f)-\alpha}\right)}{T(r,f)},$$

$$Ⓗ_r(\alpha, P_n(f)) = 1 - \limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{P_n(f)-\alpha}\right)}{T(r,f)}.$$

The suffix r in $Ⓗ_r(\alpha, P_n(f))$ denote the relative defect with respect to simple zero. Here we shall introduce absolute defect with respect to simple zeros, viz.

$$Ⓗ_a(\alpha, P_n(f)) = 1 - \limsup \frac{\bar{N}\left(r, \frac{1}{P_n(f)-\alpha}\right)}{T(r, P_n(f))}.$$

and prove relations involving these. Finally the terms S(r, f) will denote any quantity satisfying

$$S(r,f) = o(T(r,f)) \quad \text{as } r \to \infty$$

except possibly for a set of r of finite linear measure.

We first prove,

<u>Theorem 3.1</u> :

Let f and g be two meromorphic functions and $g(o) \neq 0$. Then

$$N\left(r, \frac{f}{g}\right) - N\left(r, \frac{g}{f}\right) = N(r,f) + N\left(r, \frac{1}{g}\right) - N(r,g) - N\left(r, \frac{1}{f}\right).$$

Proof -

By Jensen's formula we have on using (1,7) and (1.8) of [12, 4] in(1.5) of [12, 3] , $\log|f(0)|$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta - N(r, 1/f) + N(r,f).$$

Therefore,

$$-N(r,f) + N\left(r, \frac{1}{f}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta - \log|f(o)|. \quad \ldots (3.1)$$

But by hypothesis we have $g(o) \neq 0$ and therefore, we can change f to f/g in (3.1) and obtain

$$N\left(r, \frac{g}{f}\right) - N\left(r, \frac{f}{g}\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(re^{i\theta})}{g(re^{i\theta})}\right| d\theta - \log|f(o)| + \log|g(o)|$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|\, d\theta - \log|f(o)| + \log|g(o)|$$

$$= \left[\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta - \log |f(o)| \right] -$$

$$- \left[\frac{1}{2\pi} \int_0^{2\pi} \log |g(re^{i\theta})| \, d\theta - \log |g(o)| \right]$$

$$= \left[N(r, \frac{1}{f}) - N(r,f) \right] - \left[N(r, \frac{1}{g}) - N(r,g) \right] ,$$

and hence finally we get

$$N(r, \frac{f}{g}) - N(r, \frac{g}{f}) = N(r,f) + N(r, \frac{1}{g}) - N(r,g) - N(r, \frac{1}{f})$$

which completes the proof.

<u>Theorem 3.2</u> :

Let $f(z)$ be a meromorphic function. Then for any monomial $P_n(f)$, we have $Ⓗ_r(\alpha, P_n) \leqslant 2 - Ⓗ(\infty, f) - \delta(o, f)$.

For the proof, we shall need the following lemma :

<u>Lemma 3.1</u> :

Let

$$P_n(f) = (f)^{l_o} (f')^{l_1} \ldots ((f)^{(k)})^{l_k}$$

where $l_o + l_1 + \ldots + l_k = n$, be a monomial of degree n. Then

$$P_n'(f) = P_n(f) \left\{ l_o \frac{f'}{f} + l_1 \frac{f''}{f'} + \ldots + l_k \frac{f^{(k+1)}}{f^{(k)}} \right\} .$$

Proof of lemma 3.1 :

We have

$$P_n(f) = (f)^{l_o} (f')^{l_1} \dots (f^{(k)})^{l_k}.$$

This implies

$$P_n'(f) = l_o (f)^{l_o-1} (f') (f')^{l_1} \dots (f^{(k)})^{l_k} +$$

$$+ (f)^{l_o} l_1 (f')^{l_1-1} (f'') (f'')^{l_2} \dots (f^{(k)})^{l_k} +$$

$$+ \dots + (f)^{l_o}(f')^{l_1} \dots (f^{(k-1)})^{l_{k-1}} l_k (f^{(k)})^{l_k-1} (f^{(k+1)}).$$

That is

$$P_n'(f) = l_o \frac{f'}{f} (f)^{l_o} (f')^{l_1} \dots (f^{(k)})^{l_k} +$$

$$+ l_1 \frac{f''}{f'} (f)^{l_o} (f')^{l_1} \dots (f^{(k)})^{l_k} +$$

$$+ \dots + l_k \frac{f^{(k+1)}}{f^{(k)}} (f)^{l_o} (f')^{l_1} \dots (f^{(k)})^{l_k} .$$

$$= P_n(f) \left\{ l_o \frac{f'}{f} + l_1 \frac{f''}{f'} + \dots + l_k \frac{f^{(k+1)}}{f^{(k)}} \right\} .$$

Proof of Theorem 3.2 :

Clearly

$$\frac{\alpha}{f^n} = \frac{P_n}{f^n} - \frac{P_n - \alpha}{P_n'} \cdot \frac{P_n'}{f^n} . \qquad \dots (3.2)$$

But by lemma 3.1

$$\frac{P_n'}{f^n} = \frac{P_n}{f^n}\left\{ l_o \frac{f'}{f} + l_1 \frac{f''}{f'} + \dots + l_k \frac{f^{(k+1)}}{f^{(k)}} \right\} .$$

By Milloux's theorem

$$m(r, \frac{f^{(i)}}{f^{(j)}}) = S(r,f) \quad \text{for } j < i .$$

And as in (2.1), even if $P_n(f)$ contains terms in f, we get

$$m(r, \frac{P_n(f)}{f^n}) = S(r,f). \qquad \dots (3.3)$$

It now easily follows that

$$m(r, \frac{P_n'}{f^n}) = S(r,f). \qquad \dots (3.4)$$

Therefore, from (3.2), (3.3), (3.4) we get

$$m(r, \frac{\alpha}{f^n}) \leqslant m(r, \frac{P_n-\alpha}{P_n'}) + S(r,f)$$

$$= T(r, \frac{P_n-\alpha}{P_n'}) - N(r, \frac{P_n-\alpha}{P_n'}) + S(r,f).$$

And so, by Nevanlinna's first fundamental theorem we obtain

$$m(r, \frac{\alpha}{f^n}) \leqslant T(r, \frac{P_n'}{P_n-\alpha}) - N(r, \frac{P_n-\alpha}{P_n'}) + S(r,f) .$$

which is nothing but

$$m\left(r, \frac{\alpha}{f^n}\right) \leqslant N\left(r, \frac{P_n'}{P_n-\alpha}\right) - N\left(r, \frac{P_n-\alpha}{P_n'}\right) + m\left(r, \frac{P_n'}{P_n-\alpha}\right) +$$

$$+ S(r,f).$$

which yields on using Milloux's theorem,

$$m\left(r, \frac{\alpha}{f^n}\right) \leqslant N\left(r, \frac{P_n'}{P_n-\alpha}\right) - N\left(r, \frac{P_n-\alpha}{P_n'}\right) + S(r, P_n-\alpha) + S(r,f).$$

But $S(r, P_n - \alpha) = S(r,f)$ and so

$$nm\,(r, 1/f) \leqslant N\left(r, \frac{P_n'}{P_n - \alpha}\right) - N\left(r, \frac{P_n-\alpha}{P_n'}\right) + S(r,f).$$

That is

$$nT(r, 1/f) \leqslant \left[N\left(r, \frac{P_n'}{P_n-\alpha}\right) - N\left(r, \frac{P_n-\alpha}{P_n'}\right)\right] + nN(r,1/f)+S(r,f).$$

which gives with the use of theorem 3.1

$$nT(r,f) \leqslant N(r, P_n') + N\left(r, \frac{1}{P_n-\alpha}\right) - N(r, P_n - \alpha) -$$

$$- N\left(r, \frac{1}{P_n'}\right) + nN\left(r, \frac{1}{f}\right) + S(r,f).$$

and hence

$$nT(r,f) \leqslant \left[\bar{N}\left(r, \frac{1}{P_n-\alpha}\right) - N_o\left(r, \frac{1}{P_n'}\right)\right] + N(r, P_n') -$$

$$- N(r,P_n-\alpha) + nN\left(r, \frac{1}{f}\right) + S(r,f),$$

where $N_o\left(r, \frac{1}{P'_n}\right)$ are formed by those zeros of P'_n which are not the zeros of $P_n - \alpha$.

But $N(r, P_n - \alpha) = N(r, P_n)$, and so

$$nT(r,f) \leq \bar{N}\left(r, \frac{1}{P_n-\alpha}\right) - N_o\left(r, \frac{1}{P'_n}\right) + N(r, P'_n) - - N(r,P_n) + nN\left(r, \frac{1}{f}\right) + S(r,f).$$

It now easily follows that

$$nT(r,f) \leq \bar{N}\left(r, \frac{1}{P_n-\alpha}\right) - N_o\left(r, \frac{1}{P'_n}\right) + \bar{N}(r, P_n) + + nN\left(r, \frac{1}{f}\right) + S(r,f).$$

Since $\bar{N}(r, P_n) = \bar{N}(r,f)$ and since $N_o\left(r, \frac{1}{P'_n}\right) \geq 0$, we obtain

$$nT(r,f) \leq \bar{N}\left(r, \frac{1}{P_n-\alpha}\right) + \bar{N}(r,f) + nN\left(r, \frac{1}{f}\right) + S(r,f).$$

Dividing throughout by $T(r,f)$ and then taking limit superior as $r \to \infty$ we get

$$n \leq \limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{P_n-\alpha}\right)}{T(r,f)} + \limsup_{r \to \infty} \frac{\bar{N}(r,f)}{T(r,f)} +$$

$$+ \; n \limsup_{r \to \infty} \frac{N(r, 1/f)}{T(r,f)} + \limsup_{r \to \infty} \frac{S(r, f)}{T(r,f)} .$$

That is

$$n \leq \left[1 - \Theta_r (\alpha, P_n)\right] + \left[1 - \Theta (\infty, f)\right] +$$

$$+ \; n \left[1 - \delta(o, f)\right] ,$$

which on simplification gives

$$\Theta_r (\alpha, P_n) \leq 2 - \Theta (\infty, f) - \delta (o, f) .$$

This completes the proof of the theorem.

In [28] Theorem 4, A.P.Singh has mentioned the Theorem 3.3. However, he has not given the proof of that Theorem. Here we give a detailed proof of that theorem. Thus we shall prove,

<u>Theorem 3.3</u>

Let f(z) be a meromorphic function. Let each zero of f(z) have multiplicity $\geq$ n. Then for all positive integers k and $a \neq o$,

$$n \, \Theta_r^{(k)} (a, f) \leq (n + k + 1) - n\left[\Theta (\infty, f) + (k+1) \, \delta (a, f)\right]$$

<u>Proof of Theorem 3.3</u> :

Consider the identity

$$\frac{1}{f-a} = \frac{1}{a} \left[\frac{f^{(k)}}{f-a} - \frac{f^{(k)} - a}{f^{(k+1)}} \cdot \frac{f^{(k+1)}}{f-a} \right] .$$

Then

$$m\left(r, \frac{1}{f-a}\right) \leqslant m\left(r, \frac{1}{a}\right) + m\left(r, \frac{f^{(k)}}{f-a}\right) + m\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) +$$

$$+ m\left(r, \frac{f^{(k+1)}}{f-a}\right) + S(r,f).$$

And so by Milloux's theorem we get

$$m\left(r, \frac{1}{f-a}\right) \leqslant m\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) + S(r,f)$$

$$= T\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) - N\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) + S(r,f)$$

which yields on using the first fundamental theorem of Nevalinna,

$$m\left(r, \frac{1}{f-a}\right) \leqslant T\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) - N\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) + S(r,f)$$

$$= m\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) + N\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) - N\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) +$$

$$+ S(r,f).$$

Using Theorem 3.1 and Milloux's theorem, one easily gets

$$m\left(r, \frac{1}{f-a}\right) \leq N(r, f^{(k+1)}) + N\left(r, \frac{1}{f^{(k)}-a}\right) - N(r, f^{(k)}-a) -$$

$$- N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f).$$

Adding $N\left(r, \frac{1}{f-a}\right)$ on both the sides and using first fundamental theorem of Nevanlinna, the above inequality reduces to

$$T(r,f) \leq N\left(r, \frac{1}{f-a}\right) + N(r, f^{(k+1)}) + N\left(r, \frac{1}{f^{(k)}-a}\right) -$$

$$- N(r, f^{(k)}-a) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f).$$

But $N(r, f^{(k)}-a) = N(r, f^{(k)})$ and

$$N(r, f^{(k+1)}) - N(r, f^{(k)}) = \bar{N}(r, f^{(k)})$$

and so

$$T(r,f) = N\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-a}\right) - N_a\left(r, \frac{1}{f^{(k+1)}}\right) +$$

$$+ \bar{N}(r, f^{(k)}) + S(r,f)$$

where $N_a\left(r, \frac{1}{f^{(k+1)}}\right)$ is formed by those

Zeros of $f^{(k+1)}$ which are not the zeros of $f^{(k)}-a$. Thus

$$T(r,f) \leq N\left(r, \frac{1}{f-a}\right) - N_a\left(r, \frac{1}{f^{(k+1)}}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-a}\right) +$$

$$+ \bar{N}(r, f^{(k)}) + S(r,f)$$

$$\leq N_o\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-a}\right) + \bar{N}(r,f^{(k)}) + S(r,f),$$

where $N_o\left(r, \frac{1}{f-a}\right)$ is formed by all zeros of $f(z)-a$ taken with proper multiplicity if the multiplicity $\leq k+1$ and each zero of multiplicity $\geq K+2$ being counted $(k+1)$ times only.

Now, $\bar{N}(r,f^{(k)}) = \bar{N}(r,f)$ and so

$$T(r,f) \leq N_o\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-a}\right) + \bar{N}(r,f) + S(r,f).$$

But,

$$n\, N_o\left(r, \frac{1}{f-a}\right) \leq (k+1)\, N\left(r, \frac{1}{f-a}\right). \qquad \ldots (3.5)$$

Since on the left hand side of the inequality (3.5) each zero is counted atmost $n(k+1)$ times whereas on the right hand side each zero is counted atleast $n(k+1)$ times. Hence

$$T(r,f) \leq \left(\frac{k+1}{n}\right) N\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-a}\right) + \bar{N}(r,f) + S(r,f).$$

Dividing throughout by $T(r,f)$ and then taking limit superior as $r \to \infty$ of both the sides we get

$$1 < \left(\frac{k+1}{n}\right) \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)} + \limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{f^{(k)}-a}\right)}{T(r,f)} +$$

$$+ \limsup_{r \to \infty} \frac{\bar{N}(r,f)}{T(r,f)} + \limsup_{r \to \infty} \frac{S(r,f)}{T(r,f)},$$

which yields

$$1 < \left(\frac{k+1}{n}\right) \left[1 - \delta(a,f)\right] + \left[1 - \Theta_r^{(k)}(a,f)\right] + \left[1 - \Theta(\infty,f)\right]$$

which on rearrangement gives

$$n\Theta_r^{(k)}(a,f) < (n+k+1) - \left[n\Theta(\infty,f) + (k+1)\delta(a,f)\right]$$

as desired.

<u>Theorem 3.4</u> :

Let $f(z)$ be a meromorphic function and $P_n(f)$ be a monomial of degree n not containing f. Then

$$3\delta_r(\infty, P_n(f)) \leq (n+3) + n\delta(\infty,f) - 2n\delta(0,f).$$

<u>Proof</u> -

Consider the following identity.

$$\frac{a}{f^n} = 1 - \frac{f^n - a}{P_n(f)} \cdot \frac{P_n(f)}{f^n}, \quad a \neq 0.$$

Then

$$T(r, \frac{a}{f^n}) \leqslant T(r, \frac{f^n-a}{P_n(f)}) + T(r, \frac{P_n(f)}{f^n}) + S(r,f)$$

$$\leqslant T(r, \frac{f^n}{P_n(f)}) + T(r, \frac{a}{P_n(f)}) + T(r, \frac{P_n(f)}{f^n}) +$$

$$+ S(r,f).$$

Using first fundamental theorem of Nevanlinna, we get

$$nT(r,f) \leqslant T(r, \frac{P_n(f)}{f^n}) + T(r, \frac{P_n(f)}{a}) + T(r, \frac{P_n(f)}{f^n}) + S(r,f).$$

$$\leqslant 2T(r, \frac{P_n(f)}{f^n}) + T(r,P_n(f)) + T(r, \frac{1}{a}) + S(r,f)$$

$$= 2m(r, \frac{P_n(f)}{f^n}) + 2N(r, \frac{P_n(f)}{f^n}) + m(r,P_n(f)) +$$

$$+ N(r, P_n(f)) + S(r,f).$$

Using (2.1) we at once get

$$nT(r,f) \leqslant 2N(r, \frac{P_n(f)}{f^n}) + m(r, \frac{P_n(f)}{f^n} \cdot f^n) +$$

$$+ N(r, P_n(f) + S(r,f).$$

That is

$$nT(r,f) \leq 2N(r, P_n(f)) + 2N(r, \frac{1}{f^n}) + m(r, \frac{P_n(f)}{f^n}) +$$

$$+ m(r,f^n) + N(r, P_n(f)) + S(r,f).$$

So, once again using (2.1) we obtain

$$nT(r,f) \leq 3N(r, P_n(f)) + 2nN(r, \frac{1}{f}) + nm(r,f) + S(r,f);$$

and hence

$$n\left[T(r,f) - m(r,f)\right] \leq 3N(r,P_n(f)) + 2nN(r, \frac{1}{f}) + S(r,f),$$

which gives

$$nN(r,f) \leq 3N(r, P_n(f)) + 2nN(r, \frac{1}{f}) + S(r,f).$$

Dividing throughout by $T(r,f)$ and then taking limit superior as $r \to \infty$ of both the sides and after adjusting the terms, we will get

$$3\delta_r(\infty, P_n(f)) \leq n+3 + n\delta(\infty, f) - 2n\delta(o,f).$$

which completes the proof.

Note :

If $a \neq \infty$ then putting $n=1$ and $P_n(f) = f^{(k)}$ we get Theorem 2 of [28] .

Theorem 3.5 :

Let $f(z)$ be a meromorphic function. Let

$$P_n(f) = a(z)(f')^{l_1} (f'')^{l_2} \ldots (f^{(k)})^{l_k}$$

where $l_1 + l_2 + \dots + l_k = n$;

be a monomial of degree n not containing f. Let each zero of f(z) have multiplicity $\geq$ m. Then

$$m \, \textcircled{H}_r (1, P_n) \leq (k + m + 1) - \left[(k+1) \, \delta(o, f) + m \textcircled{H} (\infty, f) \right] .$$

Proof :-

Consider the identity

$$\frac{1}{f^n} = \frac{P_n(f)}{f^n} - \frac{P_n(f)-1}{P_n'(f)} \cdot \frac{P_n'(f)}{f^n} ,$$

from which it follows that

$$m\left(r, \frac{1}{f^n}\right) \leq m\left(r, \frac{P_n}{f^n}\right) + m\left(r, \frac{P_n(f)-1}{P_n'(f)}\right) + m\left(r, \frac{P_n'(f)}{f^n}\right) + $$

$$+ S(r,f).$$

But by inequality (2.1) and (3.4) we respectively have

$$m\left(r, \frac{P_n}{f^n}\right) = S(r,f)$$

and $$m\left(r, \frac{P_n'}{f^n}\right) = S(r,f).$$

Using these results the above inequality gets converted into

$$m\left(r, \frac{1}{f^n}\right) \leq m\left(r, \frac{P_n-1}{P_n'}\right) + S(r,f)$$

$$= T(r, \frac{P_n-1}{P'_n}) - N(r, \frac{P_n - 1}{P'_n}) + S(r,f).$$

And so by first fundamental theorem

$$m(r, \frac{1}{f^n}) \leqslant T(r, \frac{P'_n}{P_n-1}) - N(r, \frac{P_n-1}{P'_n}) + S(r,f)$$

$$= N(r, \frac{P'_n}{P_n-1}) - N(r, \frac{P_n-1}{P'_n}) + m(r, \frac{P'_n}{P_n-1}) + S(r,f).$$

With the use of Theorem 3.1 this reduces to

$$nm(r, \frac{1}{f}) \leqslant N(r, P'_n) + N(r, \frac{1}{P_n-1}) - N(r, P_n-1) -$$

$$- N(r, \frac{1}{P'_n}) + S(r, P_n-1) + S(r,f).$$

Adding both the sides by $nN(r, \frac{1}{f})$ and using the fact that $S(r, P_n-1) = S(r,f)$, we get

$$nT(r, \frac{1}{f}) \leqslant nN(r, \frac{1}{f}) + N(r, P'_n) + N(r, \frac{1}{P_n-1}) -$$

$$- N(r, P_n-1) - N(r, \frac{1}{P'_n}) + S(r,f).$$

which gives, on using $N(r, P_n-1) = N(r, P_n)$,

$$nT(r,f) \leqslant nN(r, \frac{1}{f}) + N(r, P'_n) - N(r, P_n) +$$

$$+ N(r, \frac{1}{P_n-1}) - N(r, \frac{1}{P'_n}) + S(r,f).$$

That is

$$nT(r,f) \leq nN(r, \frac{1}{f}) + \bar{N}(r, P_n) + \left\{ \bar{N}(r, \frac{1}{P_n-1}) - \right.$$

$$\left. - N_o(r, \frac{1}{P'_n}) \right\} + S(r,f).$$

where $N_o(r, \frac{1}{P'_n})$ are formed by taking those zeros of P'_n which are not the zeros of $P_n - 1$. Thus

$$nT(r,f) \leq nN_a(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{P_n-1}) + \bar{N}(r, P_n) + S(r,f),$$

where $N_a(r, \frac{1}{f})$ is formed by all zeros of $f(z)$ taken with proper multiplicity if the multiplicity is $\leq k+1$ and each zero of multiplicity $\geq k+2$ being counted $(k+1)$ times only where k is as in hypothesis.

Now,

$$\bar{N}(r, P_n) = \bar{N}(r,f)$$

and therefore above inequality becomes

$$nT(r,f) \leq nN_a(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{P_n-1}) + \bar{N}(r,f) + S(r,f).$$

But

$$mN_a(r, \frac{1}{f}) \leq (k+1) N(r, \frac{1}{f}),$$

Since on the left hand side of above inequality each zero is counted atmost $m(k+1)$ times whereas on the right hand side each zero is counted atleast $m(k+1)$ times. Hence,

$$nT(r,f) \leq \frac{k+1}{m} N(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{P_n - 1}) + \bar{N}(r,f) + S(r,f).$$

Dividing throughout by $T(r,f)$ and then taking limit superior as $r \to \infty$ of both the sides, we get

$$n \leq \left(\frac{k+1}{m}\right) \limsup_{r \to \infty} \frac{N(r, \frac{1}{f})}{T(r,f)} + \limsup_{r \to \infty} \frac{\bar{N}(r, \frac{1}{P_n - 1})}{T(r,f)} +$$

$$+ \limsup_{r \to \infty} \frac{\bar{N}(r,f)}{T(r,f)} + \limsup_{r \to \infty} \frac{S(r,f)}{T(r,f)} .$$

That is

$$n \leq \left(\frac{k+1}{m}\right) \left[1 - \delta(o, f)\right] + \left[1 - \Theta_r(1, P_n)\right] +$$

$$+ \left[1 - \Theta(\infty, f)\right] .$$

After simplification it gives

$$m\, \Theta_r(1, P_n) \leq (k+m+1) - \left[(k+1)\, \delta(o,f) + m\, \Theta(\infty, f)\right]$$

which is what we wanted to show.

<u>Note</u> :

Putting $P_n = f^{(k)}$ i.e. a monomial of degree 1, we get

$$m\, \Theta_r(1, f^{(k)}) \leq (m + k + 1) - \left[(k+1)\, \delta(o,f) + m\, \Theta(\infty, f)\right]$$

which is Theorem 4 of [28] .

Theorem 3.6 :

Let $f(z)$ be a transcendental meromorphic function. Let $P_n(f)$ be a monomial of degree n and not containing f. Then

$$\Theta_r(0, P_n) + \Theta_r(b, P_n) + \Theta_r(c, P_n)$$

$$\leqslant 2 + \delta_r(0, P_n) - n\delta(a,f);$$

where a, b, c are distinct finite numbers and $b \neq 0$, $C \neq 0$,

Proof :

Since $P_n(f)$ does not contain f, it follows as in (2.1) that

$$m\left(r, \frac{1}{(f-a)^n}\right) \leqslant m\left(r, \frac{1}{P_n(f)}\right) + S(r,f).$$

Thus

$$m\left(r, \frac{1}{(f-a)^n}\right) \leqslant T\left(r, \frac{1}{P_n(f)}\right) - N\left(r, \frac{1}{P_n(f)}\right) + S(r,f).$$

And so by Nevanlinna's first fundamental theorem

$$m\left(r, \frac{1}{(f-a)^n}\right) \leqslant T(r, P_n(f)) - N\left(r, \frac{1}{P_n(f)}\right) + S(r,f). \quad ...(3.7)$$

Also by Nevanlinna's second fundamental theorem, since $S(r, P_n(f)) = S(r,f)$ we have

$$T(r, P_n(f)) \leq \bar{N}\left(r, \frac{1}{P_n(f)}\right) + \bar{N}\left(r, \frac{1}{P_n(f)-b}\right) +$$

$$+ \bar{N}\left(r, \frac{1}{P_n(f)-c}\right) + S(r,f).$$

Making use of this, inequality (3.7) gets converted into

$$m\left(r, \frac{1}{(f-a)^n}\right) \leq \bar{N}\left(r, \frac{1}{P_n(f)}\right) + \bar{N}\left(r, \frac{1}{P_n-b}\right) + \bar{N}\left(r, \frac{1}{P_n-c}\right) -$$

$$- N\left(r, \frac{1}{P_n}\right) + S(r,f).$$

Adding $N\left(r, \frac{1}{(f-a)^n}\right)$ on both the sides, we get

$$T\left(r, \frac{1}{(f-a)^n}\right) \leq N\left(r, \frac{1}{(f-a)^n}\right) + \bar{N}\left(r, \frac{1}{P_n}\right) + \bar{N}\left(r, \frac{1}{P_n-b}\right) +$$

$$+ \bar{N}\left(r, \frac{1}{P_n-c}\right) - N\left(r, \frac{1}{P_n}\right) + S(r,f).$$

Thus

$$nT(r,f) \leq nN\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{P_n}\right) + \bar{N}\left(r, \frac{1}{P_n-b}\right) +$$

$$+ \bar{N}\left(r, \frac{1}{P_n-c}\right) - N\left(r, \frac{1}{P_n}\right) + S(r,f).$$

From which it easily follows that

$$n \leq n \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r,f)} + \limsup_{r\to\infty} \frac{\bar{N}(r, \frac{1}{P_n})}{T(r,f)} +$$

$$+ \limsup_{r\to\infty} \frac{\bar{N}(r, \frac{1}{P_n-b})}{T(r,f)} + \limsup_{r\to\infty} \frac{\bar{N}(r, \frac{1}{P_n-c})}{T(r,f)} -$$

$$- \limsup_{r\to\infty} \frac{N(r, \frac{1}{P_n})}{T(r,f)} + \limsup_{r\to\infty} \frac{S(r,f)}{T(r,f)} .$$

which is nothing but

$$n \leq n \left[1 - \delta(a,f)\right] + \left[1 - \circledH_r(0, P_n)\right] + \left[1 - \circledH_r(b, P_n)\right] +$$

$$+ \left[1 - \circledH_r(c, P_n)\right] - \left[1 - \delta_r(0, P_n)\right] .$$

After simplification, finally, we get

$$\circledH_r(0, P_n) + \circledH_r(b, P_n) + \circledH_r(c, P_n)$$

$$\leq 2 + \delta_r(0, P_n) - n\delta(a,f)$$

which is our required theorem.

Remark -

We once again observe that, putting n = 1, our theorem reduces to Theorem 6 of [28].

Now we come to another interesting result.

Theorem 3.7 :

Let $f(z)$ be a meromorphic function. Let as earlier, $P_n(f)$ be a monomial of degree n and not containing f. Then for all integers $p \geq 1$ and $a_i (i = 1, 2, \ldots, p)$ finite, distinct and non-zero complex numbers,

$$\sum_{i=1}^{p} \Theta_r (a_i, P_n) + \Theta_r (o, p_n) + \Theta (\infty, f) \leq z + p \left\{ \delta_r (0, P_n) - n\delta (o, f) \right\} .$$

Proof

By Nevanlinna's first fundamental theorem we have

$$T(r, f^n) = T\left(r, \frac{1}{f^n}\right) + S(r,f)$$

$$= m\left(r, \frac{1}{f^n}\right) + N\left(r, \frac{1}{f^n}\right) + S(r,f)$$

$$= m\left(r, \frac{P_n}{f^n} \cdot \frac{1}{P_n}\right) + N\left(r, \frac{1}{f^n}\right) + S(r,f).$$

So,

$$T(r, f^n) \leq m\left(r, \frac{P_n}{f^n}\right) + m\left(r, \frac{1}{P_n}\right) + N\left(r, \frac{1}{f^n}\right) + S(r,f)$$

reduces on using (2.1),

$$T(r, f^n) \leq m\left(r, \frac{1}{P_n}\right) + N\left(r, \frac{1}{f^n}\right) + S(r,f)$$

$$= N\left(r, \frac{1}{f^n}\right) + T\left(r, \frac{1}{P_n}\right) - N\left(r, \frac{1}{P_n}\right) + S(r,f)$$

and hence

$$T(r,f^n) \leq N\left(r, \frac{1}{f^n}\right) + T(r,P_n) - N\left(r, \frac{1}{P_n}\right) + S(r,f).$$

This gives

$$pT(r,f^n) \leq pN\left(r, \frac{1}{f^n}\right) + pT(r, P_n) - pN\left(r, \frac{1}{P_n}\right) + S(r,f). \quad ..(3.8)$$

Next, by Nevanlinna's Second Fundamental Theorem we obtain

$$pT(r,P_n) \leq \bar{N}(r,P_n) + \bar{N}\left(r, \frac{1}{P_n}\right) + \sum_{i=1}^{p} \bar{N}\left(r, \frac{1}{P_n-a_i}\right) + S(r,f). \quad \ldots (3.9)$$

But,

$$\bar{N}(r,P_n) = \bar{N}(r,f).$$

Therefore by (3.8) and (3.9) we have

$$p.nT(r,f) \leq p.n\, N\left(r, \frac{1}{f}\right) + \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{P_n}\right) + \sum_{i=1}^{p} \bar{N}\left(r, \frac{1}{P_n-a_i}\right) - pN\left(r, \frac{1}{P_n}\right) + S(r,f).$$

It now easily follows on dividing by T(r,f) that

$$p.n \leq p\, n \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f}\right)}{T(r,f)} + \limsup_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} + \limsup_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{P_n}\right)}{T(r,f)} + \limsup_{r\to\infty} \frac{\sum_{i=1}^{p} \bar{N}\left(r, \frac{1}{P_n-a_i}\right)}{T(r,f)} -$$

...

$$- p \limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{P_n}\right)}{T(r,f)} + \limsup_{r \to \infty} \frac{S(r,f)}{T(r,f)} .$$

This yields,

$$pn \leqslant pn \left[1 - \delta(o,f)\right] + \left[1 - \Theta(\infty, f)\right] +$$

$$+ \left[1 - \Theta_r(o, P_n)\right] + p - \sum_{i=1}^{p} \Theta_r(a_i, P_n) -$$

$$- p \left[1 - \delta_r(o, P_n)\right] .$$

After proper adjustment and cancellation of some terms, at the end, we get

$$\sum_{i=1}^{p} \Theta_r(a_i, P_n) + \Theta_r(o, P_n) + \Theta(\infty, f)$$

$$\leqslant 2 + p\left[\delta_r(o, P_n) - n\delta(o,f)\right]$$

which we wanted to show.

Remark :

As an immediate consequence is Theorem 7 of [28] which is obtained by putting n = 1 in the above theorem.

We now prove,

Theorem 3.8 :

Let f(z) be a meromorphic function and $P_n(f)$ be a monomial of degree n and not containing f. Then,

$$\sum_{i=1}^{p} \textcircled{H}(a_i, f) + \textcircled{H}(o, f) + 2\,\textcircled{H}(\infty, f) + \sum_{j=1}^{q} \textcircled{H}_r(b_j, P_n) + \textcircled{H}_r(o, P_n) \leq 4 + q\left\{\delta_r(o, P_n) - n\delta(o, f)\right\},$$

where a_i are non-zero, finite, distinct and $b_j \neq o$, for any j $(j = 1, 2, \ldots, q)$.

Proof -

By inequality (3.8) we have

$$qT(r, f^n) \leq qN\left(r, \frac{1}{f^n}\right) + qT(r, P_n) - qN\left(r, \frac{1}{P_n}\right) + S(r, f). \quad \ldots (3.10)$$

Next, by Nevanlinna's second Fundamental Theorem, we obtain

$$qT(r, P_n) \leq \bar{N}(r, P_n) + \bar{N}\left(r, \frac{1}{P_n}\right) + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{P_n - b_j}\right) + S(r, f). \quad \ldots (3.11)$$

With this inequality (3.10) becomes

$$qT(r, f^n) \leq \bar{N}(r, P_n) + \bar{N}\left(r, \frac{1}{P_n}\right) + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{P_n - b_j}\right) + qN\left(r, \frac{1}{f^n}\right) - qN\left(r, \frac{1}{P_n}\right) + S(r, f).$$

But

$$\bar{N}(r, P_n) = \bar{N}(r, f)$$

and therefore

$$nqT(r,f^n) \leq \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{P_n}\right) + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{P_n - b_j}\right) +$$

$$+ qN\left(r, \frac{1}{f^n}\right) - qN\left(r, \frac{1}{P_n}\right) + S(r,f). \quad \ldots(3.12)$$

Also, by Second Fundamental Theorem, we have

$$pT(r,f) \leq \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{f}\right) + \sum_{i=1}^{p} \bar{N}\left(r, \frac{1}{f-a_i}\right) +$$

$$+ S(r,f). \quad \ldots (3.13)$$

Adding (3.12) and (3.13) we get

$$(p + nq)\, T(r,f) \leq 2\bar{N}(r,f) + \bar{N}\left(r, \frac{1}{f}\right) + nq\, N\left(r, \frac{1}{f}\right) +$$

$$+ \bar{N}\left(r, \frac{1}{P_n}\right) - qN\left(r, \frac{1}{P_n}\right) + \sum_{i=1}^{p} \bar{N}\left(r, \frac{1}{f-a_i}\right) +$$

$$+ \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{P_n - b_j}\right) + S(r,f).$$

Dividing both the sides by $T(r,f)$ and then taking limit inferior as $r \to \infty$, we can have

$$p + nq \leq \liminf_{r \to \infty} \left\{ \frac{2\bar{N}(r,f)}{T(r,f)} + \frac{\bar{N}\left(r, \frac{1}{f}\right)}{T(r,f)} + \frac{nqN\left(r, \frac{1}{f}\right)}{T(r,f)} + \right.$$

$$+ \frac{\bar{N}\left(r, \frac{1}{P_n}\right)}{T(r,f)} - \frac{qN\left(r, \frac{1}{P_n}\right)}{T(r,f)} +$$

$$+ \sum_{i=1}^{q} \frac{\bar{N}\left(r, \frac{1}{f-a_i}\right)}{T(r,f)} + \sum_{j=1}^{q} \frac{\bar{N}\left(r, \frac{1}{P_n-b_j}\right)}{T(r,f)} +$$

$$\left. + \frac{S(r,f)}{T(r,f)} \right\} .$$

Since $\liminf_{r \to \infty} \left(-q \frac{N\left(r, \frac{1}{P_n}\right)}{T(r,f)} \right) = - \limsup_{r \to \infty} \frac{qN\left(r, \frac{1}{P_n}\right)}{T(r,f)}$

it easily follows that

$$p + nq \leqslant 2 \limsup_{r \to \infty} \frac{\bar{N}(r,f)}{T(r,f)} + \limsup_{r \to \infty} \frac{\bar{N}(r, 1/f)}{T(r,f)} +$$

$$+ nq \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{f}\right)}{T(r,f)} + \limsup_{r \to \infty} \frac{\bar{N}\left(r, \frac{1}{P_n}\right)}{T(r,f)} -$$

$$- q \limsup_{r \to \infty} \frac{N\left(r, \frac{1}{P_n}\right)}{T(r,f)} + \limsup_{r \to \infty} \sum_{i=1}^{p} \frac{\bar{N}\left(r, \frac{1}{f-a_i}\right)}{T(r,f)} +$$

...

$$+ \limsup_{r \to \infty} \sum_{j=1}^{q} \frac{\bar{N}\left(r, \frac{1}{P_n - b_j}\right)}{T(r,f)} + \limsup_{r \to \infty} \frac{S(r,f)}{T(r,f)} .$$

This gives,

$$p+nq \leqslant 2\left[1 - \text{Ⓗ}(\infty,f)\right] + \left[1 - \text{Ⓗ}(o,f)\right] + nq\left[1-\delta(o,f)\right] +$$

$$+\left[1 - \text{Ⓗ}_r(o, P_n)\right] - q\left[1 - \delta_r(o,P_n)\right] +$$

$$+ \sum_{i=1}^{p}\left[1 - \text{Ⓗ}(a_i, f)\right] + \sum_{j=1}^{q}\left[1 - \text{Ⓗ}_r(b_j,P_n)\right] .$$

After simplification finally it reduces to

$$\sum_{i=1}^{p} \text{Ⓗ}(a_i,f) + \text{Ⓗ}(o,f) + 2\,\text{Ⓗ}(\infty,f) + \sum_{j=1}^{q} \text{Ⓗ}_r(b_j,P_n) +$$

$$+\text{Ⓗ}_r(o, P_n)$$

$$\leqslant 4 + q\left[\delta_r(o,P_n) - n\delta(o,f)\right]$$

as desired.

<u>Theorem 3.9</u> :

Let f(z) be a meromorphic function. Then

$$\sum_{j=o}^{q+1} \text{Ⓗ}_r^{(k)}(b_j,f) \leqslant 4 - \sum_{a\in\bar{C}} \text{Ⓗ}(a,f) + q\left[\delta_r^{k}(o,f) - \right.$$

$$\left. - \delta(o,f)\right]$$

where b_j's are distinct, $b_o = 0$, $b_1 = \infty$.

Proof :

By Theorem 8 of [28], we have

$$\sum_{i=1}^{p} \Theta(a_i, f) + \Theta(o, f) + 2\,\Theta(\infty, f) +$$

$$+ \sum_{j=1}^{q} \Theta_r^{(k)}(b_j, f) + \Theta_r^{(k)}(o, f)$$

$$\leqslant 4 + q\left[\delta_r^{(k)}(o, f) - \delta(o, f)\right].$$

Therefore, on making $p \to \infty$ and observing that $\{a \,/\, \Theta(a) > 0\}$ is countable, it follows that

$$\Theta(a, f) + \Theta(\infty, f) + \sum_{j=1}^{q} \Theta_r^{(k)}(b_j, f) + \Theta_r^{(k)}(o, f)$$

$$\leqslant 4 + q\left[\delta_r^{(k)}(o, f) - \delta(o, f)\right].$$

Now

$$\Theta(\infty, f) = \Theta_r^{(k)}(\infty, f),$$

and hence

$$\sum_{a \in \overline{C}} \Theta(a, f) + \sum_{j=0}^{q} \Theta_r^{(k)}(b_j, f) + \Theta_r^{(k)}(o, f)$$

$$\leqslant 4 + q\left[\delta_r^{(k)}(o, f) - \delta(o, f)\right].$$

Now, denoting zero by b_{q+1}, the above inequality takes the form

$$\sum_{j=0}^{q+1} \textcircled{H}_r^{(k)} (b_j, f) \leq 4 + q \left[\delta_r^{(k)} (o, f) - \delta (o, f) \right] - $$

$$- \sum_{a \in \tau} \textcircled{N} (a, f).$$

which proves the theorem.

Finally we prove one more theorem on monomials.

<u>Theorem 3.10</u> :

Let $f(z)$ be a meromorphic function and let $a_i (i = 1, 2, \ldots, p)$ and b_j $(j = 1, 2, \ldots, q)$ be finite complex numbers distinct within each set and such that $b_j \neq 0$ for any j. Further, let $P_n(f)$ be a monomial of degree n and not containing f. Then,

$$\sum_{j=1}^{q} \textcircled{H}_r (b_j, P_n) + \textcircled{H}_r (0, P_n) + \textcircled{H} (\infty, f) + nq \sum_{i=1}^{p} \delta (a_i, f)$$

$$\leq q + 2.$$

<u>Proof</u> -

Let $$F(z) = \sum_{i=1}^{p} \frac{1}{(f(z) - a_i)^n}$$

Then by [21, 24] we have

$$\sum_{i=1}^{p} m \left(r, \frac{1}{(f-a_i)^n}\right) \leq m (r, F) + S(r, f)$$

$$= m \left(r, \frac{F}{P_n} \cdot P_n\right) + S(r, f)$$

...

$$\leq m(r, FP_n) + m\left(r, \frac{1}{P_n}\right) + S(r,f)$$

$$\leq \sum_{i=1}^{p} m\left(r, \frac{P_n}{(f-a_i)^n}\right) + m\left(r, \frac{1}{P_n}\right) + S(r,f)$$

$$= m\left(r, \frac{1}{P_n}\right) + S(r,f).$$

Adding $\sum_{i=1}^{p} N\left(r, \frac{1}{(f-a_i)^n}\right)$ on both the sides, we get

$$\sum_{i=1}^{p} m\left(r, \frac{1}{(f-a_i)^n}\right) + \sum_{i=1}^{p} N\left(r, \frac{1}{(f-a_i)^n}\right)$$

$$\leq m\left(r, \frac{1}{P_n}\right) + \sum_{i=1}^{p} N\left(r, \frac{1}{(f-a_i)^n}\right) + S(r,f),$$

which gives on using Nevanlinna's first fundamental theorem

$$npT(r,f) \leq T(r,P_n) + \sum_{i=1}^{p} N\left(r, \frac{1}{(f-a_i)^n}\right) + S(r,f).$$

And so

$$npqT(r,f) \leq qT(r,P_n) + q\sum_{i=1}^{p} N\left(r, \frac{1}{(f-a_i)^n}\right) + S(r,f) \quad \ldots (3.14)$$

But by Nevanlinna's Second Fundamental Theorem, we have

$$qT(r,P_n) \leq \bar{N}(r,P_n) + \bar{N}\left(r, \frac{1}{P_n}\right) + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{P_n-b_j}\right) +$$

$$+ S(r,f). \quad \ldots (3.15)$$

With the use of (3.15) inequality (3.14) becomes

$$npqT(r,f) \leq \bar{\bar{N}}(r, P_n) + \bar{N}\left(r, \frac{1}{P_n}\right) + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{P_n - b_j}\right) + $$

$$+ q \sum_{i=1}^{p} N\left(r, \frac{1}{(f-a_i)^n}\right) + S(r,f).$$

But $\bar{\bar{N}}(r, P_n) = \bar{\bar{N}}(r,f)$

and so

$$npqT(r,f) \leq \bar{\bar{N}}(r,f) + \bar{N}\left(r, \frac{1}{P_n}\right) + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{P_n - b_j}\right) + $$

$$+ q \sum_{i=1}^{p} N\left(r, \frac{1}{(f-a_i)^n}\right) + S(r,f).$$

Dividing both the sides by $T(r,f)$ and then taking limit superior as $r \to \infty$, we get

$$npq \leq \limsup_{r\to\infty} \frac{\bar{\bar{N}}(r,f)}{T(r,f)} + \limsup_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{P_n}\right)}{T(r,f)} + $$

$$+ \limsup_{r\to\infty} \sum_{j=1}^{q} \frac{\bar{N}\left(r, \frac{1}{P_n - b_j}\right)}{T(r,f)} + $$

$$+ nq \limsup_{r\to\infty} \sum_{i=1}^{p} \frac{N\left(r, \frac{1}{f-a_i}\right)}{T(r,f)} + \limsup_{r\to\infty} \frac{S(r,f)}{T(r,f)}$$

which is nothing but

$$npq \leqslant \left[1 - \textcircled{H}(\infty, f)\right] + \left[1 - \textcircled{H}_r(o, b_n)\right] +$$

$$+ \sum_{j=1}^{q} \left[1 - \textcircled{H}_r(b_j, P_n)\right] + nq \sum_{i=1}^{p} \left[1 - \delta(a_i, f)\right] .$$

Simplification of the above inequality finally gives,

$$\sum_{j=1}^{q} \textcircled{H}_r(b_j, P_n) + \textcircled{H}_r(o, P_n) + \textcircled{H}(\infty, f) +$$

$$+ nq \sum_{i=1}^{p} \delta(a_i, f) \leqslant q + 2,$$

which completes the proof of the theorem.

o0o