CHAPTER - III

GROWTH OF DIFFERENTIAL POLYNOMIALS

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Growth of differential polynomials

Let f(z) be a meromorphic function. As mentioned in Chapter II, $\pi_n(f)$ will denote a homogeneous differential polynomial of degree n. That is a finite sum of the form

$$a(z)(f)^{l_0}\ (f')^{l_1}\ \ldots\ldots\ (f^{(k)})^{l_k}$$

where $l_0 + l_1 + \ldots + l_k = n$ and a(z) is meromorphic function satisfying $T(r, a(z)) = S(r,f)$ as $r \to \infty$ where by $S(r,f)$ we mean any quantity satisfying $S(r,f) = O(T(r,f))$ as $r \to \infty$ if f is of finite order and $S(r,f) = o(T(r,f))$ outside an exceptional set of finite linear measure if f is of infinite order. Throughout we assume that the homogeneous differential polynomial is such that it does not become zero. The term $\delta(\alpha,f)$, $\Theta(\alpha,f)$ etc. being as defined at the beginning of the Chapter II.

We now prove our results.

Theorem 3.1 : For any transcendental meromorphic function of finite order

$$\Delta(\pi_n(f),0)\ \liminf_{r\to\infty} \frac{T(r,\pi_n(f))}{T(r,f)} \geq n \sum_{i=1}^{\infty} \delta(a_i) \quad \ldots (3.1)$$

and $(\Delta(\pi_n(f), 0))(n_t (1+q - q \Theta(\infty))) \geqslant n \sum_{i=1}^{\infty} \delta(a_i).$

$$\cdot (1 + \Delta(\pi_n(f),0) - \delta(\pi_n(f), 0)) \quad \ldots (3.2)$$

where $|a_i| < \infty$ and $\delta(a_i) > 0$

and $\pi_n(f)$ is a homogeneous differential polynomial of degree n, not containing f for the proof of this theorem we shall need the following two lemmas :

<u>Lemma 3.1</u> : Let f(z) be a transcendental meromorphic function and $a_1, a_2 \ldots a_q$, $q > 2$ be distinct finite complex numbers. Then

$$n \sum_{i=1}^{q} m(r, a_i, f) < T(r, \pi_n(f)) - N(r, \frac{1}{\pi_n(f)}) + \\ + S(r,f) \quad \ldots (3.3)$$

where $S(r,f) = o(T(r,f))$ as $\rightarrow \infty$ through all values if f is of finite order and $S(r,f) = o(T(r,f))$ as $r \rightarrow \infty$ outside a set of finite linear measure otherwise. And $\pi_n(f)$ is a non-zero homogeneous differential polynomial of degree n not containing f. That is $\pi_n(f) = \sum a(z)(f')^{l_1} \ldots (f^{(k)})^{l_k}$

where $l_1 + l_2 + \ldots + l_k = n$

<u>Proof</u> : Let $F(z) = \sum_{i=1}^{q} \frac{1}{(f-a_i)^n}$

Then we have

$$n \sum_{i=1}^{q} m(r, a_i, f) \leq m(r, F) + O(1)$$

$$= m\left(r, \frac{F\pi_n(f)}{\pi_n(f)}\right) + O(1)$$

$$\leq m(r, F\pi_n(f)) + m\left(r, \frac{1}{\pi_n(f)}\right) + O(1)$$

$$= \sum_{i=1}^{q} m\left(r, \frac{\sum a(z)(f')^{l_1} \dots (f^{(k)})^{l_k}}{(f - a_i)^n}\right) +$$

$$+ m\left(r, \frac{1}{\pi_n(f)}\right) + O(1)$$

$$= \sum_{i=1}^{q} m\left(r, \sum a(z) \left(\frac{f'}{f-a_i}\right)^{l_1} \dots \left(\frac{f^{(k)}}{f-a_i}\right)^{l_k}\right)$$

$$+ m\left(r, \frac{1}{\pi_n(f)}\right) + O(1)$$

$$= m\left(r, \frac{1}{\pi_n(f)}\right) + S(r,f)$$

$$= T\left(r, \frac{1}{\pi_n(f)}\right) - N\left(r, \frac{1}{\pi_n(f)}\right) + S(r,f)$$

$$= T(r, \pi_n(f)) - N\left(r, \frac{1}{\pi_n(f)}\right)$$

Therefore

$$n \sum_{i=1}^{q} m(r, a_i, f) \leq T(r,\pi_n(f)) - N(r, \frac{1}{\pi_n(f)}) + S(r,f)$$

<u>Lemma 3.2</u> : Let f be a transcendental meromorphic function of finite order, then

$$\limsup_{r \to \infty} \frac{T(r,\pi_n(f))}{T(r,f)} \leq nt\,(1 + q - q\ \Theta(\infty)) \qquad \ldots (3.4)$$

where $\pi_n(f)$ is a homogeneous differential polynomial of degree n, not containing f.

<u>Proof</u> : $T(r,\pi_n(f)) = m(r, \pi_n(f)) + N(r, \pi_n(f))$

$$\leqslant m(r, \frac{\pi_n(f)}{f^n}) + m(r,f^n) + N(r,\pi_n(f))$$

$$= m(r, \frac{\Sigma a(z)(f')^{l_1} \ldots (f^{(k)})^{l_k}}{f^{l_1 + \ldots + l_k}}) + m(r,f^n)$$

$$+ N(r, \pi_n(f))$$

$$= m(r, \Sigma a(z) (\frac{f'}{f})^{l_1} \ldots (\frac{f^{(k)}}{f})^{l_k})$$

$$+ m(r,f^n) + N(r, \pi_n(f))$$

But $m(r, \Sigma a(z)(\frac{f'}{f})^{l_1} \ldots (\frac{f^{(k)}}{f})^{l_k}) = S(r,f)$.

and so

$$T(r, \pi_n(f)) \leq m(r,f^n) + N(r, \pi_n(f)) + S(r,f).$$

Now without any loss of generality let $\pi_n(f)$ consist of t terms say $\pi_n(f) = \phi_1(f) + \phi_2(f) + \ldots + \phi_t(f)$ where each $\phi_i(f)$ $(1 \leq i \leq t)$ is a monomial in the derivatives of f but not containing f and of degree n.

And so

$$T(r, \pi_n(f)) \leq m(r,f^n) + N(r, \phi_1(f)) + \ldots$$

$$\ldots + N(r, \phi_t(f)) + S(r,f)$$

$$\leq m(r,f^n) + (n\,N(r,f) + k_1 n \bar{\bar{N}}(r,f))$$

$$+ (nN(r,f) + k_2 n \bar{\bar{N}}(r,f)) + \ldots\ldots$$

$$+ (n\,N(r,f) + k_t n \bar{\bar{N}}(r,f)) + S(r,f)$$

where k_i is the highest derivatives in the corresponding monomials $\phi_i(f)$ $(1 \leq i \leq t)$. Let q be the highest derivative occurring in the homogeneous differential polynomial (so that $k_1\ k_2 \ldots k_t \leq q$)

Therefore

$$T(r,\pi_n(f)) \leq m(r,f^n) + t\,n\,N(r,f) + tqn \bar{\bar{N}}(r,f)$$

$$\leq nm(r,f) + tnN(r,f) + tqn \bar{\bar{N}}(r,f)$$

$$\leq tn\,m(r,f) + tn\,N(r,f) + tqn \bar{\bar{N}}(r,f)$$

$$= tn\,T(r,f) + tqn\bar{\bar{N}}(r,f)$$

Now dividing by $T(r,f)$ and taking limit superior of the above inequality we get

$$\limsup_{r \to \infty} \frac{T(r,\pi_n(f))}{T(r,f)} \leq t\,n + t\,q\,n\,(1 - \Theta(\infty))$$

$$= tn + tqn - tqn\ \Theta(\infty)$$

Therefore

$$\limsup_{r \to \infty} \frac{T(r,\pi_n(f)}{T(r,f)} \leq n\,t(1 + q - q\ \Theta(\infty))$$

<u>Proof of theorem 3.1</u> :

From lemma (3.1)

$$n \sum_{i=1}^{q} m(r,\ a_i, f) \leq m\left(r, \frac{1}{\pi_n(f)}\right) + S(r,f)$$

Dividing by T(r,f) and taking limit inferior on both sides as $r \to \infty$ it easily follows that

$$n \sum_{i=1}^{q} \delta(a_i) \leq \limsup_{r \to \infty} \frac{m\left(r, \frac{1}{\pi_n(f)}\right)}{T(r,\pi_n(f))} \ \liminf_{r \to \infty} \frac{T(r,\ \pi_n(f))}{T(r,f)}$$

$$= \Delta(\pi_n(f),\ 0)\ \liminf_{r \to \infty} \frac{T(r,\ \pi_n(f)}{T(r,f)}\ .$$

Since above inequality is true for $q > 2$ letting $q \to \infty$ we get

$$n \sum_{i=1}^{\infty} \delta\ (a_i) \leq \Delta(\pi_n(f),\ 0)\ \liminf_{r \to \infty} \frac{T(r,\ \pi_n(f)}{T(r,f)}$$

This proves (3.1)

For the proof of (3.2) we add $n \sum_{i=1}^{q} N(r,a_i,\ f)$ to both sides of result of Lemma 3.1 to get

$$nqT(r,f) < T(r, \pi_n(f)) - N(r, \frac{1}{\pi_n(f)}) + n \sum_{i=1}^{q} N(r,a_i,f)$$

$$+ S(r,f)$$

Dividing by T(r,f) to both the sides and taking limit superior as $r \to \infty$ we obtain

$$nq < \limsup_{r \to \infty} \frac{T(r, \pi_n(f)}{T(r,f)} - \limsup_{r \to \infty} \frac{N(r, \frac{1}{\pi_n(f)})}{T(r,\pi_n(f)}$$

$$- \liminf_{r \to \infty} \frac{T(r,\pi_n(f)}{T(r,f)} + n \sum_{i=1}^{q} (1-\delta/a_i).$$

Since $\Delta(\pi_n(f), 0) > 0$ by (3.1) multiplying by $\Delta(\pi_n(f), 0)$ and using (3.1) and lemma 3.2 we obtain

$$\Delta(\pi_n (f),0)\, nq < \Delta(\pi_n(f),0) \limsup_{r \to \infty} \frac{T(r,\pi_n(f))}{T(r,f)}$$

$$- \Delta(\pi_n(f),0) \limsup_{r \to \infty} \frac{N(r, \frac{1}{\pi_n(f)})}{T(r,\pi_n(f)} \cdot$$

$$\cdot \liminf_{r \to \infty} \frac{T(r,\pi_n(f)}{T(r,f)} + (n \sum_{i=1}^{q} (1-\delta(a_i)))$$

$$\Delta (\pi_n (f), 0).$$

And so

$$\Delta(\pi_n(f), 0)nq \leqslant \Delta(\pi_n(f),0)\ nt\ (1 + q-q\ \Theta(\infty))$$

$$- (1 - \delta(\pi_n(f), 0))\ n \sum_{i=1}^{\infty} \delta(a_i) +$$

$$+ (n \sum_{i=1}^{q} 1 - \delta(a_i))\ \Delta(\pi_n(f), 0).$$

Thus

$$\Delta(\pi_n(f), 0)\ n\ q + (1 - \delta(\pi_n(f), 0))\ n \sum_{i=1}^{q} \delta(a_i)$$

$$\leqslant nt\ (1 + q - q\ \Theta(\infty))\ \Delta(\pi_n(f), 0)$$

$$+(n \sum_{i=1}^{\infty} (1 - \delta(a_i)))\ \Delta(\pi_n(f), 0)$$

from which it follows that

$$\Delta(\pi_n(f), 0)\ nq + (1 - \delta(\pi_n(f), 0))\ n \sum_{i=1}^{q} \delta(a_i)$$

$$\leqslant nt\ (1+q - q\ \Theta(\infty))\Delta(\pi_n(f), 0) +$$

$$+ nq\Delta(\pi_n(f), 0) - (n \sum_{i=1}^{\infty} \delta(a_i))\ \Delta(\pi_n(f),0)$$

Rearranging and letting $q \to \infty$ we get

$$(1 + \Delta(\pi_n(f), 0) - \delta(\pi_n(f), 0))\ n \sum_{i=1}^{\infty} \delta(a_i)$$

$$\leqslant n\ t(1 + q -q\ \Theta(\infty))\ (\Delta(\pi_n(f), 0))$$

which proves the theorem.

<u>Theorem 3.2</u> : If $f(z)$ is a meromorphic function of finite order with $\Theta(0,f) = \Theta(\infty, f) = 1$ and $\pi_n(f)$ is a

homogeneous differential polynomial in f of degree n which does not reduce to a constant then

$$T(r, \pi_n(f)) \sim n\,T(r,f) \quad \text{as } r \to \infty$$

Proof : We have $\Theta(0,f) = 1 = \Theta(\infty, f)$

therefore $$\Theta(0,f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, \frac{1}{f})}{T(r,f)}$$

$$= 1$$

$$\Theta(\infty, f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} = 1$$

Therefore $\bar{N}(r, \frac{1}{f}) = S(r,f)$

and $\bar{N}(r,f) = S(r,f)$

and we know that from theorem 1 of [3]

$$n(1 - m\alpha) \leq \liminf_{r\to\infty} \frac{T(r,\pi_n)}{T(r,f)} \leq \limsup_{r\to\infty} \frac{T(r,\pi_n)}{T(r,f)} \leq n(1+m\alpha)$$

where $$\alpha = \limsup_{r\to\infty} \frac{\bar{N}(r,f) + \bar{N}(r, \frac{1}{f})}{T(r,f)}$$

Therefore $\alpha = 0$

Therefore from above inequality we get

$$n \leq \liminf_{r\to\infty} \frac{T(r,\pi_n(f))}{T(r,f)} \leq \limsup_{r\to\infty} \frac{T(r,\pi_n(f))}{T(r,f)} \leq n$$

Thus $\lim_{r \to \infty} \frac{T(r, \pi_n(f))}{T(r,f)} = n,$

Consequently $T(r, \pi_n(f)) \sim n\, T(r,f)$ as $r \to \infty$

This completes the proof.

Before giving an application of theorem 3.2 we shall need the following definition :

By a homogeneous differential polynomial of degree n not containing f we shall mean a finite sum of the form

$$\pi_n(f) = \sum a(z)\, (f')^{l_1}\, (f'')^{l_2} \ldots (f^{(k)})^{l_k}$$

where $l_1 + \ldots + l_k = n$ and $a(z)$ is meromorphic function satisfying $T(r, a(z)) = S(r,f)$.

We now give the application.

<u>Theorem 3.3</u> : If $f(z)$ is an entire function of finite order with $\sum_{\alpha \neq \infty} \delta(\alpha, f) = 1$ then

$$\delta(0, \pi_n(f)) = 1.$$

where $\pi_n(f)$ is a non-zero homogeneous differential polynomial of degree n not containing f.

<u>Proof</u> : set $F(z) = \sum_{\nu=1}^{q} \frac{1}{(f(z) - \alpha_\nu)^n}$

then as in the thesis of A.A. Mudalgi [8]

$$n\sum_{\upsilon=1}^{q} m\left(r, \frac{1}{f-\alpha_\upsilon}\right) \leqslant m(r,F) + o(1)$$

$$= m\left(r, \frac{F\pi_n(f)}{\pi_n(f)}\right) + o(1)$$

$$\leqslant m\left(r, \sum_{\upsilon=1}^{q} \frac{\pi_n(f)}{(f-\alpha_\upsilon)^n}\right) + m\left(r, \frac{1}{\pi_n(f)}\right)$$

$$+ o(1)$$

$$= m\left(r, \sum_{\upsilon=1}^{q} a(z) \frac{(f')^{l_1}(f'')^{l_2}\dots(f^{(k)})^{l_k}}{(f-\alpha_\upsilon)^n}\right) +$$

$$+ m\left(r, \frac{1}{\pi_n(f)}\right) + o(1)$$

$$\leqslant \sum_{\upsilon=1}^{q} m(r,a(z)) + m\left(r, \left(\frac{f'}{f-\alpha_\upsilon}\right)^{l_1}\right) + \dots$$

$$\dots + m\left(r, \frac{1}{\pi_n(f)}\right) + o(1)$$

Using Milloux's results and the fact that $m(r,a(z) \leqslant T(r,a(z)) =$

$= S(r,f)$

We obtain

$$n\sum_{\upsilon=1}^{q} m\left(r, \frac{1}{f-\alpha_\upsilon}\right) \leqslant m\left(r \frac{1}{\pi_n(f)}\right) + S(r,f)$$

Dividing by $T(r,f)$ and taking limit inferior we get

$$\liminf_{r \to \infty} \frac{n \sum_{\nu=1}^{q} m\left(r, \frac{1}{f-\alpha_\nu}\right)}{T(r,f)} \leqslant \liminf_{r \to \infty} \frac{m\left(r, \frac{1}{\pi_n(f)}\right)}{T(r,\pi_n(f))} \cdot$$

$$\cdot \limsup_{r \to \infty} \frac{T(r, \pi_n(f))}{T(r,f)}$$

Therefore $n \sum_{\nu=1}^{q} \delta(\alpha_\nu, f) \leq n\, \delta(0, \pi_n(f))$

Consequently $\sum_{\nu=1}^{q} \delta(\alpha_\nu, f) \leqslant \delta(0, \pi_n(f))$

Making $q \to \infty$ we obtain

$$\sum_{\nu=1}^{\infty} \delta(\alpha_\nu, f) \leq \delta(0, \pi_n(f))$$

Since the set $\{\delta(\alpha,f) / \delta(\alpha, f) > 0\}$ is countable it follows that

$$\sum_{\nu=1}^{\infty} \delta(\alpha_\nu, f) = \sum_{\alpha \neq \infty} \delta(\alpha, f)$$

And so

$$\sum_{\alpha \neq \infty} \delta(\alpha,f) \leq \delta(0, \pi_n(f))$$

But by hypothesis $\sum_{\alpha \neq \infty} \delta(\alpha, f) = 1$

And so

$$1 \leqslant \delta(0, \pi_n f) \leqslant 1$$

Thus $\delta(0, \pi_n(f)) = 1$.

Our next theorem finds relation between deficient values of entire functions with that of its derivative. Thus we prove :

<u>Theorem 3.4</u> : If f(z) is an entire function of finite order then

$$\sum_{a \neq \infty} \delta(a, f) \leq \delta(0, f^{(k)})$$

<u>Proof</u> : Set $F(z) = \sum_{\nu=1}^{q} \frac{1}{f(z)-a_\nu}$ then by [7, 33]

$$\sum_{\nu=1}^{q} m\left(r, \frac{1}{f(z)-a_\nu}\right) \leq m(r, F(z)) + O(1)$$

$$= m\left(r, \frac{Ff'}{f'}\right) + O(1)$$

$$\leq m\left(r, \sum \frac{f'}{f-a_\nu}\right) + m\left(r, \frac{1}{f'}\right) + O(1)$$

$$= m\left(r, \frac{1}{f'}\right) + o(T(r,f)$$

$$= m\left(r, \frac{f^{(k)}}{f' f^{(k)}}\right) + o(T(r,f))$$

$$\leq m\left(r, \frac{f^{(k)}}{f'}\right) + m\left(r, \frac{1}{f^{(k)}}\right) + o(T(r,f))$$

$$= m\left(r, \frac{1}{f^{(k)}}\right) + S(r,f') + o(T(r,f))$$

by Milloux's theorem.

Also since $S(r,f') = S(r,f)$ it follows that

$$\sum_{\nu=1}^{q} m(r, \frac{1}{f(z)-a_\nu}) \leqslant m(r, \frac{1}{f^{(k)}}) + S(r,f) \qquad \ldots (3.5)$$

Now dividing by $T(r,f^{(k)})$ and taking limit inferior (3.5) becomes

$$\liminf_{r \to \infty} \frac{\sum_{\nu=1}^{q} m(r, \frac{1}{f(z)-a_\nu})}{T(r,f)} \leqslant \liminf_{f \to \infty} \frac{m(r, \frac{1}{f^{(k)}})}{T(r,f^{(k)})}$$

Consequently

$$\sum_{\nu=1}^{q} \delta(a_\nu, f) \leqslant \delta(0, f^{(k)}) .$$

Making $q \to \infty$ we get

$$\sum_{\nu=1}^{\infty} \delta(a_\nu, f) \leqslant \delta(0, f^{(k)}).$$

since the set of values of a for which $\delta(a,f) > 0$ is countable, it now follows that

$\sum_{\nu \neq \infty} \delta(a_\nu, f) \leqslant \delta(0, f^{(k)})$. This prove the theorem.

Remark : Putting $k = 1$ we obtain Theorem 4.6 of W.K.Hayman [7, 104] . Another result dealing with the Nevanlinna characteristic of f and the Nevanlinna characteristic of $f^{(k)}$ is the following.

Theorem 3.5 : If f(z) is a meromorphic function of finite order, then

$$\liminf_{r \to \infty} \frac{T(r,f)}{T(r,f^{(k)})} \geqslant \frac{1}{k+1}$$

<u>Proof</u> : If f(z) is a meromorphic function of finite order we have

$$\liminf_{r \to \infty} \frac{T(r,f)}{T(r,f')} \geqslant \frac{1}{2}$$

Clearly $N(r, f^{(k)}) = N(r,f) + k\,\bar{N}(r,f)$

$$\leqslant N(r,f) + k\,N(r,f)$$

Thus

$$N(r,f^{(k)} \leqslant (k+1)\,N(r,f) \qquad \ldots (3.6)$$

Now consider $m(r,f^{(k)})$.

$$m(r, f^{(k)}) = m\left(r, \frac{f^{(k)}}{f} . f\right)$$

$$\leqslant m\left(r, \frac{f^{(k)}}{f}\right) + m(r,f)$$

$$= m(r, f) + S(r,f).$$

And so

$$m(r,f^{(k)}) \leqslant (k+1)\,m(r,f) + S(r,f) \qquad \ldots (3.7)$$

Therefore combining (3.6) and (3.7) we get

$$m(r, f^{(k)}) + N(r, f^{(k)}) \leqslant (k+1)\,T(r,f) + S(r,f)$$

Thus

$$T(r, f^{(k)}) \leqslant (k+1)\,T(r,f) + S(r,f)$$

which in turn yields

$$\liminf_{r \to \infty} \frac{T(r,f)}{T(r, f^{(k)})} \geq \frac{1}{k+1} \quad \text{as required.}$$

In the other direction we have

<u>Theorem 3.6</u> : Let f(z) be a meromorphic function of order ρ , $(0 < \rho < \infty)$ Then

$$\liminf_{r \to \infty} \frac{T(r,f)}{T(r,f^{(k)})} < \infty$$

<u>Proof</u> : It is known (See for e.g. [17]) that

$$T(r,g) < A_g \, T(kr, g')$$

where $k > 1$ and $r > 0$

Therefore $T(r,f^{(k-1)}) < A_{f^{(k-1)}} \, T(kr, f^{(k)}$

Also

$$T(r, f^{(k-2)}) < A_{k-2} \, T(kr, f^{(k-1)}$$

$$< A_{k-2} \, (A_{k-1} \; T(k^2 r, f^{(k)})$$

$$= B_{k-2} \, T(k^2 r, f^{(k)})$$

where B_{k-2} is a constant depending on $f^{(k-1)}(0)$ $f^{(k-2)}(0)$.

In general

$$T(r, f^{(k-p)}) < B_{k-p} \, T(k^p r, f^{(k)})$$

and

$$T(r, f') < B_1 \, T(k^{k-1} r, f^{(k)}),$$

$$T(r,f) < B_0 T(k^k r, f^{(k)})$$

$$= B_0 T(\alpha r, f^{(k)})$$

where $\alpha = (k)^k$

Thus

$$T(r,f) \leqslant B_0 (\alpha r)^{\rho(\alpha r)}$$

$$\sim B_0 \alpha^{\rho} r^{\rho(r)}$$

$$= B_0 \alpha^{\rho} T(r, f^{(k)})$$

using proximate order for a sequence.

Therefore $\liminf_{r \to \infty} \dfrac{T(r,f)}{T(r,f^{(k)})} < \infty.$

This proves the theorem.

We next prove

<u>Theorem 3.7</u> : Let $f(z)$ be a non constant meromorphic function, then

$$m(r, \infty) + n \sum_{\nu=1}^{q} m(r, p_\nu) \leqslant (n+1)T(r,f) - N_1(r) + S(r,f)$$

where $P(z)$ is a polynomial of degree n,

$$N_1(r) = (n+1) N(r,f) + N\left(r, \frac{1}{\emptyset_n}\right) - N(r, \emptyset_n)$$

$$S(r) = m\left(r, \frac{\emptyset_n}{f^n}\right) + m\left(r, \sum_{\nu=1}^{q} \frac{\emptyset_n}{(f - p_\nu(z))^n}\right)$$

$$+ ng \log^+ \frac{3q}{\delta} + n \log 2 + \log \frac{1}{|f'(0)|}$$

where ϕ_n is a monomial of degree n in derivative of f but not containing f.

<u>Proof</u> : The construction of $|P_i - P_j| \geqslant \delta$ given below is as in [11] which we give here for sake of completeness. Let q be any positive integer > 2 and consider any q Polynomials P_i, $1 \leqslant i \leqslant q$ and let $P_i \in B(1)$ where B(1) be the set of all polynomials in z of degrees at most $1 \geqslant 0$. Let $A = (a_1, a_2 \dots a_N)$ be the finite set of coefficients associated with these q polynomials, the a_i's being distinct. Then for $i \neq j$ $P_i - P_j$ is a polynomial whose highest degree term is $(a_\lambda - q_\mu) z^k$ or $a_\lambda z^k$. Here $\lambda \neq \mu$ and $0 \leqslant k \leqslant 1$, $a_\lambda, a_\mu \in A$. Therefore $|P_i - P_j| \sim |a_\lambda - a_\mu| r^k$ or $|P_i - P_j| \sim |a_\lambda| r^k$ as $r \to \infty$. Let $\delta = \text{Min} \{|a_\lambda|, |a_\lambda - a_\mu|\}$ Then for $1 \leqslant i < j \leqslant q$, we have $|P_i - P_j| \geqslant \delta$ for $r \geqslant r_o$ uniformly in z

$$\text{Set } F(z) = \sum_{\nu=1}^{q} \frac{1}{(f(z) - P_\nu(z))^n}$$

Suppose for some ν, $|f(z) - P_\nu(z)| < \frac{\delta}{3q}$ ... (3.8)

Then for $\mu \neq \nu$

$$\begin{aligned} |f(z) - P_\mu| &\geqslant |P_\mu - P_\nu| - |P_\nu - f(z)| \\ &\geqslant \delta - \frac{\delta}{3\,q} \\ &\geqslant \frac{2}{3}\,\delta. \end{aligned}$$

Therefore for $\mu \neq \nu$

$$\frac{1}{|f(z) - P_\mu|} \leq \frac{3}{2\delta} \leq \frac{1}{2q \;\; |f(z) - P_\nu(z)|} \qquad \ldots (3.11)$$

Consider

$$|F(z)| \geq \frac{1}{(f(z) - P_\nu)^n} - \sum_{\mu \neq \nu} \frac{1}{(f(z)-P_\mu)^n}$$

$$= \frac{1}{|f(z)-P_\nu|^n} - \sum_{\mu \neq \nu} \frac{1}{2^n q^n \, |f(z) - P_\nu|^n} \quad \text{using(3.9)}$$

$$= \frac{1}{|f(z) - P_\nu|^n} \left\{ 1 - \frac{q-1}{2^n \; q^n} \right\}$$

$$\geq \frac{1}{|f(z) - P_\nu|^n} \cdot \frac{1}{2^n}$$

since $1 \geq \frac{1}{2^n} + \frac{1}{2^n}$ for $n \geq 1$ and

$$1 - \frac{q-1}{2^n \; q^n} \geq 1 - \frac{q^n}{2^n \; q^n} = 1 - \frac{1}{2^n}$$

which gives $1 - \frac{q-1}{2^n \; q^n} \geq \frac{1}{2^n}$.

Hence

$$\log^+ |F(z)| \geq \log^+ \frac{1}{|f(z)-P_\nu|^n} - n \log 2$$

$$= \sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z)-P_\mu|^n} - \sum_{\mu \neq \nu} \log^+ \frac{1}{|f(z)-P_\mu|^n} - n \log 2 \qquad \ldots (3.10)$$

But since $\mu \neq \nu$,

$$|f - P_\mu| \geq |P_\mu - P_\nu| - |f - P_\nu|$$

$$\geq \delta - \delta/3q$$

$$= \frac{(3q-1)\delta}{3q}$$

$$\geq \frac{\delta}{3q}$$

we have

$$\log^+ \frac{1}{|f - P_\mu|^n} \leq \log^+ \left(\frac{3q}{\delta}\right)^n .$$

Therefore

$$\sum_{\mu \neq \nu} \log^+ \frac{1}{|f - P_\mu|^n} \leq (q-1) \log^+ \left(\frac{3q}{\delta}\right)^n$$

$$\leq nq \log^+ \left(\frac{3q}{\delta}\right)$$

Hence from (3.10) we have

$$\log^+ |F(z)| \geq \sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z) - P_\mu|^n} - nq \log^+ \frac{3q}{\delta} - n \log 2 \qquad \ldots (3.11)$$

Next we consider the case when

$$|f(z) - P_\nu| \geq \frac{\delta}{3q} \quad \text{for all } \nu .$$

Then we have

$$\log^+ \frac{1}{|f(z) - P_\nu|^n} \leq \log^+ \left(\frac{3q}{\delta}\right)^n$$

and so

$$\sum_{\nu=1}^{q} \log^{+} \frac{1}{|f(z)-P_{\nu}|^{n}} \leqslant nq \log^{+} \frac{3q}{\delta}$$

This shows that R.H.S. of (3.11) is negative. But L.H.S. of (3.11) is non-negative and therefore (3.11) is trivially true in this case and it is true in all cases.

Multiplying (3.11) both the sides by $1/2\pi$ and integrating over $[0, 2\pi]$ we get

$$m(r,F) \geqslant \sum_{\nu=1}^{q} m\left(r, \frac{1}{(f-P_{\nu})^{n}}\right) - nq \log^{+} \frac{3q}{\delta} - n \log 2.$$

And so

$$m(r,F) \geqslant n \sum_{\nu=1}^{q} m(r,P_{\nu}) - n\, q \log^{+} \frac{3q}{\delta} - n \log 2. \qquad \ldots (3.12)$$

Thus

$$m(r,F) = m\left(r, \frac{1}{f^{n}} \frac{f^{n}}{\phi_{n}} \phi_{n} F\right), \text{ which yields}$$

$$m(r,F) \leqslant m\left(r, \frac{1}{f^{n}}\right) + m\left(r, \frac{f^{n}}{\phi_{n}}\right) + m(r, \phi_{n} F) \qquad \ldots (3.13)$$

But from Nevanlinna's first fundamental theorem we have

$$T(r,f) = T\left(r, \frac{1}{f}\right) + \log |f(0)|$$

and so

$$m\left(r, \frac{f^{n}}{\phi_{n}}\right) = m\left(r, \frac{\phi_{n}}{f^{n}}\right) + N\left(r, \frac{\phi_{n}}{f^{n}}\right) - N\left(r, \frac{f^{n}}{\phi_{n}}\right) + \log \left| \frac{f^{n}(0)}{\phi_{n}(0)} \right|$$

and

$$m(r, \frac{1}{f^n}) = T(r,f^n) - N(r, \frac{1}{f^n}) + \log \frac{1}{|f^n(0)|}$$

Therefore (3.13) will finally yield

$$m(r,F) \leq nT(r,f) - n\,N(r, \frac{1}{f}) + n \log \frac{1}{|f(0)|}$$

$$+ m(r, \frac{\phi_n}{f^n}) + N(r, \frac{\phi_n}{f^n}) - N(r, \frac{f^n}{\phi_n})$$

$$+ m(r, \phi_n F) + \log \left| \frac{f^n(0)}{\phi_n(0)} \right|$$

Combining this inequality with (3.12) gives

$$n \sum_{\nu=1}^{q} m(r, P_\nu) - nq \log^+ \frac{3q}{\delta} - n \log 2 \leq m(r,F)$$

$$\leq n\,T(r,f) - nN(r, \frac{1}{f}) + n \log \frac{1}{|f(0)|} + m(r, \frac{\phi_n}{f^n})$$

$$+ N(r, \frac{\phi_n}{f^n}) - N(r, \frac{f^n}{\phi_n}) + m(r, \phi_n F) + \log \left| \frac{f^n(0)}{\phi_n(0)} \right| .$$

Therefore

$$n \sum_{\nu=1}^{q} m(r, P_\nu) + m(r,f) \leq m(r,f) + m(r,F) + nq \log^+ \frac{3q}{\delta} +$$

$$+ n \log 2$$

$$\leq n\,T(r,f) - n\,N(r, \frac{1}{f}) + N(r, \frac{\phi_n}{f^n})$$

$$- N(r, \frac{f^n}{\phi_n}) + m(r, \frac{\phi_n}{f^n}) + m(r, \phi_n F) +$$

$$+ \log \frac{1}{|\phi_n(0)|} + T(r,f) - N(r,f) + nq\log^+ \frac{3q}{\delta} + $$

$$+ n \log 2$$

Thus

$$n \sum_{\nu=1}^{q} m(r, P_\nu) + m(r,f) \leq nT(r,f) - n N(r, \frac{1}{f}) + N(r,\phi_n)$$

$$+ N(r, \frac{1}{f^n}) - N(r,f^n) - N(r, \frac{1}{\phi_n}) + \log \left| \frac{1}{\phi_n(0)} \right|$$

$$+ T(r,f) - N(r,f) + nq \log^+ \frac{3q}{\delta} + n \log 2$$

$$+ m(r, \frac{\phi_n}{f^n}) + m(r, \phi_n F)$$

$$\leq 2(n+1) T(r,f) + N(r, \phi_n) - N(r, \frac{1}{\phi_n})$$

$$- n + 1 N(r,f) + S(r,f).$$

And so

$$n \sum_{\nu=1}^{q} m(r, P_\nu) + m(r,f) \leq (n+1)T(r,f) - \{(n+1)N(r,f) -$$

$$- N(r, \phi_n) + N(r, \frac{1}{\phi_n}) \} + S(r).$$

which finally yields

$$m(r,f) + n \sum_{\nu=1}^{q} m(r, P_\nu) \leq (n+1)T(r,f) - N_1(r) + S(r)$$

where $N_1(r) = (n+1) N(r,f) + N(r, \frac{1}{\phi_n}) - N(r, \phi_n)$

and $S(r) = m(r, \frac{\phi_n}{f^n}) + m(r, \sum_{\nu=1}^{q} \frac{\phi_n}{(f - P_\nu(z))^n}) +$

$$+ ng \log^{+} \frac{3g}{\delta} + n \log 2 + \log \frac{1}{|\phi_n(0)|}$$

Remark : If $P_{\nu}(z)$ is constant and n=1 then we get theorem 2.1 of Hayman [7, 31] .

We now give proofs of two theorems stated without proof by R. Parthasarathy [12]

Theorem 3.8 : Let f(z) be an entire function of order ρ $(0 < \rho < \infty)$ for which

$$\limsup_{r \to \infty} \frac{\log M(r,f)}{r^{\rho} L(r)} = \alpha . \quad \ldots (3.14)$$

and $$\limsup_{r \to \infty} \frac{N(r, 1/f)}{r^{\rho} L(r)} = \beta. \quad \ldots (3.15)$$

Let $\psi(z)$ be a homogeneous differential polynomial of degree $p \geq 1$ with all the coefficients a(z) entire. Then for every complex number w except possibly for w = 0

$$\limsup_{r \to \infty} \frac{\bar{N}(r, w, \psi)}{r^{\rho} L(r)} \geq p \left(\frac{\alpha}{h(\rho)} - \beta \right) \quad \ldots (3.16)$$

$$\limsup_{r \to \infty} \frac{\bar{n}(r, w, \psi)}{r^{\rho} L(r)} \geq \rho \, P \left(\frac{\alpha}{h(\rho)} - \beta \right) \quad \ldots (3.17)$$

and $\delta(0, f) + \Theta(w, \psi) \leq 1$ ... (3.18)

where $h(\rho) = \left\{\rho + (1 + \rho^2)^{1/2}\right\} \left\{\frac{1+(1+\rho^2)^{1/2}}{\rho}\right\}$, $\rho > 0$

Proof : Let $\epsilon > 0$ be given. Then by (3.15) we have

$$N(r, \frac{1}{f}) < (\beta + \epsilon)\, r^{\rho} L(r) \text{ for all } r \geqslant r_0. \text{ Also}$$

by (3.14)

$$\log M(r,f) > (r - \epsilon) r^{\rho} L(r) \quad \text{for a sequence of } r \to \infty.$$

Using Lemma 3.1 and the fact that f(z) is entire, we obtain

$$P \{1 + o(1)\} \; T(r,f) < PN(r, \frac{1}{f}) + \bar{N}(r, w, \psi)$$

$$< P(\beta + \epsilon) r^{\rho} L(r) + \bar{N}(r, w, \psi)$$

for all $r \geqslant r_0$.

Also for $\lambda > 1$

$$T(r,f) \geqslant \frac{\lambda - 1}{\lambda + 1} \log M(r/\lambda, f).$$

Thus we have for a sequence of $r \to \infty$

$$T(r,f) > (\alpha - \epsilon) \frac{\lambda - 1}{\lambda + 1} \left(\frac{r}{\lambda} \right) L\left(\frac{r}{\lambda} \right)$$

and

$$\frac{\bar{N}(r, w, \psi)}{r^{\rho} L(r)} > \frac{P}{\lambda^{\rho}} \{1 + o(1)\} (\alpha - \epsilon) \left(\frac{\lambda - 1}{\lambda + 1} \right) \frac{L(\frac{r}{\lambda})}{L(r)} - P(\beta + \epsilon).$$

Since $\frac{L(r/\lambda)}{L(r)} \to 1$ as $r \to \infty$ we get

$$\limsup_{r \to \infty} \frac{\bar{N}(r, w, \psi)}{r^{\rho} L(r)} \geqslant \frac{1}{\lambda^{\rho}} \left(\frac{\lambda - 1}{\lambda + 1} \right) P\alpha - P\beta.$$

The maximum value of $\frac{1}{\lambda^{\rho}} \left(\frac{\lambda - 1}{\lambda + 1} \right)$ is easily seen to be

$\dfrac{1}{h(\rho)}$ and hence

$$\limsup_{r\to\infty} \frac{\bar{N}(r, w, \psi)}{r^{\rho} L(r)} \geq P\left(\frac{\alpha}{h(\rho)} - \beta\right).$$

from Lemma 1 of [12] we have

$$\limsup_{r\to\infty} \frac{\bar{n}(r, w, \psi)}{r^{\rho} L(r)} \geq \limsup_{r\to\infty} \frac{\bar{N}(r, w, \psi)}{r^{\rho} L(r)}$$

Therefore

$$\limsup_{r\to\infty} \frac{\bar{n}(r, w, \psi)}{r^{\rho} L(r)} \geq \rho P\left(\frac{\alpha}{h(\rho)} - \beta\right).$$

Again by (3.14) we have for $r \geq r_0$

$$P\{1 + o(1)\} < P\frac{N(r, 1/f)}{T(r,f)} + \frac{\bar{N}(r, w, \psi)}{T(r,f)}.$$

Since $\psi(z)$ is entire $N(r, \psi/w) = 0$ and by lemma 2 of [12]

$$m(r, \psi/w) \leq m\left(r, \frac{\psi}{wf^{p}}\right) + m(r, f^{p})$$

$$= S(r,f) + pm(r,f)$$

Hence by $T(r, \psi/w) < \{P + o(1)\}\ T(r,f)$

Thus we have for all $r \geq r_0$

$$P\{1 + o(1)\} < P\frac{N(r, 1/f)}{T(r, f)} + P\frac{\bar{N}(r, w, \psi)}{T(r, \psi)}(1+o(1))$$

Letting $r \to \infty$ we get

$$P \leq P \limsup_{r \to \infty} \frac{N(r, 1/f)}{T(r, f)} + P \limsup_{r \to \infty} \frac{\bar{N}(r, w, \psi)}{T(r, \psi)} .$$

And so

$P \leq P\,(1 - \delta(0, f)) + P\,(1 - \Theta(w, \psi))$, which on simplification yields

$$\delta(0, f) + \Theta(w, \psi) \leq 1.$$

This completes the proof.

<u>Theorem 3.9</u> : Let f(z) be a meromorphic function of order ρ $(0 < \rho < \infty)$ for which

$$\limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho} L(r)} = a \qquad \ldots (3.19)$$

$$\text{and } \limsup_{r \to \infty} \frac{\frac{1}{P}\bar{N}(r, f) + N(r, 1/f)}{r^{\rho} L(r)} = b. \qquad \ldots (3.20)$$

Let $\psi(z)$ be a homogeneous differential polynomial of degree P (≥ 1) in f. Then for every complex number w, except possibly for w = 0

$$\limsup_{r \to \infty} \frac{\bar{N}(r, w, \psi)}{r^{\rho} L(r)} \geq P\,(a - b)$$

$$\limsup_{r \to \infty} \frac{\bar{n}(r, w, \psi)}{r^{\rho} L(r)} \geq P\,\rho\,(a-b)$$

and $\delta(0, f) + \frac{1}{P}\,\Theta(\infty, f) + (k+1)\,\Theta(w, \psi) \leq k+1+\frac{1}{P}$

where k is the order of the highest derivative occurring in ψ (z).

Proof : We have from (3.20), given $\varepsilon > 0$ with $0 < |w| < \infty$

$$\frac{1}{p}\ \bar{N}(r,f) + N\left(r, \frac{1}{f}\right) < (b+\varepsilon)\ r^{\rho} L(r)$$

for all $r \geq r_0$.

Theorem 3.10 : Let f(z) be a meromorphic function and $\pi_n(f)$ be a homogeneous differential polynomial of degree n.

Let $\dfrac{T(r, \pi_n(f)}{T(r,f)} \to \alpha$ as $r \to \infty$ where $\alpha \geq n$ then

$$\Theta(\infty, f) \leq 1 + \frac{1}{m} - \frac{\alpha}{pmn}$$

where m is the highest derivative occurring in $\pi_n(f)$ and p is the number of terms in π_n (f).

Proof :

Let $\dfrac{T(r, \pi_n(f))}{T(r,f)} \to \alpha$ as $r \to \infty$ where $\alpha \geq n$.

Now $m(r, \pi_n(f)) = m\left(r, \dfrac{\pi_n(f)}{f^n}\right) + m(r, f^n)$ ... (3.23)

And so

$$m\left(r, \frac{\pi_n(f)}{f^n}\right) = m\left(r, \frac{\sum a(z)(f')^{l_1} \ldots (f^{(k)})^{l_k}}{f^{l_1+l_2 \ldots +l_k}}\right)$$

$$\leq \sum_{1}^{p} m(r, a(z)) + m\left(r, \left(\frac{f'}{f}\right)^{l_1}\right) + \ldots\ldots\ldots$$

$$\ldots\ m\left(r, \left(\frac{f^{(k)}}{f}\right)^{l_k}\right)$$

$$\leqslant \sum_{1}^{p} \; l_1 \, m(r, \frac{f'}{f}) + l_2 m(r, \frac{f''}{f}) + \ldots + l_k \, m(r, \frac{f^{(k)}}{f})$$

$$+ S(r,f)$$

Since $m(r, a(z)) \leqslant T(r,a(z) = S(r,f)$.

And so using Milloux's theorem it follows that

$$m(r, \frac{\pi_n(f)}{f^n}) \leqslant \sum_{1}^{p} \; l_1 S(r,f) + l_2 \, S(r,f) + \ldots + l_k \, S(r,f)$$

$$= np \, S(r,f)$$

$$= S(r,f).$$

Therefore

$$m(r, \pi_n \, (f)) \leqslant m(r,f^n) + S(r,f) \quad \text{by (3.23)}$$

$$= nm \, (r,f) + S(r,f)$$

Thus

$$m(r, \pi_n(f)) \leqslant Pnm \, (r,f) + S(r,f) \qquad \ldots (3.24)$$

Also

$$N(r,\pi_n(f)) = N(r, \sum a(z) \, (f')^{l_1} (f'')^{l_2} \ldots (f^{(k)})^{l_k}$$

$$= \sum_{1}^{p} N(r, a(z)) + l_1 \, N(r, f') + \ldots$$

$$\ldots + l_k \, N(r, f^{(k)})$$

$$N(r, \pi_n(f)) \leqslant \sum_{1}^{p} l_1 \, (N(r,f) + \bar{N} \, (r,f)) + l_2 \, (N(r,f) +$$

$$+ 2\bar{N} \, (r,f)) + l_k \, (N(r,f) + k \, \bar{N}(r,f)) + S(r,f)$$

Since $N(r, a(z)) \leqslant T(r,a(z)) = S(r,f)$

And so

$$N(r, \pi_n(f)) \leqslant \sum_{1}^{p} (l_1 + l_2 + \dots + l_k) N(r,f) + l_1 \bar{N}(r,f) +$$

$$+ 2l_2 \bar{N}(r,f) + \dots + kl_k \bar{N}(r,f) + S(r,f)$$

$$\leqslant \sum_{1}^{p} n N(r,f) + l_1 k\bar{N}(r,f) + l_2 k \bar{N}(r,f) + \dots$$

$$\dots + l_k k\bar{N}(r,f)$$

$$= \sum_{1}^{p} n N(r,f) + n k \bar{N}(r,f) + S(r,f)$$

$$= p n N(r,f) + pnm \bar{N}(r,f) + S(r,f)$$

where p denotes the number of terms in the homogeneous differential polynomial and m is the highest derivative of differential polynomial and $l_1 + l_2 + \dots + l_k = n$ and where n is the degree of differential polynomial.

Therefore

$$N(r, \pi_n(f)) \leqslant Pn N(r,f) + pnm \bar{N}(r,f)$$

That is

$$N(r, \pi_n(f)) \leqslant Pn (N(r,f) + m \bar{N}(r,f)). \qquad \dots (3.25)$$

Combining (3.24) and (3.25) we get

$$T(r, \pi_n(f)) \leqslant P n T(r,f) + P nm \bar{N}(r,f) + S(r,f).$$

Since $T(r, \pi_n(f)) / T(r,f) \to \alpha$ it follows that

$$\alpha T(r,f) \leqslant Pn T(r,f) + Pnm \bar{N}(r,f) + S(r,f)$$

$$(\alpha - Pn) T(r,f) \leqslant pnm \bar{N}(r,f) + S(r,f) \qquad \dots (3.26)$$

Dividing (3.26) by T(r,f) and taking limit superior we get

$$(\alpha - Pn) \leqslant \limsup_{r \to \infty} \frac{Pnm\, \bar{\bar{N}}\,(r,f)}{T\,(r,f)} + \limsup_{r \to \infty} \frac{S(r,f)}{T(r,f)}$$

Thus

$$(\alpha - Pn) \leqslant Pnm\,(1 - \Theta\,(\infty, f)\,)$$

Consequently

$$Pn\,m\ \Theta\,(\infty, f) \leqslant P\,n\,m + Pn - \alpha.$$

And so

$$\Theta\,(\infty,f) \leqslant 1 + \frac{1}{m} - \frac{\alpha}{Pmn}.$$

<u>Remark</u> : If m = 1, n = 1 and P = 1 then $\pi_n(f) = f'$ and so $\Theta\,(\infty, f) \leqslant 2 - \alpha$ which is theorem 3 of S.K.Singh and V.N.Kulkarni [17]

We finally end the Chapter by giving some application of Nevanlinna theory to differential equations.

<u>Theorem 3.11</u> : The differential equation

$$a_1\,(z)\,(f(z)\,)^n\,p\,(f) + \pi_{n-k}\,(f) = 0 \qquad \ldots (3.27)$$

where $a_1\,(z) \not\equiv 0$ and $1 \leq k \leq n$ has no transcendental meromorphic solution f(z) satisfying $N(r,f) = S(r,f)$ where $\pi_{n-k}\,(f)$ is a non-zero homogeneous differential polynomial of degree n - k and P(f) is any non-zero differential polynomial and a(z) are meromorphic functions satisfying $T(r,a\,(z)\,) = S(r,f)$.

<u>Proof</u> :Suppose there exists a transcendental meromorphic

function f satisfying (3.27) such that $N(r,f) = S(r,f)$ then

$$(f)^n P(f) = \frac{-\pi_{n-k}(f)}{a_1}$$

Hence by lemma (3.5)

$m(r, P(f)) = S(r,f)$.

Also $N(r, P(f)) = N(r, \Sigma a(z)(f)^{l_0} (f')^{l_1} .. (f^{(k)})^{l_k}$

$= S(r,f)$.

Therefore

$$T(r, P(f)) = S(r,f) \qquad \dots (3.28)$$

Also from (3.27) we get

$$(f)^n = \frac{-\pi_{n-k}(f)}{a_1 P(f)}$$

And hence by Nevanlinna's first fundamental theorem

$$n\,T(r,f) \leqslant T(r, \pi_{n-k}(f)) + T(r,P(f)) + T(r,a_1) + O(1)$$

$$= T(r, \pi_{n-k}(f)) + S(r,f) \quad \text{by (3.28)} \quad \dots$$

Also since $N(r,f) = S(r,f)$ we have

$$T(r, \pi_{n-k}(f) = m(r, \pi_{n-k}(f)) + S(r,f)$$

$$\leqslant m\left(r, \frac{\Sigma a(z)(f)^{l_0} (f')^{l_1} \dots (f^{(k)})^{l_k}}{f^{n-k}}\right)$$

$$+ m(r, f^{n-k}) + S(r,f)$$

$$= m\left(r, \ \Sigma a(z) \left(\frac{f'}{f}\right)^{l_1} \ldots \left(\frac{f^{(k)}}{f}\right)^{l_k}\right)$$

$$+ \ m\left(r, \ f^{n-k}\right) + S(r,f)$$

$$\leqslant \ m\left(r, \ a(z)\right) + l_1 \ m\left(r, \frac{f'}{f}\right) + \ldots.$$

$$+ \ l_k m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \ f^{n-k}\right) + S(r,f)$$

$$= m\left(r, \ f^{n-k}\right) + S\,(r,f)$$

$$T(r, \ \pi_{n-k}\,(f)\,) \leqslant (n-k)\ m(r,f) + S(r,f)$$

$$= (n-k)\ T(r,f) + S(r,f)$$

$n\ T(r,f) \leqslant (n-k)\ T(r,f) + S(r,f)$

This is a contradiction. Hence the theorem

<u>Remark</u> : Putting $P(f) = f'$ and $k = 1$ we obtain theorem 3 of G.P.Barker and A.P.Singh [1].

We have then on using lemma 4

$$\{P + o(1)\} \quad T(r,f) \ < \ P(b + \varepsilon)\, r^{\rho} \ \ L(r) + \bar{N}(r, w, \ \psi(z))$$

for all $r \geqslant r_0$

On dividing by r^{ρ} $L(r)$ and letting $r \to \infty$ and using (3.21) we get

$$\limsup_{r \to \infty} \frac{\{P+o(1)\} \quad T(r,f)}{r^{\rho} \ \ L(r)} \quad \limsup_{r \to \infty} \ P(b+\epsilon) \ +$$

$$+ \limsup_{r \to \infty} \frac{\bar{N}\ (r, \ w, \ \psi(z)\ )}{r^{\rho} \ \ L(r)}$$

and so

$$\{P+o(1)\}\ a \leq P(b+\varepsilon) + \limsup_{r \to \infty} \frac{\bar{N}(r, w, \psi(z))}{r^{\varrho} L(r)}$$

which yields

$$Pa \leqslant Pb + \limsup_{r \to \infty} \frac{\bar{N}(r, w, \psi(z))}{r^{\varrho} L(r)}$$

Thus

$$\limsup_{r \to \infty} \frac{\bar{N}(r, w, \psi(z))}{r^{\varrho} L(r)} \geqslant P(a-b) \qquad \ldots (3.21)$$

from Lemma 1 of [12] we have

$$\limsup_{r \to \infty} \frac{\bar{n}(r, w, \psi)}{r^{\varrho} L(r)} \geqslant \limsup_{r \to \infty} \frac{\bar{N}(r, w, \psi)}{r^{\varrho} L(r)} \ldots (3.22)$$

Combining (3.21) and (3.22) we get

$$\limsup_{r \to \infty} \frac{\bar{n}(r, w, \psi)}{r^{\varrho} L(r)} \geqslant p\,\varrho\,(a-b)$$

Using lemma 3 of [12] we obtain

$$T\left(r, \frac{\psi(z)}{w}\right) \leqslant P(k+1)\, T(r,f) + S(r,f)$$

Also by lemma 4 of [12] we have

$$P\,T(r,f) < P\,N(r, 1/f) + \bar{N}(r,f) + \bar{N}(r,w, \psi(z)) + S(r,f)$$

And so

$$P < P \frac{(r, 1/f)}{T(r,f)} + \frac{\bar{N}(r,f)}{T(r,f)} + \{P(k+1)+o(1)\} \frac{\bar{N}(r,w,\psi(z)}{T(r, w, \psi)} \cdot$$

$$\cdot \frac{T(r, w, \psi)}{T(r, f)} + o(1)$$

Letting $r \to \infty$ we get

$$P < P\ [1 - \delta(o,f)] + [1 - \Theta(\infty, f)] +$$

$$+ (P(k+1))\ (1 - \Theta(w, \Psi(z)))$$

which on simplification yields

$$\delta(o,f) + \frac{1}{p}\Theta(\infty, f) + (k+1)\Theta(w, \Psi(z)) \leq k+1 + \frac{1}{p} .$$

<u>Lemma 3.3</u> Let $\Psi(z)$ be a homogeneous differential polynimial of degree P in the meromorphic function f(z) then

$$PT(r,f) < PN(r, \frac{1}{f}) + \bar{N}(r,f) + \bar{N}(r, 1, \frac{\Psi}{w}) + S(r,f)$$

<u>Proof</u> : Working as in theorem 3.2 of Hayman [7, 57] we get

$$m\left(r, \frac{1}{\frac{\Psi(z)}{w}}\right) < \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{\frac{\Psi(z)}{w} - 1}\right) - N_0\left(r, \frac{1}{(\frac{\Psi}{w})'}\right) +$$

$$+ S(r,f)$$

Also

$$PT(r,f) = T(r,f^p) = T(r, \frac{1}{f^p}) + O(1)$$

$$= m(r, \frac{1}{f^p}) + N(r, \frac{1}{f^p}) + O(1)$$

$$\leq m\left(r, \frac{\Psi(z)/w}{f^p}\right) + m\left(r, \frac{1}{\frac{\Psi(z)}{w}}\right) +$$

$$+ PN(r, 1/f) + O(1).$$

And so

$$PT(r,f) \leq m\left(r, \frac{1}{\frac{\Psi(z)}{w}}\right) + PN(r, 1/f) + S(r,f)$$

since $m\left(r, \frac{\Psi(z)/w}{f^P}\right) = S(r,f)$.

Thus

$$PT(r,f) \leq PN\left(r, \frac{1}{f}\right) + \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{\frac{\Psi(z)}{w} - 1}\right) - N_0\left(r, \frac{1}{\left(\frac{\Psi}{w}\right)'}\right) + S(r,f)$$

since $N_0\left(r, \frac{1}{(\Psi/w)'}\right) \geq 0$, it follows that

$$PT(r,f) < PN(r, 1/f) + \bar{N}(r,f) + \bar{N}(r, w, \Psi(z)) + S(r,f)$$

which completes the Lemma.

<u>Note</u> :

(i) $p(f) \not\equiv 0$ is essential, since, if $P(f) \equiv 0$ then there exists transcendental solutions of (3.27) satisfying $N(r,f) = S(r,f)$. For example consider $f(z) = e^z$ and $\pi_{n-1}(f) = \pi_1(f) = f - f' = e^z - e^z = 0$.

Thus e^z is a solution of (3.27) and $N(r,e^z) = 0 = S(r,f)$.

(ii) Also the condition $N(r,f) = S(r,f)$ in the above theorem is essential.

Since consider the equation $2f^3 - (f'' + f) + 0$, that is $2f^2 f - (f'' + f) = 0$ then the above has $f(z) = \sec z$ as its solution and clearly $N(r,f) \neq S(r,f)$

(iii) $(f)^n - (f)^n = 0$ for any function f trivially shows that k should be greater than or equal to 1 in our theorem.

G.P.Barker and A.P.Singh in [1] have proved the following theorem.

Theorem : No transcendental meromorphic function with $N(r,f) = S(r,f)$ can satisfy an equation $a_1(z)(f(z))^n P(f) + a_2(z)p(f) + a_3 = 0$ where $a_1(z) \not\equiv 0$, n is positive integer and $P(f)$ is a monomial of degree $\geqslant 1$.

It looks reasonable to except that the above theorem should hold for homogeneous differential polynomials also instead of only monomials. But as the number of terms in a differential polynomial though finite, may be large, we have not been able to prove this result. However, if we put a restrictions on the number of terms in a homogeneous differential polynomial then we have the following theorem :

Theorem 3.12 : No transcendental meromorphic function f with $N(r,f) = S(r,f)$ can satisfy an equation of the form

$$a_1(z)(f(z))^n \pi_k(f) + a_2(z)\pi_k(f) + a_3(z) = 0, \quad \ldots (3.29)$$

$n \geqslant 1$, where $a_1(z) \not\equiv 0$ and $\pi_k(f)$ is a non-zero homogeneous differential polynomial of degree k having p terms, where p & K satisfy the relation $(p - 1)k < n$.

For the proof of the above theorem we shall need the following lemmas of [1]

Lemma 3.4 : If f is meromorphic and not constant in the plane, if $g(z) = f(z))^n + P_{n-1}(f)$, where $P_{n-1}(f)$ is a differential polynomial of degree almost n-1 in f and if

$N(r,f) + N(r, 1/g) = S(r,f)$ then $g(z) = (h(z))^n$,

$h(z) = f(z) + \frac{1}{n} a(z)$ and $(h(z))^{n-1} a(z)$ is obtained by substituting h(z) for f(z), h'(z) for f'(z) etc. in terms of degree n-1 in $P_{n-1}(f)$.

Lemma 3.5 : If f(z) is meromorphic and transcendental in the plane and that $(f(z))^n P(z) = Q(z)$ where P(z), Q(z) are differential polynomials in f(z) and degree of Q(z) is atmost n. Then $m(r, P(z)) = S(r,f)$ as $r \to \infty$

Proof of theorem 3.12 :

Case (i) we first consider the case $n \geqslant 2$ suppose (3.31) holds clearly $a_3 \not\equiv 0$, for otherwise either f is a relational or $T(r,f) = S(r,f)$ and both of which are not possible.

Now from (3.31) we get

$$(f)^n + \frac{a_2}{a_1} = - \frac{a_3}{a_1 \pi_k(f)} = G(z) \text{ say}$$

Then

$$N(r, \frac{1}{G}) = N(r, \frac{a_1 \pi_k(f)}{a_3}) = S(r,f).$$

Also $N(r,f) = S(r,f)$.

Therefore by Lemma (3.4)

$$G = (f)^n$$

which yields $a_2 \equiv 0$. Thus equation (3.29) becomes

$$(f)^n \pi_k(f) = -\frac{a_3}{a_1}$$

and hence $T(r, (f)^n \pi_k(f)) = S(r,f)$. ... (3.30)

Now let

$$\emptyset(f) = f^n \pi_k(f)$$

$$= f^n \left\{ \sum_1^p (f)^{l_0} (f')^{l_1} \ldots (f^t)^{l_t} \right\}$$

where $l_0 + l_1 + \ldots + l_t = k$.

Therefore $$\frac{1}{f^n} = \frac{1}{\emptyset(f)} \left\{ \sum_1^p (f)^{l_0} (f')^{l_1} \ldots (f^t)^{l_t} \right\}.$$

Thus

$$\frac{1}{f^{n+k}} = \frac{1}{\emptyset(f)} \left\{ \sum_1^p \left(\frac{f'}{f}\right)^{l_1} \ldots \left(\frac{f^t}{f}\right)^{l_t} \right\}$$

Applying Nevanlinna's first fundamental theorem and that $T(r, \emptyset) = S(r,f)$ we obtain

$$(n+k)\, T(r,f) \leq \sum_1^p \left\{ l_1 T\left(r, \frac{f'}{f}\right) + \ldots + l_t T\left(r, \frac{f^{(t)}}{f}\right) \right\} + S(r,f).$$

Using Milloux's theorem [7, 55] it now follows that

$$(n+k)\, T(r,f) \leq \sum_1^p l_1 N\left(r, \frac{f'}{f}\right) + \ldots + l_t N\left(r, \frac{f^{(t)}}{f}\right) + $$

$$+ S(r,f).$$

But $N(r,f) = S(r,f)$ and so

$$N(r, \frac{f^{(t)}}{f}) \leqslant N(r, f^{(t)}) + N(r, \frac{1}{f})$$

$$\leqslant (t+1)\, N(r, f) + N(r, \frac{1}{f})$$

$$= N(r, \frac{1}{f}) + S(r,f)$$

Therefore

$$(n+k)\, T(r,f) \leqslant \sum_{1}^{p} \{l_1\, N(r, \frac{1}{f}) + \dots + l_t\, N(r, \frac{1}{f})\} + S(r,f)$$

$$= \sum_{1}^{p} (l_1 + l_2 + \dots + l_t\} \, N(r, \frac{1}{f}) + S(r,f)$$

$$= \sum_{1}^{p} (k - l_o)\, N(r, \frac{1}{f}) + S(r,f)$$

$$= p(k - l_o)\, N(r, \frac{1}{f}) + S(r,f)$$

Thus

$$(n+k)T(r,f) \leqslant p(k-l_o)\, T(r,f) + S(r,f)$$

$$\leqslant pk\, T(r,f) + S(r,f)$$

This is a contradiction since $n + k > pk$.

Case (ii) : We now consider the case $n = 1$. when $n = 1$, the hypothesis implies $p = 1$ and so $\pi_k\,(f)$ becomes a monomial. This particular case has been considered by G.P.Barker and A.P.Singh. We give their proof for sake of completeness.

Let $F = f + \frac{a_2}{a_1}$

then $\pi_k\,(f) = Q(F)$ where $Q(F)$ is a differential polynomial in F. Then (3.31) can be written as $FQ(f) = \frac{-a_3}{a_1}$

and hence by Lemma (3.5)

$$m(r, Q(F)) = S(r,f) = S(r,f)$$

$$N(r, Q(F)) = S(r,f)$$

Now $N(r, Q(F)) = N(r, \pi_k(f))$

$$= N(r, \sum a(z)(f)^{l_0} \ldots (f^{(t)})^{l_t}$$

$$\leqslant N(r, a(z)) + N(r, (f)^{l_0}) + \ldots.$$

$$\ldots + N(r, (f^{(t)})^{l_t}$$

$$= N(r, a(z)) + l_0 N(r,f) + \ldots..$$

$$\ldots + l_t N(r,f)$$

$$= S(r,f) + S(r,f) + \ldots + S(r,f)$$

and so

$N(r, Q(F)) = S(r,f)$. Also

$m(r, Q(F)) = S(r,f)$

Therefore

$$T(r, Q(F)) = S(r,f)$$

from which it follows that

$$T(r,f) = S(r,f)$$

This is a contradiction. This proves the theorem.

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