CHAPTER - I

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A Brief Biography of Rolf Nevanlinna.

Rolf Nevanlinna was born on 22nd October, 1895. He came from a Swedish speaking finnish family containing Soldiers, Scientists and Engineers. Rolf's father Otto Wilhelm was a teacher.

When Rolf went to School in 1902 he moved straight into the second class, since he could already read and write. Rolf did well at secondary school and matriculated near the top of his class. Perhaps the best teacher was his own father who taught him mathematics and physics in the final year of school. He also learned German and French.

His chief interest were classics and mathematics in that order. He had a wonderful feeling for music, he belonged to quartet in Helsinki and became chairman of the Sibelius Academy.

Between school and University he read Lindelof's 'Introduction to higher analysis' and did all the problems. In 1913 he went to Helsinki University, where Lindeloff (who was a cousin of Rolf's father) was the outstanding scientist. He helped Rolf with advice and criticism. In 1915 Rolf felt he ought to go to Germany to get military training in a battalion established there for Finnish freedom fighters. In 1918-1919 he wrote his thesis. He put a great deal of effort into presenting his work in an optimal way and then afterwards he wrote and rewrote his manuscript.

Rolf married with Mary on 4th June 1919, the day Rolf got his doctorate. When Rolf graduated in 1919, there were no jobs open in Universities, so he became a school teacher. He was always an enthusiatic teacher. He taught mathematics to his children.

In 1920 Landau invited Rolf to join him in Göttingen and he went there in 1924. During these years Rolf started to develop the theory which bears his name. He became a Docent at the University of Helsinki in 1922 and professor in 1926 and it was only then that he stopped teaching in school.

Rolf always enjoyed lecturing and teaching on a personal basis. He prepared his lectures in outline and felt that there should be some room for improvisation around a fixed theme. Sometimes at the blackboard a connection with other areas would occur to him. Later on he would write up his lectures and sometimes turn them into books.

In 1924 Rolf visited Göttingen. Here he met Hilbert, Landau, Courant and Noether. Rolf refused the offer to succeed Weyl at Zürich. In 1936-1937 he was again in Göttingen as visiting Professor. During the second world war Rolf developed a method for reviewing ballistic tables.

In 1941 Rolf became Rector of Helsinki University and discussed with Mannerheim how soldiers could continue to study during quiet times at the front and what can be done after the war. In 1944 a bomb hit the building he was working in and he also found an unexploded bomb. Suddenly he was a hero.

In 1945 he was asked to resign his post as Rector no doubt because of his proGerman sympathies. In October 1946 Rolf went again to Zürich. There he met many mathematicians and also the Physicist. In 1948 he became one of the 12 members of the newly established Finish Academy. Rolf continued to be guest professor at Zürich for the next 15 years. After the war Rolf's interest began to turn to calculus of variations and applications to physics. He was also concerned in getting the first computer to Finland and to establish computer science as a University subject.

From 1959 to 1962 Rolf was President of the I.M.U. Nevanlinna was fairly conservative in his views. He did not think that sets formed the best introduction to mathematics at school. He did not like to see children spoilt, feeling that they would find life hard later. But he took an optimistic view in general of the future of Finland and the world.

Rolf never needed much sleep. In the late 1930's he used to rise 4, join his family at 10 till lunch time, then work again till 7, and spend the evening with his family. In later life also he was bright and full of zest at 11 or 12 P.M.

He was calm and peaceful at the end. He died on 28th May, 1980.

Considering the tremendous importance of Nevanlinna theory, recognition came relatively slowly to Rolf but his last 30 years were very full of honours. He had honorary doctorates from Heidelberg (1936), Bucharest (1942), Giessen (1952), Berlin (1955), Jyväskyla (1969), Glasgrow (1969) Uppsala (1974) and Istanbul (1976). He received the international Wihuri prize for scientists and Artists in 1958 and the Henrik Steffens prize for Nordic culture in 1967. He was elected to Honorary Membership of the London Mathematical Society in 1959. He was also an honorary member of the following institutions, Finnish Academy of Science and Letters (1975), Deutsche Akademie (1938), Finnish Mathematical Society (1955), Swiss Mathematical Society (1962); Society of Actuaries of Finland (1965), Teachers of Mathematics and Physics (1965), Göttingen Academy (1967), Royal Swedish Academy (1967) (Foreign Members); Danish Academy (1967)(Foreign Member) Leopoldina (1967), Hungarian Academy (1970) correspondent of the Institute de France (1967). He was an Honorary Fellow of Göttingen University (1937) and Honorary Professor of Zürich University (1948). He was an Honorary member of the Sibelius Academy (1978) and Honorary president of the Finnish Cultural Foundation.

He had the Grand cross of the order of the white Rose of Finland and he was commander, First class, of the order of Lion of Finland. He had the cross of Liberty, Second Class without swords, for merit during the war 1939-40.

He continued to write a large number of papers and books throughout his life on many different topics ranging from ballistics to education. His book on Riemann Surfaces is one of the best accounts of the subject. He wrote an excellent elementary textbook with paatero.

He wrote over 200 research articles, a detailed reference of which can be found in [8] . His last published paper was in 1980 written at the age of 85.

Introduction to Nevanlinna theory.

Nevanlinna theory originates from a general formula of the two brothers F. and R. Nevanlinna, by which they were developing a general method for the investigation of meromorphic functions. This formula includes both the poisson formula and Jensen formula as special cases and in its most important form it expresses the logarithm of the modulus of an arbitrary meromorphic function by the boundary values of the function along a concentric circle around the origin and the zeros and poles of the function inside this circle.

Nevanlinna theory was created at the moment when Rolf Nevanlinna gave the formula an ingenious interpretation. This happened around 1924. The most general result of Nevanlinna theory can be summerized by saying that the distribution of the solutions to the equation $g_1(z) = a_1$ is extremely uniform for almost all values of a_1; there can only exist a small minority of values which the function takes relatively rarely. The investigation of these exceptional values constitutes the main task of value distribution theory in the sense of Nevanlinna. The earlier value distribution theory before Nevanlinna can be traced back to the year 1876, when K. Weierstrass showed that in the vicinity of an isolated essential singularity a meromorphic function $g_1(z)$ approaches every given value a_1 arbitrarily closely. In 1879 E. Picard proved that the surprising fact that a meromorphic function takes in the vicinity of an isolated essential singularity every finite or infinite value a_1 with two exceptions at the most. Points which are not taken are now called picard exceptional values, of the function. The results which were found by the mathematicians E. Laguerre, H. Poincare, J. Hadamard, E.Borel and others revealed that in spite of the possible existence of picard exceptional values the distribution of zeros or, more generally, the distribution of a-points of an entire function is controlled, at least in some sense, by the growth behaviour of the maximum modulus function

$$M(r, g_1) = \max_{|z| = r} |g_1(z)|$$

which has the function of a transcendental analogue of the degree of a polynomial. This approach of early value distribution theory breaks down, however, if $g_1(z)$ is meromorphic, since then $M(r, g_1)$ becomes infinite if $g_1(z)$ has a pole on the circle $|z| = r$. In discussing the meromorphic function

$g_1(z)$ we can no longer use the maximum modulus as a convenient tool for expressing the rate of growth of function. Therefore, in Nevanlinna theory the role of log $M(r,f_1)$ is taken by an increasing real valued function $T(r,g_1)$, the "Nevanlinna characteristic function" which is associated to the given meromorphic function $g_1(z)$. A great deal of work had been done in establishing the relationship between distribution of values and growth when Rolf Nevanlinna created his epoque making theory. This theory, which applies to entire functions as well as to meromorphic functions, even improved tremendously the earlier value distribution theory of entire functions. There have been many attempts to extend the Nevanlinna theory in several directions. One of these is known as theory of holomorphic or meromorphic curves. This theory was initiated by H. and J. Weyl in 1938. The most difficult problem of this extension, the proof of the defect relation for holomorphic curves, was solved by Ahlfors.

Recently a very modern treatment of this theory was given by H. Wu. The theory of holomorphic curves by Weyl-Ahlfors was further extended in a very general way to a higher dimensional theory first by W. Stoll and then in a different direction by H.I.Levine, S.S.Chern, R.Bott and other authors. In 1972 introducing once more fascinating new ideas the Ahlfors-Weyl theory was extended in a different direction, more regarding to algebraic geometry by J.Carlson and P. Griffiths to equidimensional holomorphic mappings.

Hans J.W. Ziegler extended the formalism of Nevanlinna theory to systems of $n \geqslant 1$ meromorphic functions $g_1(z)$, $g_2(z) \ldots g_n(z)$ in a way, which is fundamentally different from the theory of holomorphic or meromorphic curves of Weyl-Ahlfors and its higher dimensional generalization. As in Nevanlinna theory again the starting point is a generalization of formula of poisson-Jensen. Nevanlinna, which he discovered in 1964, when he was trying to extend the Nevanlinna formalism to the simultaneous solutions of systems of n equations.

$$
\begin{aligned}
&g_1(z) = a_1 \\
&g_2(z) = a_2 \\
&\vdots \\
&g_n(z) = a_n, \; z \in C, \quad a_1 \in C, \ldots, a_n \in C
\end{aligned}
$$

where $w_j = g_j\,(z)$, $j = 1, 2 \ldots n$ are $n \geqslant 1$ meromorphic functions.

He succeeded in extending formally both the main theorems of Nevanlinna theory together with the Nevanlinna deficiency relation. Although the above system of equations has only solutions for points $a = (a_1, \ldots a_n) \in g(c)$, a set which is rather thin for $n > 1$, these results seemed to be quite interesting. However, one difficult main problem was still to solve, the problem of finding the true geometric meaning of the extended quantities, a problem which was proposed to him by Helmul Grunsky and by Rolf Nevanlinna in 1967-1968. He finally achieved the result and he represented this result in [23, ix]

The study of exceptional values of entire functions started with the famous theorems, of Picard and Borel. Picard's theorem was a generalization of weierstrass's theorem who proved that in the neighbourhood of an isolated essential singularity an analytic function comes arbitrarily close to every complex value. In fact Picards proved that in the neighbourhood of an isolated essential singularity an analytic function comes not only "close" to a, it assumes every value a infinity of times except possibly for one value of a. As a particular case if f(z) is a transcendental entire function, then f(z) - a has infinity of zeros except possibly for one value of a. This exceptional value is called exceptional value picard and is normally denoted by e.v.p.

For stating the Borel's theorem, we shall need some terminology.
If f(z) is an entire function, then we shall define as usual the order of f, (denoted by ρ) by

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r,f)}{\log r}$$

where $M(r,f) = \max_{|z| = r} |f(z)|$. And the exponent of convergence of a-points of f(z) denoted by $\rho_1(a)$ is defined to be

$$\rho_1(a) = \limsup_{r \to \infty} \frac{\log^+ n(r,a)}{\log r}$$

where $n(r,a)$ denotes the number of zeros of $f(z)-a$ in $|z| \leqslant r$.

Hadamard in 1893 proved that $\rho_1(a) \leqslant \rho$ for all a. Borel in 1897 provedthat if $f(z)$ is an entire function of finite order then $\rho_1(a) = \rho$ except possible for one value of a and if this exception occurs then ρ must be an integer. This exception is now called as exceptional value Borel and we denote if by e.v.B. If the given function is of finite order then, the theorem of Borel includes the theorem of picard as a particular case.

Borel later generalized his theorem for infinite order, He introduced a variable order $\eta(r)$ and showed that for certain categories of entire functions,

$$\limsup_{r \to \infty} \frac{\log^+ n(r,a)}{\eta(r) \log r} = 1 \text{ except possibly for}$$

one value of a in which case the left hand side of the above inequality is less than one.

Picard's and Borel's theorem have been proved in more general context of meromorphic functions by R.Nevanlinna. The second fundamental theorem of Nevanlinna furnishes a very simple proof of picard and Borel theorems. Valiron has extended Borel's theorem in a sector. He proved that if $f(z)$ is a meromorphic function then there exists a direction D (which he calls Borel's direction) such that in every angle containing that line in its interior the exponent of convergence of the zeros of $f(z) - a$ is equal to the order of the function for all values of a except possibly for two (for $a = \infty$ the zeros of $f(z)-a$ are to be replaced by the poles of $f(z)$).

Let $f(z)$ be entire function. If $n(r,a) = O(1)$ then obviously a is e.v.p. (exceptional value in the sense of Picard), and if $\rho_1(a) < \rho$ then a is e.v.B. (in the sense of Borel).

$$\text{Let } \delta(\alpha) = 1 - \limsup_{r \to \infty} \frac{N(r,\alpha)}{T(r)}$$

If $\delta(\alpha) > 0$, we say that α is e.v.N. (exceptional value in the sense of Nevanlinna). Let E denote the set of non-decreasing functions $\phi(x)$ such that

$$\int_A^{\infty} \frac{1}{x\,\phi(x)}\, dx < \infty$$

Then it is known that for entire functions $f(z)$ of non-integral and zero order and for a class of functions of integral order including all functions of maximum or minimum types

$$\liminf_{r \to \infty} \frac{\log M(r,f)}{n(r,\alpha)\,\phi(r)} = 0 \text{ for every } \alpha \text{ and every}$$

$\phi(x) \in E$. Hence it is natural to define $\alpha (0 \leq |\alpha| < \infty)$ as e.v.E. for $f(z)$.

$$\text{if} \quad \liminf_{r \to \infty} \frac{\log M(r,f)}{n(r,\alpha)\,\phi(r)} > 0 \text{ for some } \phi(x) \in E.$$

Then it is known (See [18]) that e.v.P. $\Rightarrow$ e.v.B. $\Rightarrow$ c.v.E $\Rightarrow$ e.v.N.

We can also define e.v.P., e.v.B., e.v.N. for a meromorphic function f(z) also where α can take the value ∞. In that case $N(r,\alpha)$ will be associated with the poles of f(z). Similarly we can define α $(0 \leq |\alpha| < \infty)$ to be e.v.E. for a meromorphic function f(z) if $\liminf_{r \to \infty} \frac{T(r,f)}{n(r,\alpha)\phi(r)} > 0$ for some $\phi(x) \in E$. If f(x) is a meromorphic function then again we get e.v.P. $\Rightarrow$ e.v.E. $\Rightarrow$ e.v.N. But in this case e.v.B. $\Rightarrow$ e.v.N. is not true. As a matter of fact Valiron [22] has shown by an example that if α is e.v.B. for a meromorphic function, it may not be e.v.N.

Exceptional values came to be studied in their relation to asymptotic values. S.M.Shah in 1952 proved that if f(z) is an entire function of order ρ $(0 < \rho < \infty)$ having α as e.v.E. then the number of finite asymptotic values of f(z) is ρ each being α itself (Two asymptotic values are called different if they are separated by apath along which $f(z) \to \infty$). An entire function of finite order ρ possesses at most 2ρ finite asymptotic values. This is known as Denjoy's conjecture. This was proved by Ahlfors. But in case of meromorphic function above result is not true. Valiron has constructed a meromorphic function of finite order having infinity of asymptotic values which form a non-enumerable set.

Nevanlinna conjectured that if α is e.v.N. for an entire or a meromorphic function then α must be an asymptotic value. For a meromorphic function the conjecture was proved to be false in 1941 by Madame Laurent Schwartz. For an entire function of infinite order it was proved to be false by W.K. Hayman and for an entire function of finite order it was proved to be false by A.A. Goldberg. Also he disproved another conjecture namely if 0 is e.v.N. for an entire function of finite order then the order of the function cannot exceed 2λ where λ is the lower order.

With some additional hypothesis the conjecture of Nevanlinna is true. A Edrei and W.H.J. Fuchs have proved that if f(z) is an entire function of finite order and if $\sum_i \delta(a_i) = 2$ then each deficient value of f(z) is also an asymptotic value. A Edrei has proved that if we replace the condition $\sum_i \delta(a_i) = 2$ by some other smooth condition then the restriction that f(z) must be of finite order can be removed and each deficient value will be asymptotic value.

S.K.Singh and S.M.Shah have studied exceptional values in another context also. For an entire function f(z) of finite order it is well known that $\log M(r,f) \sim \log M(r,f')$. Hence it is reasonable to conjecture that for an entire function of finite order

$$T(r,f) \sim T(r,f')$$

Nevanlinna actually conjectured that for an entire function

$T(r,f) \sim T(r,f')$ and for meromorphic function either $T(r,f') \sim T(r,f)$ or $T(r,f') \sim 2T(r,f)$. These conjecture have neither been proved nor disproved. S.K.Singh and S.M. Shah have proved that if f(z) is a meromorphic function of finite order such that $\delta(\alpha) = \delta(\infty) = 1, \quad \alpha \neq \infty$

then $T(r,f') \sim T(r,f)$

and if $\delta(\alpha_1) = \delta(\alpha_2) = 1, \quad \alpha_1 \neq \alpha_2$

$|\alpha_1| < \infty, \quad |\alpha_2| < \infty$

then $T(r,f') \sim 2T(r,f)$

Later S.K.Singh and S.M.Shah improved these results further.

If $|\alpha_i| < \infty \quad \sum_{1}^{\infty} \delta(\alpha_i) = 2$ then $T(r,f') \sim 2T(r,f)$ where f(z) is a meromorphic function of finite order. A pflugar proved that if f(z) is an entire function of finite order ρ such that $\sum_{i} \delta(\alpha_i) = 2$. Then ρ must be integer. S.K.Singh and S.M.Shah proved that if f(z) is a meromorphic function of finite order such that $\delta(\alpha_1) = 1 \quad \sum_{i=2}^{\infty} \delta(\alpha_i) = 1$

where $\{\alpha_i\}$, $i = 1,2,3...$ are constants finite or infinite different from each other then ρ must be a positive integer.

Before proceeding further we shall need some definitions dealing with asymptotic values:

If f(z) is an entire function and γ is a curve starting from $Z = 0$ and proceeding towards infinity and if $f(z) \to a$ (a finite) as $z \to \infty$ along γ, we say that a is an asymptotic value for f(z). Also γ is called an asymptotic path. From a well-known theorem see for e.g. [19] we know that if γ_1 and γ_2 are two asymptotic paths along which f(z) tends to a and to be respectively, then there must exist a path γ_3 lying between γ_1 and γ_2 such that $f(z) \to \infty$ as $z \to \infty$ along γ_3, where γ_3 is called a path of infinite determination. For illustration consider

<u>Example 1</u> : Let $f(z) = e^z$ since $e^z \to 0$ as $z \to \infty$ along negative real axis, so 0 is an asymptotic value for e^z.

<u>Example 2</u> : Let $f(z) = e^{z^2}$ Here $f(z) \to 0$ as $z \to \infty$ along the imaginary axis in the upper half plane as well as in the lower half plane. Also along the positive real axis; $e^{z^2} \to \infty$ as $z \to \infty$. Hence there exists a path of infinite determination separating the two asymptotic paths, (i) the imaginary axis in the upper half plane and (ii) in the lower half plane. Hence e^{z^2} has two asymptotic values each being 0. It can be seen that it has no other asymptotic value. The function $f(z) = e^z$ has only one namely 0 as the asymptotic value.

<u>Example 3</u> : $f(z) = \int_0^z e^{-t^q}\, dt$ (q is an integer $\geqslant 1$)

has $a_0, a_1, a_2 \ldots a_{q-1}$ for its asymptotic values where

$$a_k = \exp\left(\frac{2\pi i k}{q}\right) \int_0^q e^{-t^q}\, dt \quad (k = 0, 1, 2 \ldots q-1).$$ Since

the order of an entire function is the same as the order of its derivative, and $f'(z) = e^{-z^q}$ is of order q, hence f(z) is also of order q. In the three examples given above the entire functions have exactly the same number of asymptotic values as their orders. Thus using above analogy one can say that an entire function of order ρ can have at most ρ asymptotic values. But as such this is not true.

For consider the following example :

$f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$ is an entire function of order $\rho = 1/2$, $f(z) \to 0$ as $z \to \infty$ along the positive real axis. Hence $\dfrac{\sin\sqrt{z}}{\sqrt{z}}$ has 2ρ asymptotic values.

Denjoy conjectured that an entire function of order ρ $(0 < \rho < \infty)$ has atmost 2ρ asymptotic values. This conjecture was proved by L.V. Ahlfors, and that this is best possible result can be seen from the example $\dfrac{\sin\sqrt{z}}{\sqrt{z}}$

NOTATIONS AND TERMINOLOGY, SOME RESULTS.

Nevanlinna theory originates from general formula

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi$$

$$+ \sum_{i=1}^{m} \log\left|\frac{R(z-a_i)}{R^2 - \bar{a}_i z}\right| - \sum_{j=1}^{n} \log\left|\frac{R(z-b_j)}{R^2 - \bar{b}_j z}\right| \quad \ldots(1.1)$$

where f(z) is meromorphic in $|z| \leq R$ $(0 < R < \infty)$ and

a_i ($i = 1$ to m) are the zeros and b_j($j=1$ to n) the poles of $f(z)$ in $|z| < R$ and $f(z) \neq 0, \infty$. The case when there are no zeros or poles is usually called poisson's formula. The case when $z = 0$ is called Jensen's formula.

We define

$$\log^+ x = \log x, \qquad \text{if } x \geqslant 1$$

$$\log^+ x = 0, \qquad \text{if } 0 \leqslant x < 1$$

Clearly if $x > 0$, $\log x = \log^+ x - \log^+ \frac{1}{x}$

Thus

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left|f(Re^{i\phi})\right| \, d\phi = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f(Re^{i\phi})\right| \, d\phi -$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|\frac{1}{f(Re^{i\phi})}\right| \, d\phi$$

We denote

$$m(R_1 f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(Re^{i\phi})\right| \, d\phi \qquad \ldots(1.2)$$

and

$$N(R,f) = \sum_{j=1}^{n} \log\left|\frac{R}{bj}\right| = \int_0^R n(t,f) \frac{dt}{t} \qquad \ldots(1.3)$$

$$N(R,\frac{1}{f}) = \sum_{i=1}^{m} \log\left|\frac{R}{a_i}\right| = \int_0^R n(t,\frac{1}{f}) \frac{dt}{t} \qquad \ldots(1.4)$$

where $n(t,f)$ denotes number of poles of $f(z)$ in $|z| < t$ and

$n(t, \frac{1}{f})$ denotes number of zeros of $f(z)$ in $|z| < t$ with this notation and $z = 0$ (1.1) becomes

$$\log |f(0)| = m(R,f) - m(R, \frac{1}{f}) + N(R,f) - N(R, \frac{1}{f})$$

or

$$m(R,f) + N(R,f) = m(R, \frac{1}{f}) + N(R, \frac{1}{f}) + \log |f(0)|$$

We write

$$T(R,f) = m(R,f) + N(R,f) \qquad \ldots(1.5)$$

Thus Jensen's formula becomes

$$T(R,f) = T(R, \frac{1}{f}) + \log |f(0)| \qquad \ldots(1.6)$$

where the term or function $T(R,f)$ is called the Nevanlinna's characteristic function of $f(z)$ and $m(R,f)$ is a sort of averaged magnitude of $\log |f|$ on arcs of $|z| = R$ where $|f|$ is large. The term $N(R,f)$ relates to the poles. Function $T(R,f)$ plays a cardinal role in the whole theory of meromorphic functions.

Now let $a_1, a_2, \ldots a_q$ are any complex numbers then

$$\log^+ \left| \prod_{\nu=1}^{q} a_\nu \right| \leqslant \sum_{\nu=1}^{q} \log^+ |a_\nu|$$

and

$$\log^+ \left| \sum_{\nu=1}^{q} a_\nu \right| \leqslant \log^+ (q \max_{\nu=1, \ldots q} |a_\nu|)$$

$$\leqslant \sum_{\nu=1}^{q} \log^+ |a_\nu| + \log q.$$

By applying these inequalities to q meromorphic functions $f_1(z), \dots f_q(z)$ and using (1.2) we obtain

$$m\left(r, \sum_{\nu=1}^{q} f_\nu (z)\right) \leqslant \sum_{\nu=1}^{q} m\,(r, f_\nu (z)\,) + \log q$$

$$m(r, \prod_{=1}^{q} f_\nu (z)) \leqslant \sum_{\nu=1}^{q} m\,(r, f_\nu (z)\,).$$

If f(z) is the sum or product of the functions $f_\nu (z)$, then the order of a pole of f(z) at a point z_o is at most equal to the sum of the orders of the poles of the $f_\nu (z)$ at z_0
Thus

$$N(r, \sum_{\nu=1}^{q} f_\nu (z)\,) \leqslant \sum_{\nu=1}^{q} N(r, f_\nu (z)\,)$$

$$N(r, \prod_{\nu=1}^{q} f_\nu (z)\,) \leqslant \sum_{\nu=1}^{q} N(r, f_\nu (z)\,)$$

Using (1.5) we deduce

$$T(r, \sum_{\nu=1}^{q} f_\nu (z)\,) \leqslant \sum_{\nu=1}^{q} T(r, f_\nu (z)\,) + \log q$$

$$T(r, \prod_{\nu=1}^{q} f_\nu (z)\,) \leqslant \sum_{\nu=1}^{q} T(r, f_\nu (z)\,)$$

The whole of Nevanlinna's theory of meromorphic function is based on two fundamental theorems, known as the first fundamental theorem and second fundamental theorem of

Nevanlinna. We now state the Nevanlinna's first fundamental theorem.

Nevanlinna's first fundamental theorem :

If a is any complex number then

$$m(R, \frac{1}{f-a}) + N(R, \frac{1}{f-a}) = T(R,f) - \log |f(0)-a| + \varepsilon(a,R)$$

where $|\varepsilon(a,R)| \leqslant \log^+ |a| + \log 2$

If we allow R to vary, the first fundamental theorem can be written as

$$m(R,a) + N(R,a) = T(R) + O(1) \qquad \ldots(1.7)$$

for every a finite or infinite. The term $m(R,a)$ refers to the average, Smallness in a certain sense of f-a, on the circle $|z| = R$ the term $N(R,a)$ to the number of roots of the equation $f(z) = a$ in $|z| < R$. For any a the sum of these two terms is the same apart from a bounded term.

Thus first fundamental theorem of Nevanlinna provides an upper bound to the number of roots of the equation $f(z)=a$, valid for all a. Also it follows that the sum $m + N$ is largely independent of a.

For simplicity we write $m(R,a)$, $N(R,a)$ $n(R,a)$, $T(R)$ instead of $m(R, \frac{1}{f-a})$, $N(R, \frac{1}{f-a})$ $n(R, \frac{1}{f-a})$, $T(R,f)$ if a is finite, and $m(R, \infty)$ $N(R, \infty)$, $n(R, \infty)$ instead of $m(R, f)$ N, (R, f), $n(R, f)$.

We will now give cartan's identity and convexity theorem.

Theorem : Suppose that $f(z)$ is meromorphic in $|z| < R$, then

$$T(r,f) = \frac{1}{2\pi} \int_0^{2\pi} N(r, e^{i\theta})\, d\theta + \log^+ |f(0)|$$

$$(0 < r < R)$$

As consequences of the above are the following :

Corollary 1 : The Nevanlinna characteristic $T(r,f)$ is an increasing convex function of $\log r$ for $0 < r < R$.

Corollary 2 : We have in all cases

$$\frac{1}{2\pi} \int_0^{2\pi} m(r, e^{i\theta})\, d\theta \leqslant \log 2$$

Remark :

(i) Corollary 1 was originally proved by R. Nevanlinna by a different method.

(ii) Corollary 2 shows that $m(r, a)$ is bounded in the average on the circle $|a| = 1$ and a corresponding result holds on any other circle. Thus if $T(r,f)$ is large, $m(r,a)$ is bounded and $N(r,a)$ is nearly equal to $T(r)$ for most a in a certain sense.

(iii) Also it is easily seen that for entire, $T(r,f)$ has many properties similar to $\log M(r,f)$ and so it is but natural to define the order of a meromorphic function by

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} .$$

The type σ for an entire function of order ρ $(o < \rho < \infty)$ is defined by

$$\sigma = \limsup_{r \to \infty} \frac{\log M(r,f)}{r^{\rho}}$$

we say σ is of mean type, minimal type or maximal type according as $0 < \sigma < \infty$, $\sigma = 0$ or $\sigma = \infty$.

Consequently the type τ of a meromorphic function f of order ρ is defined by

$$\tau = \limsup_{r \to \infty} \frac{T(r,f)}{r^{\rho}}$$

And as in the entire case we say that

f is of minimal type iff $\tau = 0$.

f is of mean type if $0 < \tau < \infty$.

f is of maximal type if $\tau = \infty$.

As mentioned above for an entire function $\log M(r,f)$ and $T(r,f)$ have similar properties. This is due to the fact that for an entire function f and $0 < r < R$

$$T(r,f) \leqslant \log^{+} M\ (r,f) \leqslant \frac{R+r}{R-r}\ T\ (R,f)$$

See for instance [7, 18] .

The above also helps in proving that for an entire function,

$$\sigma = 0 \quad \text{iff} \quad \tau = 0$$

$$\sigma = \infty \quad \text{iff} \quad \tau = \infty$$

$$0 < \sigma < \infty \quad \text{iff} \quad 0 < \tau < \infty$$

Let us note that the last case does not imply $\sigma = \tau$.

Although the first fundamental theorem provides an upper bound to the number of roots of the equation $f(z) = a$ it is unable to tell that which term is more important either $m(r,a)$ or $N(r,a)$.

The second fundamental theorem of Nevanlinna shows that in general it is the term $N(r,a)$ which is dominant in the sum $m + N$ and further that in $N(r,a)$ we do not decrease the sum much if multiple roots are counted simply. Thus for most values of a the equation $f(z) = a$ has nearly as many roots as the first fundamental theorem permits and moreover, the majority of these roots are simple.

<u>Nevanlinna's second fundamental theorem</u> :

Let $f(z)$ be non constant meromorphic function of order ρ . Let $a_1, a_2, \ldots a_q$ $(q \geqslant 3)$ be distinct finite or infinite complex numbers. Then

$$(q-2)\ T(r,f) < \sum_{i=1}^{q} N(r,a_i) - N_1(r) + S(r,f)$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N(r, 1/f') + 2N\ (r,f) - N(r,f')$$

and

$$S(r,f) = m(r,\frac{f'}{f}) + m(r, \sum_{\nu=1}^{q} \frac{f'}{(f-a_\nu)}) - q \log^+ \frac{3q}{\delta} +$$

$$+ \log 2 + \log \frac{1}{|f'(0)|}$$

with modifications of $f(0) = \infty$, or $f'(0) = 0$ the quantity $S(r,f)$ will in general play the role of an unimportant error term.

Nevanlinna's second fundamental theorem was originally proved for constants $a_1, a_2, \ldots a_q$. Later Nevanlinna himself proved that his theorem is true even for small meromorphic functions $a_i(z)$ instead of only constants, but with a restriction that $q = 3$. Thus Nevanlinna's theorem for small functions was:

If $f(z)$ is meromorphic and admissible in $|z| < R_0$ and $a_1(z)$, $a_2(z)$, $a_3(z)$ are distinct meromorphic functions satisfying for $\nu = 1,2$ and 3

$$T(r,a_\nu(z)) = o\ (T(r,f))\ \text{as}\ r \to R_0$$

then $$(1 + o(1))\, T(r,f) \leq \sum_{\nu=1}^{3} \bar{N}\left(r, \frac{1}{f-a_\nu(z)}\right) + S(r,f)$$

as $r \to R_0$, where $S(r,f)$ is as defined earlier.

It was an open problem whether this theorem is true for the general case. Only very recently C.F.Osgood [10] announced that he has solved the problem, the proof of which as he mentioned, will be given at a later date.

We now define the deficiency

$$\delta(a) = \delta(a,f) = \liminf_{r \to \infty} \frac{m(r,a)}{T(r,f)}$$

$$= 1 - \limsup_{r \to \infty} \frac{N(r,a)}{T(r,f)}$$

where

$$N(r,a) = N(r,a,f) = \int_0^r \frac{n(t,a) - n(0,a)}{t}\, dt + n(0,a) \log r$$

and

$n(t,a) = n(t,a,f) =$ the number of roots of the equation $f(z) = a$ in $|z| \leq t$, multiple roots being counted with their multiplicity

$$\Theta(a) = \Theta(a,f) = 1 - \limsup_{r \to \infty} \frac{\bar{N}(r,a)}{T(r,f)}$$

and

$$\theta(a) = \theta(a,f) = \liminf_{r \to \infty} \frac{N(r,a) - \bar{N}(r,a)}{T(r,f)}$$

where

$$\bar{N}(r,a) = \bar{N}(r,a,f) = \int_0^r \frac{\bar{n}(t,a) - \bar{n}(o,a)}{t}\,dt$$

$$+ \bar{n}(o,a) \log r$$

and

$\bar{n}(t,a) = \bar{n}(t,a,f) =$ the number of distinct roots of $f(z)$ in $|z| \leq t$.

The quantity $\delta(a)$ is called the deficiency of the value a and $\theta(a)$ is called the index of multiplicity.

We now give an easy consequence of Nevanlinna's second fundamental theorem, that is deficiency relation.

<u>Nevanlinna's theorem on deficient values</u> :

If $f(z)$ is admissible in $|z| < R_o$. Then the set of values a for which $\Theta(a) > 0$ is countable and we have, on summing over all such values a

$$\sum_a (\delta(a) + \theta(a)) \leqslant \sum_a \Theta(a) \leqslant 2$$

To illustrate the simplicity of its application we give below the proof of Picard's theorem viz, if $f(z)$ is a transcendental meromorphic function, then $f(z) - a$ has infinity of zeros for all $a \in \bar{C}$ except possibly for two values of a. For, suppose there are three values a_1, a_2, a_3

say such that $f(z) - a_i$ $(i = 1,2,3)$ have only finitely many zeros. Then $N(r,a_i) = o(\log r)$ $i = 1,2,3$. And so $\delta(a_i)=1$ for $i = 1,2,3$ consequently

$\sum_{\alpha \in \mathbb{C}} \delta(\alpha) \geqslant 3$, contradicting the above result that

$$\sum_{\alpha \in \mathbb{C}} \delta(\alpha) \leqslant 2$$

Similarly proof of Borel's theorem can also be given very elegantly. See for instance [19] . We list below some of the interesting results proved by on deficient value $\delta(a,f)$.

R. Nevanlinna and A. Pfluger have shown that for integral functions of finite order ρ , equality is possible in $\sum \delta(a,f) \leq 2$ only if ρ is a positive integer.

A.A.Goldberg has shown that it is enough to consider functions with positive zeros and negative poles. Goldberg was also the first to construct meromorphic functions, with infinitely many deficient values. The result due to Teichmüller and Goldberg states that f(z) has no deficient values other than a.

A meromorphic function f(z) of positive order ρ may possess a completely arbitrary countable set of deficient values. If $\rho = 0$ then f(z) may possess at most one deficient value.

Now question comes whether an integral function f(z) of finite order ρ may possess infinitely many deficient values ? This was proved by Arkeljan.

If $\rho \leqslant \frac{1}{2}$ then function cannot have any finite deficient values, but nothing is known in general for $\rho > \frac{1}{2}$.

Edrei and Fuchs have shown that if f(z) has all its zeros on a finite number of straight lines then it can have atmost a finite number of deficient values.

Another interesting result regarding the deficient values was its inverse problem viz. if for some δ_i and θ_i with $\sum (\delta_i + \theta_i) \leqslant 2$, then does there exist a meromorphic function f(z) with $\delta(\alpha_i, f) = \delta_i$ and $\Theta(\alpha_i, f) = \Theta_i$. This was solved in affirmative in 1977 by D. Drasin [2] who proved the following :

Let sequences $\{\delta_i\}$ $\{\Theta_i\}$ $(1 \leqslant i < N \leqslant \infty)$ of non-negative numbers be assigned such that $0 < \delta_i + \Theta_i \leqslant 1$ $(1 \leqslant i < N)$, $\sum(\delta_i + \Theta_i) \leqslant 2$, together with a sequence $\{a_i\}$ $(1 \leqslant i < N)$ of distinct complex numbers. Then there exists a meromorphic function f(z) having $\delta(a_i, f) = \delta_i$, $\Theta(a_i, f) = \Theta_i$ $1 \leqslant i < N$, and $\delta(a,f) = \Theta(a,f) = 0$ for $a \notin \{a_i\}$. Further if $\phi(r)$ is a positive increasing function with $\phi(r) \to \infty$ as $r \to \infty$, the function f(z) may be chosen so that its Nevanlinna characteristic satisfies $T(r,f) < r^{\phi(r)}$ for all large r.

We now state another interesting theorem due to Nevanlinna, which states as follows :

Theorem : Suppose that $f_1(z)$, $f_2(z)$ are meromorphic in the plane and let $E_j(a)$ be the set of points z such that $f_j(z)=a$ $(j = 1,2)$. Then if $E_1(a) = E_2(a)$ for five distinct values of a, $f_1(z) \equiv f_2(z)$ or f_1, f_2 are both constant. This theorem is called Nevanlinna's uniqueness theorem. Now $f_1(z) = e^z$, $f_2(z) = e^{-z}$ with $a = 0, 1, -1, \infty$ shows that here 5 can not be replaced by 4.

The results on the shared value mentioned above was done about sixty years back. Only recently properties of functions which share less than five values, have been studied. For instance G.G. Gundersen [5] proved that if f and g share four value $\{a_i\}_{i=1}^{n}$ ignoring multiplicity and $f \not\equiv g$, then outside a set of r of finite linear measure $\lim_{r\to\infty} \frac{T(r,f)}{T(r,g)} = 1$.

And if f and g share three values ighoring multiplicity then outside set of finite measure

$$\limsup_{r\to\infty} \frac{T(r,f)}{T(r,g)} \leqslant 3 \text{ and } \limsup_{r\to\infty} \frac{T(r,g)}{T(r,f)} \leqslant 3$$

and this result is best possible, since

$$f(z) = \frac{e^{3z} - 3e^{2z}}{1 - 3e^z} \quad \text{and } g(z) = \frac{e^z - 3}{1 - 3e^z}$$

share $0, 1, \infty$ and satisfy $T(r,f) \sim 3T(r,g)$ as $r \to \infty$. The sharing of values by function and its derivative was done by L.A.Rubel and C.C.Yang who proved that (See [13]) if a non-constant entire function and its derivative f' share two finite value counting multiplicities then f = f'. Later G.G.Gundersen in [6] improved this result and showed that if a non constant meromorphic function f and its derivative f' share two finite values counting multiplicities then f = f' we state some more theorems dealing with derivatives of meromorphic function. For instance it is known (See [7]) that A derivative $f^{(1)}(z)$ of a meromorphic function f(z) assumes all finite complex values with atmost one exception. On the other hand, $f(z) = \tan z \neq \pm i$, so that f(z) itself may have two exceptional values which are not assumed and if $f(z) \neq a$ for some finite a, then $f^{(1)}(z)$ assumes every finite value except possibly zero.

Another important theorem which we shall frequently use in our discussion is the Milloux theorem which states as follows :

Milloux Theorem : Let P be a positive integer and

$$\psi(z) = \sum_{\nu=0}^{p} a_{\nu}(z)\, f^{(\nu)}(z)$$

Then $m\left(r, \dfrac{\psi(z)}{f(z)}\right) = S(r,f)$

and $T(r,\psi) \leq (P + 1)\, T(r,f) + S(r,f)$.

We now give definition of differential polynomial. Let f(z) be a non constant meromorphic function in the complex plane. Let $a(z)$ be a meromorphic function in the plane satisfying $T(r, a(z)) = S(r,f)$ as $r \to \infty$, then a finite sum of the form $a(z)(f(z))^{l_0} \ldots (f^{(k)}(z))^{l_k}$ is called a differential polynomial of degree atmost n, and it is denoted by $P_n(f)$, where $l_o + l_1 + l_2 + \ldots + l_k \leq n$. And if for all terms constituting $P_n(f)$, $l_o + l_1 + \ldots + l_k = n$, then $P_n(f)$ is called a homogeneous differential polynomial of degree n.

A.P.Singh and G.P.Barker have obtained some properties of differential polynomials using Nevanlinna theory and then they have used those properties for the study of differential equation involving meromorphic functions, their derivatives and differential polynomials.

The concept of absolute defect of a with respect to the derivative f' was introduced by H. Milloux. This definition was later extended by Xiong Qing-Lai. He introduced the term

$$\delta_r^{(k)}(a,f) = 1 - \limsup_{t \to \infty} \frac{N(t, 1/f^{(k)}-a)}{T(t,f)}$$

and called it as the relative defect of the value a with respect to the derivative $f^{(k)}$. And the usual defect with respect to $f^{(k)}$ namely $\delta_a^{(k)}(a,f) = 1 - \limsup_{r \to \infty} \frac{N(r, 1/f^{(k)}-a)}{T(r,f^{(k)})}$

was called the absolute defect of the value a with respect to $f^{(k)}$. He found various relation between these two defects.

Later A.P.Singh defined the relative defects corresponding to distinct zeros and distinct poles and found various relation between these. Also he found some relation involving the relative defects corresponding to the common roots of two meromorphic functions. He used following notations and terminology.

Let $f_1(z)$, $f_2(z)$ be two non constant meromorphic functions and a be any complex number. Let $n_o(r,a)$ denote the number of common roots in the disk $|z| \leq r$ of the two equation $f_1(z) = a$ and $f_2(z) = a$ and let $\bar{n}_o(r,a)$ denote the number of common roots in the disk $|z| \leq r$ of the two equation $f_1(z) = a$ and $f_2(z) = a$, where the multiplicity is disregarded.

$$\bar{N}_o(r,a) = \int_0^r \frac{\bar{n}_o(t,a)-\bar{n}_o(0,a)}{t}\, dt + \bar{n}_o(0,a) \log r$$

$$\bar{N}_{1,2}(r,a) = \bar{N}\left(r, \frac{1}{f_1-a}\right) + \bar{N}\left(r, \frac{1}{f_2-a}\right) - 2\bar{N}_o(r,a)$$

Let $\bar{n}_o^{(k)}(r,a)$, $\bar{N}_{1,2}^{(k)}(r,a)$ etc. denote the corresponding quantities with respect to $f_1^{(k)}$ and $f_2^{(k)}$

$$\Theta_{1,2}(a) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_{1,2}(r,a)}{T(r,f_1)\, T(r,f_2)}$$

$$\Theta^{(k)}_{1,2}(a) = 1 - \limsup_{r \to \infty} \frac{\bar{N}^{(k)}_{1,2}(r,a)}{T(r,f_1) + T(r,f_2)}$$

$$\delta_{1,2}(a) = 1 - \limsup_{r \to \infty} \frac{N_{1,2}(r,a)}{T(r,f_1) + T(r,f_2)}$$

Using Nevanlinna theory A.P.Singh has proved a number of theorems on differential equations.

Another interesting result was proved by valiron that the order of $f'(z)$ does not exceed that of $f(z)$. Whittaker proved that the two orders are actually the same.

Later L.R. Sons [21] proved if $f(z)$ is transcendental meromorphic function of finite order $\rho(f)$, then $\rho(f) = \rho(\phi)$ where $\phi(z)$ is a monomial given by

$$\phi(z) = (f(z))^{l_o} (f'(z))^{l_1} \dots (f^{(k)}(z))^{l_k}$$

where $l_o \geqslant 1$, $l_k \geqslant 1$, $l_i \geqslant 0$, $1 \leqslant i \leqslant k-1$.

This theorem was later extended for certain types of homogeneous differential polynomial and for functions f of even infinite order by A.P.Singh [16] , who proved that if $f(z)$ is a transcendental meromorphic function and $\phi(z)$ is a non-zero homogeneous differential polynomial of degree n, and such that each term of $\phi(z)$ contains f as its factor, then order of f equals order of ϕ .

A theorem similar to the above with a slightly different hypothesis was proved by Bhoosnurmath and Gopalkrishna [4]. They proved : If f is a meromorphic function satisfying $\bar{N}(r,f) + \bar{N}(r, \frac{1}{f}) = S(r,f)$ and if P is a homogeneous differential polynomial in f which does not reduce to a constant, then the order of P equals the order of f and $\bar{N}(r,P) + \bar{N}(r, \frac{1}{P}) = S(r,P)$ so that $\bar{N}(r, \frac{1}{P-a}) \neq S(r,P)$ and $\Theta(a,P) = 0$ for all $a \in \bar{C} - \{0, \infty\}$. Also, no element of $\bar{C} - \{0, \infty\}$ is an e.v.B. for P for distinct zeros."

We know the following theorem proved by Hayman.

<u>Theorem</u> : If f is a meromorphic function and m is a positive integer than either f has no evp in C or $f^{(m)}$ has no evp in C except possibly zero. Later in 1977 H.S.Gopalkrishna and Subhas S. Bhoosnurmath extended the above theorem that is they have proved the theorem for differential polynomial in f of degree $n \geqslant 2$ also they have extended the above theorem for certain linear combinations in the successive derivative of f.

Using comparison function r^{ϱ} L(r), S.M.Sarangi and S.J. Patil have obtained the bounds for $m(r, \frac{1}{f_1-g})$ and $N(r, \frac{1}{f_1-g})$ which are satisfied except possibly for certain exceptional function g(z)

S.K.Singh and V.N.Kulkarni have obtained relation between $T(r,f)$ and $T(r,f')$ where $T(r,f)$ and $T(r,f')$ are the nevanlinna's characteristic functions of the meromorphic functions $f(z)$ and $f'(z)$ respectively. Also results pertaining to Nevanlinna exceptional values have been established and bounds for $K(f')$ in terms of Nevanlinna defects have been given, where

$$K(f') = \limsup_{r \to \infty} \frac{N(r,f') + N(r,1/f')}{T(r,f')}$$

R. Parthasarathy has obtained bounds for $\bar{n}(r,w,\psi)$ $\bar{N}(r,w,\phi)$. Using the comparison function $r^{\varrho} L(r)$ and assuming certain growth estimate on f, which are satisfied **except** for certain exceptional values of w.

In the present dissertation we study application of Nevanlinna theory to the differential equation, homogeneous differential polynomial and their deficient values and the derivatives of meromorphic functions. Here we have extended the several results of W.K.Hayman, S.K.Singh, V.N.Kulkarni, A.P.Singh and G.P.Barker.

The Second Chapter deals with deficient values of meromorphic functions and their derivatives, where we have extended result of W.K.Hayman viz, $\sum_{a\neq\infty} \delta(a,f) + \delta(0,f(z)-z^k) \leq 1$

Also we have obtained an upper bound for $\delta(0,\ f(z)-z^k)$ in terms of deficient values of the derivative $f^{(k)}$ of f under certain conditions imposed on the zeros and poles of f. See theorem 2.2. Besides these we have found an upper bounds of $\delta(0,\ P_n(f)\ )$ where $P_n(f)$ is a monomial of degree n, for instance we have shown in theorem 2.4.

Theorem 2.4 : If f is an entire function of finite order λ which is not a positive integer and $P_n(f)$ is a differential monomial not containing f that is,

$$P_n(f) = (f')^{l_1}\ (f'')^{l_2}\ \ldots.\ (f^{(k)}\ )^{l_k}$$

where $l_1 + l_2 + l_3 + \ldots + l_k = n$, And if $\bar{N}\ (r,f)+\bar{N}(r,\frac{1}{f}) = S(r,f)$

then

$$\delta\ (0,\ P_n) \leqslant 1 - K(\ \lambda\ )$$

where $K\ (\ \lambda\ ) \geqslant 1-\lambda$, if $0 < \lambda < 1$

$$\geqslant (q+1-\lambda\ )(\ \lambda\ -q)/2\ \lambda\ (q+1)\quad \{2 + $$

$$+ \log\ (q+1)\}\quad \text{if}\quad \lambda > 1$$

and $q = [\lambda]$

In the remaining portion of this chapter we have proved some theorems dealing with relative defects of meromorphic function (see theorem 2.6, which is an extension of theorem of A.P. Singh [14] . Finally we have ended this chapter with one more theorem dealing with the common roots of two meromorphic functions.

The third and last Chapter deals with the growth of differential polynomial. In this chapter we have shown $T(r, \pi_n(f)) \sim nT(r,f)$ under certain conditions. Also we have found relation between deficient values of entire function with that of its derivative, for instance in theorem 3.4 we have proved "If f(z) is an entire, function of finite order then

$$\sum_{a \neq \infty} \delta(a,f) \leqslant \delta(0, f^{(k)})$$

which is an extension of theorem 4.6 of W.K.Hayman. We have also obtained result dealing with Nevanlinna characteristic of f and Nevanlinna characteristic of $f^{(k)}$ in theorem 3.5 and 3.6 and have ended the Chapter by giving applications of Nevanlinna theory to differential equations.

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