CHAPTER - II

DEFICIENT VALUES OF MEROMORPHIC FUNCTIONS AND THEIR DERIVATIVES.

CHAPTER - II

Deficient values of meromorphic functions and their derivatives.

1. Notation, Terminology :

Let $f(z)$ be a transcendental meromorphic function in the complex plane. As in Chapter I we define $n(r,a) = n(r,a,f)$, the number of roots with due count of multiplicity, of the equation $f(z) = a$ in $|z| \leq r$ and by $n(r,\infty)$ the number of poles of $f(z)$ in $|z| \leq r$ $\bar{n}(r,a)$ the number of distinct roots of $f(z) = a$ in $|z| \leq r$. And if f_1 and f_2 are two non-constant meromorphic functions in the plane, then let $n_o(r,a)$ denote the number of common roots in the disk $|z| \leq r$ of the two equations $f_1(z)=a$ and $f_2(z) = a$, and let $\bar{n}_o(r,a)$ denote the number of common roots in the disk $|z| \leq r$ of the two equations $f_1(z) = a$ and $f_2(z) = a$ where the multiplicity is disregarded. Set

$$N(r,a) = N(r,a,f) = \int_0^r \frac{n(t,a) - n(0,a)}{t}\,dt + n(0,a)\log r$$

$$\bar{N}(r,a) = \bar{N}(r,a,f) = \int_0^r \frac{\bar{n}(t,a) - \bar{n}(0,a)}{t}\,dt + \bar{n}(0,a)\log r$$

$$\bar{N}_o(r,a) = \int_0^r \frac{\bar{n}_o(t,a) - \bar{n}_o(0,a)}{t}\,dt + \bar{n}_o(0,a)\log r$$

$$\bar{N}_{1,2}(r,a) = \bar{N}\left(r, \frac{1}{f_1-a}\right) + \bar{N}\left(r, \frac{1}{f_2-a}\right) - 2\bar{N}_o(r,a)$$

$$\Theta_{1,2}(a) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_{1,2}(r,a)}{T(r,f_1) + T(r,f_2)}$$

$$\Theta^{(k)}_{1,2}(a) = 1 - \limsup_{r \to \infty} \frac{\bar{N}^{(k)}_{1,2}(r,a)}{T(r,f_1) + T(r,f_2)}$$

$$\delta_{1,2}(a) = 1 - \limsup_{r \to \infty} \frac{N_{1,2}(r,a)}{T(r,f_1) + T(r,f_2)}$$

$$\Theta_o(a) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_o(r,a)}{T(r,f_1) + T(r,f_2)}$$

The term $S(r,f)$ will denote any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ except possibly for a set of r of finite linear measure.

A differential polynomial $\pi_n(f)$ is a finite sum of the form $a(z)(f(z))^{l_o} (f'(z))^{l_1} \ldots (f^{(k)}(z))^{l_k}$ where $l_0 + l_1 + l_2 + \ldots + l_k = n$. And if for all the terms constituting $\pi_n(f)$, $l_0 + l_1 + \ldots l_k = n$ then $\pi_n(f)$ is called a homogeneous differential polynomial of degree n. In case $\pi_n(f) = a(z)(f(z))^{l_0} (f'(z))^{l_1} \ldots (f^{(k)}(z))^{l_k}$ with $l_0 + l_1 + \ldots l_k = n$, $\pi_n(f)$ will be called a monomial of degree n

Theorem 2.1 : For any transcendental entire function f(z) and $k \geq 1$

$$\sum_{a \neq \infty} \delta(a,f) + \delta(0, f(z) - z^k) \leq 1$$

Proof : Let f(z) be transcendental entire function and let $a_1, a_2, \ldots a_q \in \mathbb{C}$ be distinct.

$$\text{Let } F(z) = \sum_{\nu=1}^{q} \frac{1}{f(z) - a_\nu}$$

then

$$\sum_{\nu=1}^{q} m\left(r, \frac{1}{f(z) - a_\nu}\right) \leq m(r,F) + O(1)$$

$$= m\left(r, \frac{Ff'}{f'}\right) + O(1)$$

$$= m\left(r, \sum \frac{f'}{f-a_\nu}\right) + m\left(r, \frac{1}{f'}\right) + O(1)$$

Thus, using Milloux theorem we get

$$\sum_{\nu=1}^{q} m\left(r, \frac{1}{f(z)-a_\nu}\right) \leq m\left(r, \frac{1}{f'}\right) + o\,(T(r,f)) \quad \ldots (2.1)$$

$$\leq m\left(r, \frac{1}{f^{(k)}}\right) + o\,(T(r,f))$$

for all $k \geq 1$, and outside a set of finite linear measure.

Thus, we have

$$\sum_{\nu=1}^{q} m\left(r, \frac{1}{f(z)-a_\nu}\right) \leq m\left(r, \frac{1}{f'(z)}\right) + o\left(T(r,f)\right) \quad \ldots (2.2)$$

Outside a set of finite linear measure. By app lying this result to $f(z) - z$ with $q = 1$, $a_1 = 0$ we deduce similarly that

$$m\left(r, \frac{1}{f(z)-z}\right) \leq m\left(r, \frac{1}{f'(z)-1}\right) + o\left(T(r, f(z) - z)\right)$$

$$\leq m\left(r, \frac{1}{f'(z)-1}\right) + o\left(T(r,f)\right)$$

Since $T(r,f(z) - z) = T(r,f) + O(\log r) \sim T(r,f)$

Consider $m\left(r, \frac{1}{f(z)-z^k}\right)$

Applying result 2.2 to $m\left(r, \frac{1}{f(z)-z^k}\right)$

We get

$$m\left(r, \frac{1}{f(z)-z^k}\right) \leq m\left(r, \frac{1}{f'(z)-kz^{k-1}}\right) + o\left(T(r, f(z)-z^k)\right)$$

$$= m\left(r, \frac{1}{f'(z)-kz^{k-1}}\right) + o\left(T(r,f)\right).$$

Similarly

$$m\left(r, \frac{1}{f'(z)-kz^{k-1}}\right) \leqslant m\left(r, \frac{1}{f''(z)-(k)(k-1)z^{k-2}}\right) + o(T(r,f'))$$

$$= m\left(r, \frac{1}{f''(z)-(k)(k-1)z^{k-2}}\right) + o(T(r,f)$$

and

$$m\left(r, \frac{1}{f''(z)-(k)(k-1)z^{k-2}}\right) \leqslant m\left(r, \frac{1}{f'''(z)-(k)(k-1)(k-2)z^{k-3}}\right.$$

$$+ o(T(r,f''))$$

$$= m\left(r, \frac{1}{f'''(z)-(k)(k-1)(k-2)z^{k-3}}\right)$$

$$+ \ o(T(r,f)\ ).$$

Continuing by induction we obtain

$$m\left(r, \frac{1}{f^{k-1}-(k)(k-1)\ldots z}\right) \leqslant m\left(r, \frac{1}{f^{k}(z) - (k)(k-1)\ldots z^{k-k}}\right)$$

$$+ o(T(r,f)\ ).$$

Thus,

$$m\left(r, \frac{1}{f(z)-z^{k}}\right) \leqslant m\left(r, \frac{1}{f^{(k)}(z)-k!}\right) + o(T(r,f)\ ) \quad \ldots(2.3)$$

Applying (2.1) to $f^{(k)}$ with $q = 2$ and $a_1 = 0$ $a_2=k$ we get

$$m\left(r, \frac{1}{f^{k}(z)}\right) + m\left(r, \frac{1}{f^{k}(z) - k!}\right)$$

$$\leq m\left(r, \frac{1}{f^{(k+1)}(z)}\right) + o(T(r,f^{k})$$

$$= T\left(r, \frac{1}{f^{(k+1)} z}\right) + o(T(r,f^{k}))$$

since f is entire function.

By Nevanlinna's first fundamental theorem it follows that

$$m\left(r, \frac{1}{f^{(k)}(z)}\right) + m\left(r, \frac{1}{f^{(k)}(z) - k!}\right) \leq T(r, f^{(k+1}(z)) +$$

$$+ o(T(r,f^{k})$$

$$= m(r, f^{(k+1)}(z)) + o(T(r,f^{k}))$$

$$= m\left(r, \frac{f^{(k+1)}(z)}{f^{(k)}(z)} f^{(k)}(z)\right) + o(T(r,f^{k}))$$

$$\leq m(r,f^{k}(z)) + o(T(r,f^{k}))$$ using Milloux

theorem.

Thus

$$m\left(r, \frac{1}{f^{(k)}(z)}\right) + m\left(r, \frac{1}{f^{(k)}(z)-k!}\right) \leq \{1 + o(1)\} \quad T(r,f^{k}) \ldots (2.4)$$

outside a set of finite linear measure.

On combining (2.4), (2.3) and (2.1) we deduce that outside a set of finite linear measure

$$\sum_{\nu=1}^{q} m(r, \frac{1}{f(z)-a_\nu}) + m(r, \frac{1}{f(z)-z^k})$$

$$\leqslant m(r, \frac{1}{f^k}) + m(r, \frac{1}{f^k - k!}) + o(T(r,f)).$$

And so

$$\sum_{\nu=1}^{q} m(r, \frac{1}{f(z)-a_\nu}) + m(r, \frac{1}{f(z)-z^k}) \leqslant \{1+o(1)\} \quad T(r, f^{(k)}) +$$

$$+ o(T(r,f) \qquad \ldots (2.5)$$

Dividing by $T(r,f)$ and taking limit inferior as $r \to \infty$ of (2.5) and observing that f is entire it follows that

$$\sum_{\nu=1}^{q} \delta(a_\nu, f) + \delta(0, f(z) - z^k) \leqslant 1$$

Since q is arbitrary making $q \to \infty$ we get

$$\sum_{\nu=1}^{\infty} \delta(a_\nu, f) + \delta(0, f(z) - z^k) \leqslant 1$$

Also since the $\{a_\nu / \delta(a_\nu, f) > 0\}$ is countable it follows that

$$\sum_{\nu=1}^{\infty} \delta(a_\nu, f) = \sum_{a \neq \infty} \delta(a, f)$$

Therefore

$$\sum_{a \neq \infty} \delta(a,f) + \delta(0,\ f(z) - z^k) \leq 1$$

This proves the theorem.

<u>Remark</u> :

1. Putting k = 1, we obtain inequality 4.16 of Hayman [7, 89]

2. In the above theorem we have obtained an upper bound for $\delta(0,\ f(z) - z^k)$ in terms of deficient values of f. We can also find an upper bound for $\delta(0,\ f(z) - z^k)$ in terms of deficient values of the derivatives $f^{(k)}$ of f under certain conditions imposed on the zeros and poles of f, Thus we shall prove

<u>Theorem 2.2</u> : If f(z) is transcendental meromorphic function with $N(r,f) + N(r\ \frac{1}{f}) = S(r,f)$ as $r \to \infty$ and $k \geqslant 1$ then

$$\delta\ (0,\ f(z) - z^k) \leq \delta\ (K!\ ,\ f^{(k)}\ (z))$$

<u>Proof</u> : Let f(z) be transcendental function and let $a_1, a_2, \text{---}\ a_q \in \mathbb{C}$ be distinct.

$$\text{Let } F(z) = \sum_{\nu=1}^{q} \frac{1}{f(z) - a_\nu}$$

then

$$\sum_{\nu=1}^{q} m\left(r, \frac{1}{f(z)-a_\nu}\right) \leq m(r, F) + O(1)$$

$$= m\left(r \frac{Ff'}{f'}\right) + O(1)$$

$$= m\left(r, \sum \frac{f'}{f-a_\nu}\right) + m\left(r, \frac{1}{f'}\right) + O(1).$$

Thus by using Milloux's theorem we get

$$\sum_{\nu=1}^{q} m\left(r, \frac{1}{f(z)-a_\nu}\right) \leq m\left(r, \frac{1}{f'}\right) + o\,(T(r,f)).$$

Once again by Milloux's theorem it follows that

$$\sum_{\nu=1}^{q} m\left(r, \frac{1}{f(z)-a_\nu}\right) \leq m\left(r, \frac{1}{f^{(k)}}\right) + o(T(r,f)) \quad \ldots (2.6)$$

for all $K \geq 1$

Outside a set of finite linear measure.

By applying the above result to $f(z)-z$ with $q = 1$, $a_1 = o$ we deduce similarly that

$$m\left(r, \frac{1}{f(z)-z}\right) \leq m\left(r, \frac{1}{f'(z)-1}\right) + o\,(T(r, f(z) - z))$$

and so

$$m\left(r, \frac{1}{f(z)-z}\right) \leq m\left(r, \frac{1}{f'(z)-1}\right) + o(T(r,f)) \quad \ldots (2.7)$$

Since $T(r,f(z)-z) = T(r,f) + O(\log r) \sim T(r,f)$.

Consider $m\left(r, \dfrac{1}{f(z)-z^k}\right)$

applying result (2.7) to $m\left(r, \dfrac{1}{f(z)-z^k}\right)$ we get

$$m\left(r, \frac{1}{f(z)-z^k}\right) \leqslant m\left(r, \frac{1}{f'(z)-kz^{k-1}}\right) + o(T(r,f(z)-z^k))$$

$$= m\left(r, \frac{1}{f'(z)-k\,z^{k-1}}\right) + o(T(r,f))$$

$$m\left(r, \frac{1}{f'(z)-kz^{k-1}}\right) \leqslant m\left(r, \frac{1}{f''(z)-(k)(k-1)z^{k-2}}\right) + o(T(r,f'(z)))$$

$$= m\left(r, \frac{1}{f''(z)-(k)(k-1)z^{k-2}}\right) + o(T(r,f))$$

$$m\left(r, \frac{1}{f''(z)-(k)(k-1)z^{k-2}}\right) \leqslant m\left(r, \frac{1}{f'''(z)-(k)(k-1)(k-2)z^{k-3}}\right)$$

$$+ o(T(r,f))$$

Continuing this way we get

$$m\left(r, \frac{1}{f^{(k-1)}(z)-(k)(k-1)\ldots z}\right) \leqslant m\left(r, \frac{1}{f^{(k)}(z)-(k)(k-1)\ldots z^{k-k}}\right)$$

$$+ o(T(r,f))$$

Thus

$$m\left(r, \frac{1}{f(z)-z^k}\right) \leqslant m\left(r, \frac{1}{f^{(k)}(z)-k!}\right) + o(T(r,f)) \quad \ldots (2.8)$$

Now dividing by T(r,f) to (2.8) and taking limit inferior we get

$$\liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f(z)-z^k}\right)}{T(r,f)} \leqslant \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f^{(k)}(z)-k!}\right)}{T(r,f^{(k)})} \cdot \frac{T(r,f^{(k)})}{T(r,f)} \quad \ldots (2.9)$$

But $T(r,f^{(k)}) = m(r,f^{(k)})$

$$m\left(r, \frac{f^{(k)}}{f}\right) + m(r,f)$$

$$= m(r,f) + S(r,f)$$

and so $\dfrac{T(r,f^{(k)})}{T(r,f)} \leqslant 1$ as $r \to \infty$

Thus (2.9) yields

$$\liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f(z)-z^k}\right)}{T(r,f)} \leqslant \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f^{(k)}(z)-k!}\right)}{T(r,f^{(k)})}$$

$$\limsup_{r\to\infty} \frac{T(r,f^{(k)})}{T(r,f)}$$

$$\liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f^{(k)}(z)-k!}\right)}{T(r,f^{(k)})}$$

And so

$$\delta(0, f(z)-z^k) \leq \delta(k! \;\; f^{(k)}).$$

Remark : Since for an entire function $N(r,f) = 0 = S(r,f)$ an immediate consequence of the above theorem is the following corollary.

Corollary 2.1 : If $f(z)$ is a transcendental entire function of finite order with $N(r, \frac{1}{f}) = S(r,f)$ as $r \to \infty$ Then $\delta(0, f(z) - z^k) \leq \delta(k!, f^{(k)}(z))$.

In our next theorem we shall find a bound for proximity function of the monomial of $f(z) - z^k$ with the proximity function of $\frac{1}{f^{(k)} - k!}$. Thus we shall prove :

Theorem 2.3 : If $f(z)$ is a transcendental entire function and if $P_n(f)$ denotes a monomial of degree n not containing f, that is if $P_n(f)$ is of the form

$$(f')^{l_1} (f'')^{l_2} \ldots. (f^{(k)})^{l_k}$$

where $l_1 + l_2 + \ldots. + l_k = n$, then

$$m\left(r, \frac{1}{P_n(f(z)-z^k)}\right) \leq n.\, m\left(r, \frac{1}{f^{(k)}(z)-k!}\right) + S(r,f)$$

Proof : Let $a_1, a_2 \ldots. a_q \in \mathbb{C}$ be distinct.

And let $F(z) = \sum_{\nu=1}^{q} \frac{1}{(f(z) - a_\nu)^n}$

then we have

$$n \sum_{\nu=1}^{q} m\left(r, \frac{1}{f - a_\nu}\right) \leqslant m(r,F) + O(1)$$

$$\leqslant m\left(r, \frac{FP_n(f)}{P_n(f)}\right) + O(1)$$

$$\leqslant m\left(r, \sum_{\nu=1}^{q} \frac{P_n(f)}{(f-a_\nu)^n}\right) + m\left(r, \frac{1}{P_n}\right) + O(1)$$

$$= m\left(r, \sum_{\nu=1}^{q} \frac{(f')^{l_1} \cdots (f^{(k)})^{l_k}}{(f - a_\nu)^n}\right) + m\left(r, \frac{1}{P_n}\right) + O(1)$$

$$= m\left(r, \sum_{\nu=1}^{q} \left(\frac{f'}{f-a_\nu}\right)^{l_1} \left(\frac{f''}{f-a_\nu}\right)^{l_2} \cdots \left(\frac{f^{(k)}}{f-a_\nu}\right)^{l_k}\right) + m\left(r, \frac{1}{P_n}\right) + O(1)$$

$$\leqslant \sum_{\nu=1}^{q} m\left(r, \left(\frac{f'}{f-a_\nu}\right)^{l_1} \left(\frac{f''}{f-a_\nu}\right)^{l_2} \cdots \left(\frac{f^{(k)}}{f-a_\nu}\right)^{l_k}\right) + m\left(r, \frac{1}{P_n}\right) + O(1)$$

$$\leq \sum_{\nu=1}^{q} m(r,(\frac{f'}{f-a_\nu})^{l_1}) + m(r,(\frac{f''}{f-a_\nu})^{l_2}) + \ldots$$

$$\ldots + m(r,(\frac{f^{(k)}}{f-a_\nu})^{l_k}) + m(r,\frac{1}{P_n}) + O(1)$$

$$= \sum_{\nu=1}^{q} l_1 m(r,\frac{f'}{f-a_\nu}) + l_2 m(r,\frac{f''}{f-a_\nu}) + \ldots\ldots$$

$$\ldots + l_k m(r,\frac{f^{(k)}}{f-a_\nu}) + m(r,\frac{1}{P_n}) + O(1)$$

$$= \sum_{\nu=1}^{q} l_1 S(r,(f-a_\nu)) + l_2 S(r,(f-a_\nu)) + \ldots$$

$$\ldots l_k S(r,(f-a\quad)) + m(r,\frac{1}{P_n}) +$$

$$+ O(1).$$

using Milloux's theorem.

Therefore

$$n \sum_{\nu=1}^{q} m(r,\frac{1}{f-a_\nu}) \leq q(l_1 S(r,f) + l_2 S(r,f) + \ldots.$$

$$+ l_k S(r,f)) + m(r,\frac{1}{P_n}) + O(1)$$

$$= q(l_1 + l_2 + \ldots + l_k) S(r,f)$$

$$+ m(r,\frac{1}{P_n}) + O(1)$$

$$= nq\, S(r,f) + m(r, \frac{1}{P_n}) + o(1).$$

$$= m\,(r, \frac{1}{P_n}) + S(r,f).$$

Thus we have the following result :

$$n \sum_{\nu=1}^{q} m\,(r, \frac{1}{f-a_\nu}) \leq m(r, \frac{1}{P_n(f)}) + S(r,f) \qquad \ldots (2.10)$$

By applying above result to f(z) - z with $q = 1$, $a_1 = 0$ we get

$$nm\,(r, \frac{1}{f(z)-z}) \leq m\,(r, \frac{1}{P_n(f(z)-z)}) + S(r, f(z)-z\,)$$

But $S(r, f(z) - z) = S(r,f)$.

Therefore,

$$n\, m\,(r, \frac{1}{f(z)-z}) \leq m\,(r, \frac{1}{P_n(f(z)-z}) + S(r,f)$$

where

$$P_n\,(f(z)-z) = ((f(z)-z)')^{l_1}\,((f(z)-z)'')^{l_2} \ldots.((f(z)-z)^{(k)})^{l_k}$$

$$= (f'(z)-1)^{l_1}\,(f'')^{l_2} \ldots.(f^{(k)})^{l_k}\,.$$

Therefore

$$n\, m\,(r, \frac{1}{f(z)-z}) \leq m(r, \frac{1}{(f'(z)-1)^{l_1}(f'')^{l_2} \ldots(f^{(k)})^{l_k}}) + S(r,f).$$

Now consider

$$m\left(r, \frac{1}{f(z) - z^k}\right)$$

then as in (2.10)

$$n\, m\left(r, \frac{1}{f(z)-z^k}\right) \leq m\left(r, \frac{1}{P_n(f(z)-z^k}\right) + S(r,f)$$

where

$$P_n\,(f(z) - z^{(k)} = (\,(f(z)-z^{(k)})')^{l_1}\,((f(z)-z^{(k)})'')^{l_2}\;\ldots.$$

$$\ldots\ldots\;((\,f(z) - z^{(k)}\,)^{(k)}\,)^{l_k}$$

$$= (f'-kz^{k-1})^{l_1}\,(f''-(k)(k-1)z^{k-2})^{l_2}\ldots$$

$$\ldots.\;(f^{(k)} - k!\,)^{l_k}$$

Next,

$$m\left(r, \frac{1}{P_n(f(z)-z^k)}\right) = m\left(r, \frac{1}{(f'-kz^{k-1})^{l_1}\,(f''-k(k-1)z^{k-2})^{l_2}\ldots\ldots(f^{(k)}-k!\,)^{l_k}}\right.$$

$$\leq m\left(r, \frac{1}{(f'-kz^{k-1})^{l_1}}\right) + m\left(r, \frac{1}{(f''-k(k-1)z^{k-1})^{l_2}}\right) +$$

$$\ldots + m\left(r, \frac{1}{(f^{(k)}-k!\,)^{l_k}}\right)$$

...

$$= l_1 m\left(r, \frac{1}{(f'-kz^{k-1})}\right) + l_2\, m\left(r, \frac{1}{f''-(k)(k-1)z^{k-2}}\right)$$

$$+ \ldots\ldots + l_k\, m\left(r, \frac{1}{f^{(k)} - k!}\right) \qquad \ldots (2.11)$$

Now consider the first term $m\left(r, \frac{1}{(f'-kz^{k-1})^{l_1}}\right)$ of (2.11).

$$m\left(r, \frac{1}{(f'-kz^{k-1})^{l_1}}\right) = m\left(r, \frac{((f'-kz^{k-1})')^{l_1}}{(f'-kz^{k-1})^{l_1}} \cdot \frac{1}{((f'-kz^{k-1})')^{l_1}}\right)$$

$$\leq m\left(r, \left(\frac{(f'-kz^{k-1})'}{(f'-kz^{k-1})}\right)^{l_1}\right) + m\left(r, \frac{1}{((f'-kz^{k-1})')^{l_1}}\right)$$

$$= l_1 m\left(r, \frac{(f'-kz^{k-1})'}{(f'-kz^{k-1})}\right) + l_1 m\left(r, \frac{1}{f''-k\ k-1z^{k-2}}\right)$$

$$= l_1 S(r, (f'-kz^{k-1})) + l_1 m\left(r, \frac{1}{f''-(k)(k-1)z^{k-2}}\right)$$

$$= l_1 S(r, f) + l_1 m\left(r, \frac{1}{f''-(k)(k-1)z^{k-2}}\right).$$

Therefore we get

$$m\left(r, \frac{1}{(f'-kz^{k-1})^{l_1}}\right) \leq l_1\, m\left(r, \frac{1}{f''-(k)(k-1)z^{k-2}}\right) + S(r,f)$$

$$l_1 m\left(r, \frac{1}{f''-(k)(k-1)z^{k-2}}\right) \leq l_1\, m\left(r, \frac{1}{f'''-(k)(k-1)(k-2)z^{k-3}}\right) +$$

$$+ S(r,f).$$

Continuing this way we get

$$m\left(r, \frac{1}{(f'-kz^{k-1})^{l_1}}\right) \leq l_1 m\left(r, \frac{1}{f^{(k)} - k!}\right) + S(r,f) \quad \ldots (2.12)$$

Next consider second term of (2.11)

$$m\left(r, \frac{1}{(f''-(k)(k-1)z^{k-2})^{l_2}}\right) = m\left(r, \frac{((f''-(k)(k-1)z^{k-2})')^{l_2}}{(f''-(k)(k-1)z^{k-2})^{l_2}} \cdot\right.$$

$$\left. \cdot \frac{1}{((f''-(k)(k-1)z^{k-2})')^{l_2}}\right)$$

$$\leq m\left(r, \left(\frac{(f''-(k)(k-1)z^{k-2})'}{(f''-(k)(k-1)z^{k-2})}\right)^{l_2}\right) +$$

$$+ m\left(r, \frac{1}{((f''-(k)(k-1)z^{k-2})')^{l_2}}\right)$$

$$= l_2 m\left(r, \frac{(f''-(k)(k-1)z^{k-2})'}{(f''-(k)(k-1)z^{k-2}}\right) +$$

$$+ l_2\, m\left(r, \frac{1}{(f''-k(k-1)z^{k-2})'}\right)$$

$$\leqslant \quad l_2 S(r, f''-(k)(k-1)z^{k-2}) +$$

$$+ l_2\, m\left(r, \frac{1}{(f''-(k)(k-1)z^{k-2})'}\right)$$

$$= l_2(S)(r,f) + l_2 m\left(r, \frac{1}{f''' - (k)(k-1)k-2)z^{k-3}}\right).$$

Thus we get the result of the type

$$m\left(r, \frac{1}{(f''-(k)(k-1)z^{k-2})^{l_2}}\right) \leqslant l_2 m\left(r, \frac{1}{f'''-(k)(k-1)(k-2)z^{k-3}}\right) +$$

$$+ S(r,f)$$

Continuing this way we get

$$m\left(r, \frac{1}{(f''-(k)(k-1)z^{k-2})^{l_2}}\right) \leqslant l_2 m\left(r, \frac{1}{f^{(k)}-k!}\right) + S(r,f) \ldots (2.13)$$

Next similarly considering third term of (2.11) we get

$$m\left(r, \frac{1}{(f'''-(k)(k-1)(k-2)z^{k-3})^{l_3}}\right) \leqslant l_3\, m\left(r, \frac{1}{f^{(k)}-k!}\right) + S(r,f). \quad \ldots (2.14)$$

Thus adding all such terms (2.12), (2.13), (2.14), the inequality (2.11) gives

$$m\left(r, \frac{1}{P_n(f(z)-z^{(k)})}\right) \leqslant l_1 m\left(r, \frac{1}{f^{(k)}-k!}\right) + l_2 m\left(r, \frac{1}{f^{(k)}-k!}\right) + \ldots$$

$$\ldots + l_k m\left(r, \frac{1}{f^{(k)}-k!}\right) + S(r,f)$$

$$= (l_1 + l_2 + \ldots + l_k)\left(m\left(r, \frac{1}{f^{(k)}-k!}\right)\right) + S(r,f)$$

Thus

$$m\left(r, \frac{1}{P_n(f(z)-z^{(k)})}\right) \leqslant n\, m\left(r, \frac{1}{f^{(k)}-k!}\right) + S(r,f).$$

This completes the proof.

In our next two theorems we find an upper bounds of $\delta(0, P_n(f)$ where $P_n(f)$ is a monomial in f of degree n.

Theorem 2.4 : If f is an entire function of finite order which is not a positive integer and $P_n(f)$ is a differential monomial not containing f, that is,

$$P_n(f) = (f')^{l_1} (f'')^{l_2} \dots (f^{(k)})^{l_k}$$

where $l_1 + l_2 + l_3 + \dots + l_k = n$, And if $\bar{N}(r,f)+\bar{N}(r,\frac{1}{f})=S(r,f)$

then

$$\delta(0, P_n) \leq 1 - k(\lambda)$$

where $K(\lambda) \geqslant 1-\lambda$, if $0 < \lambda < 1$

$$\geqslant (q+1-\lambda)(\lambda-q)/2\lambda(q+1)\{2+\log(q+1)\} \text{ if } \lambda > 1$$

and $q = [\lambda]$

For the proof of this theorem we shall need the following lemma [7, 101]

<u>Lemma 2.1</u> : Suppose that f(z) is meromorphic in the plane and of finite order λ where λ is not a positive integer then

$$K(f) = \limsup_{r \to \infty} \frac{N(r,f) + N(r,\frac{1}{f})}{T(r,f)} \geqslant K(\lambda)$$

where $k(\lambda) \geqslant 1-\lambda$ if $0 < \lambda < 1$

$$\geqslant (q+1-\lambda)(\lambda-q)/2\lambda(q+1)\{2+\log(q+1)\}$$

if $\lambda > 1$ and where $q = [\lambda]$

<u>Proof of theorem 2.4</u> : By theorem of Gopalkrishna and Bhoosnurmath [4]

we have $\rho(P_n(f)) = \rho(f)$

$= \lambda$

Therefore by Lemma 2.1

$$K(P_n(f)) = \limsup_{r \to \infty} \frac{N(r,P_n) + N(r, \frac{1}{P_n})}{T(r,P_n)} \geqslant K(\lambda) \quad \ldots(2.15)$$

But $N(r, P_n) = S(r,f)$ since $N(r,f) = S(r,f)$.

Therefore (2.15) becomes

$$\limsup_{r \to \infty} \frac{N(r, 1/P_n)}{T(r,P_n)} \geqslant K(\lambda)$$

And so

$$1 - \limsup_{r \to \infty} \frac{N(r, 1/P_n)}{T(r,P_n)} \leqslant 1 - K(\lambda)$$

Therefore $\delta(0, P_n) \leqslant 1 - K(\lambda)$

which completes the proof.

<u>Remark</u> : In the above theorem we have proved the result when $P_n(f)$ did not contain f as its factor. If $P_n(f)$ contains the factor f then also the above result is true. In fact in this case we get the result even for homogeneous

differential polynomials. Thus we shall prove the next theorem.

Theorem 2.5 : Let $f(z)$ be a meromorphic function of finite order λ where λ is not a positive integer satisfying $N(r,f) = S(r,f)$. Let $\pi_n(f)$ be a nonzero homogeneous differential polynomial of degree n of the form $\pi_n(f) = a_0(f)^{\nu_0}(f')^{\nu_1}\ldots$

$$\ldots (f^{(i)})^{\nu_i} + b_0 (f)^{\mu_0} (f')^{\mu_1}\ldots (f^{(j)})^{\mu_j} + \ldots\ldots$$

$$\ldots + c_0(f)^{\delta_0} (f')^{\delta_1}\ldots (f^{(k)})^{\delta_k}.$$

where $\nu_0 + \nu_1 + \ldots\ldots = \mu_0 + \mu_1 + \ldots\ldots = \delta_0 + \delta_1 + \ldots = n$

and a_0, b_0, c_0 are meromorphic functions satisfying

$T(r,a_0) = S(r,f)$, $T(r, b_0) = S(r,f)$ etc.

Then $\delta(0, \pi_n(f)) \leqslant 1 - k(\lambda)$

where $k(\lambda) \geqslant 1 - \lambda$ if $0 < \lambda < 1$

and $k(\lambda) \geqslant (q+1-\lambda)(\lambda - q)/2\lambda(q+1)\{2+\log(q+1)\}$

if $\lambda > 1$ and where $q = [\lambda]$

Proof : By theorem of A.P.Singh [16]

$$\rho(f) = \rho(\pi_n)$$

Therefore by lemma 2.1

$$K(\pi_n(f)) = \limsup_{r \to \infty} \frac{N(r,\pi_n(f)) + N(r, 1/\pi_n(f))}{T(r, \pi_n(f))} \geqslant K(\lambda) \qquad \ldots (2.16)$$

Now $N(r, \pi_n(f)) = S(r,f)$ since $N(r,f) = S(r,f)$.

And so (2.16) becomes

$$K(\pi_n(f)) = \limsup_{r \to \infty} \frac{N(r, 1/\pi_n(f))}{T(r, \pi_n(f))} \geqslant K(\lambda)$$

Therefore $1 - \delta(0, \pi_n(f)) \geqslant K(\lambda)$.

Thus $\delta(0, \pi_n(f)) \leqslant 1 - K(\lambda)$

which completes the proof.

In the remaining portion of this chapter we shall prove some theorems dealing with relative defects of meromorphic functions. We begin with the following theorem.

Theorem 2.6 : Let $f(z)$ be a transcendental meromorphic function and $P(z)$ be a polynomial in z of degree n. Then for any positive integer $K > n$

$$\delta_r^{(k)}(\infty, f) \leqslant \frac{3}{2} - \frac{1}{2}\left\{\delta(0,f) + \delta(0,f-p)\right\}$$

Proof : From the identity

$$\frac{p(z)}{f} = 1 - \frac{f-p(z)}{f^{(k)}} \cdot \frac{f^{(k)}}{f}$$

and Nevanlinna's first fundamental theorem it easily follows that

$$T(r, \frac{p(z)}{f}) \leqslant T(r, \frac{f^{(k)}}{f-p^{(z)}}) + T(r, \frac{f^{(k)}}{f}) + O(1)$$

Let $p(z) = a_0 + a_1 z + \ldots a_n z^n$ Then

$$T(r, \frac{a_0 + a_1 z + \ldots + a_n z^n}{f}) \leqslant T(r, \frac{f^{(k)}}{f-(a_0+\ldots+a_n z^n})$$

$$+N(r, \frac{f^{(k)}}{f}) + m(r, \frac{f^{(k)}}{f}) + O(1)$$

Using Milloux's theorem it follows that

$$T(r, \frac{a_0+a_1 z+\ldots+a_n z^n}{f}) \leqslant T(r, \frac{f^{(k)}}{f-(a_0+\ldots+a_n z^n}) +$$

$$+ N(r, \frac{f^{(k)}}{f}) + S(r,f)$$

$$= N(r, \frac{f^{(k)}}{f-(a_0+\ldots+a_n z^n}) +$$

$$+m(r, \frac{f^{(k)}}{f-(a_0+\ldots+a_n z^n}) + N(r, f^k) +$$

$$+ N(r, \frac{1}{f}) + S(r,f)$$

$$\leq N\left(r, \frac{f^{(k)}}{f-(a_0+\dots+a_nz^n)}\right) + N(r, f^{(k)})$$

$$+ N(r, \frac{1}{f}) + S(r, f-(a_0+a_1z+\dots a_nz^n))$$

$$+ S(r,f)$$

$$= N\left(r, \frac{f^{(k)}}{f-(a_0+\dots+a_nz^n)}\right) + N(r, f^{(k)}) +$$

$$+ N(r, \frac{1}{f}) + S(r,f)$$

$$= N\left(r, \frac{f^{(k)}}{f-p}\right) + N(r, f^{(k)}) + N(r, \frac{1}{f}) +$$

$$+ S(r,f)$$

Therefore

$$T(r,f) \leq N(r, f^{(k)}) + N(r, \frac{1}{f-p}) + N(r, f^{(k)}) + N(r, \frac{1}{f}) + S(r,f)$$

And so

$$T(r,f) \leq 2N(r, f^{(k)}) + N(r, \frac{1}{f-p}) + N(r, \frac{1}{f}) + S(r,f) \quad \dots(2.17)$$

Dividing by $T(r,f)$ and taking limit superior of (2.17) as $r \to \infty$ we get

$$1 \leq 2(1 - \delta_r^{(k)}(\infty, f)) + (1 - \delta(0, f-p)) + (1 - \delta(0,f))$$

Thus

$$1 \leq 4 - 2\,\delta_r^{(k)}(\infty, f) - \{\delta(0, f-p) - \delta(0, f)\}$$

which yields

$$2\delta_r^{(k)}(\infty, f) \leq 3 - \{\delta(0, f-p) + \delta(0, f)\}$$

and so

$$\delta_r^{(k)}(\infty, f) \leq \frac{3}{2} - \frac{1}{2}\{\delta(0, f) + \delta(0, f-p)\} \quad .$$

This proves Theorem 2.6.

Remark : A.P.Singh's theorem 2 [14] is a particular case of our theorem, which is obtained by taking $P(z) = C$, a constant polynomial.

If instead of considering all the zeros and poles of $f(z)$, we consider only the distinct zeros and poles, we obtain the following theorem.

Theorem 2.7 : Let $f(z)$ be a meromorphic function. Let K be any positive integer, then for all positive integers $P \geq 1$ and a_i $(i = 1, 2 \ldots p)$ finite distinct and non-zero complex numbers

$$\sum_{i=1}^{p} \Theta_r^{(k)}(a_i, f) \leq (P+1) - \{P\,\delta(0, f) + \Theta(\infty, f)\}$$

Proof : For the proof of this theorem we shall require inequality (2.18) mentioned below. This inequality has been proved in [14] . We give this portion in our result for sake of completeness.

From Nevanlinna's first fundamental theorem it easily follows that

$$T(r,f) \leq N(r, \frac{1}{f}) + m(r, \frac{f^{(k)}}{f}) + m(r, \frac{1}{f^{(k)}}) + S(r,f)$$

$$= N(r, \frac{1}{f}) + m(r, \frac{1}{f^{(k)}}) + S(r,f) \text{ by } [1, 55]$$

$$= N(r, \frac{1}{f}) + T(r, \frac{1}{f^{(k)}}) - N(r, \frac{1}{f^{(k)}}) + S(r,f)$$

$$= N(r, \frac{1}{f}) + T(r, f^{(k)}) - N(r, \frac{1}{f^{(k)}}) + S(r,f)$$

Thus

$$T(r,f) \leq N(r, \frac{1}{f}) + T(r, f^{(k)}) - N(r, \frac{1}{f^{(k)}}) + S(r,f)$$

and so

$$PT(r,f) \leq PN(r, \frac{1}{f}) + \bar{N}(r,f) + \bar{N}(r, \frac{1}{f^{(k)}}) +$$

$$+ \sum_{i=1}^{p} \bar{N}(r, \frac{1}{f^{(k)}-a_i}) - P\,N(r, \frac{1}{f^{(k)}}) + S(r,f) \quad \ldots (2.18)$$

$$\leqslant \quad PN(r, \frac{1}{f}) + \bar{N}(r,f) + \sum_{i=1}^{p} \bar{N}(r, \frac{1}{f^{(k)}-a_i}) + S(r,f)$$

Therefore

$$P \leqslant P(1 - \delta(0,f)) + (1 - \Theta(\infty, f)) + \sum_{i=1}^{p} (1 - \Theta_r^{(k)}(a_i, f)$$

which yields

$$0 \leqslant - P\delta(0,f) + 1 - \Theta(\infty, f) + P - \sum_{i=1}^{p} \Theta_r^k (a_i f)$$

And so

$$\sum_{i=1}^{p} \Theta_r^{(k)} (a_i, f) \leqslant (1 + P) - (\Theta(\infty, f) + P\delta(0,f))$$

This proves the theorem.

Our next two theorems give proofs of the theorems stated (without proof) by A.P.Singh in [14] and [15] .

Theorem 2.8 : Let f(z) be a meromorphic function. Let each zero of f(z) have multiplicity greater than or equal to n. Then for all positive integer K and a $\neq$ 0, ∞,

$$n \Theta_r^k (a,f) \leqslant (n + K + 1) - (n\Theta(\infty, f) + (K+1)\delta(a,f))$$

Proof :

Consider the identity

$$\frac{a}{f-a} = \frac{f^{(k)}}{f-a} - \frac{f^{(k)}-a}{f^{(k+1)}} \cdot \frac{f^{(k+1)}}{f-a}$$

$$m\left(f, \frac{a}{f-a}\right) \leqslant m\left(r, \frac{f^{(k)}}{f-a}\right) + m\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) + m\left(r, \frac{f^{(k+1)}}{f-a}\right)$$

$$= m\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) + S(r,f)$$

$$= T\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) - N\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) + S(r,f)$$

$$m\left(r, \frac{a}{f-a}\right) \leqslant m\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) + N\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) - N\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right)$$

$$+ S(r,f)$$

$$= N\left(r, \frac{f^{(k+1)}}{f^{(k)}-a}\right) - N\left(r, \frac{f^{(k)}-a}{f^{(k+1)}}\right) + S(r,f)$$

$$m\left(r, \frac{a}{f-a}\right) \leqslant N(r, f^{(k+1)}) + N\left(r, \frac{1}{f^{(k)}-a}\right) - N(r, f^{(k)}-a) -$$

$$- N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f)$$

Now

$$T(r,f) = m\left(r, \frac{a}{f-a}\right) + N\left(r, \frac{a}{f-a}\right) + S(r,f)$$

$$\leqslant N(r, f^{(k+1)}) + N\left(r, \frac{1}{f^{(k)}-a}\right) - N(r, f^{(k)}-a)$$

$$- N\left(r, \frac{1}{f^{(k+1)}}\right) + N\left(r, \frac{a}{f-a}\right) + S(r,f)$$

$$T(r,f) \leqslant \bar{N}(r, f^{(k)}) + N\left(r, \frac{1}{f^{(k)}-a}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right)$$

$$+ N\left(r, \frac{a}{f-a}\right) + S(r,f)$$

$$= \bar{N}(r,f) + \left\{ \bar{N}\left(r, \frac{1}{f^{(k)}-a}\right) - N_a\left(r, \frac{1}{f^{(k+1)}}\right) \right\}$$

$$+ N\left(r, \frac{1}{f-a}\right) + S(r,f)$$

where $N_a\left(r, \frac{1}{f^{(k+1)}}\right)$ is formed by the zeros of $f^{(k+1)}$, but not the zeros of $f^{(k)} - a$. And so

$$T(r,f) \leqslant \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{f^{(k)}-a}\right) + N_o\left(r, \frac{1}{f-a}\right) + S(r,f)$$

where $N_o\left(r, \frac{1}{f-a}\right)$ is formed by all the zeros of $f(z)-a$ taken with proper multiplicity if multiplicity $\leqslant k+1$ and each zero of multiplicity $\geqslant k+1$ being counted $k+1$ times only.

But

$$nN_o(r, \frac{1}{f-a}) \leqslant (k+1)\, N(r, \frac{1}{f-a})$$

Since by hypothesis each zero of f-a has multiplicity $\geqslant$ n.

Thus

$$T(r,f) \leqslant \frac{k+1}{n} N(r, \frac{1}{f-a}) + \bar{N}(r, \frac{1}{f^{(k)}-a}) + \bar{N}(r,f) +$$

$$+ S(r,f). \quad \ldots (2.19)$$

Dividing by T(r,f) and taking limit superior of (2.19) we get

$$1 \leqslant \frac{k+1}{n}(1-\delta(a,f)) + (1 - \Theta_r^{(k)}(a,f)) + (1 - \Theta(\infty,f))$$

$$n \leqslant (k+1)(1-\delta(a,f)) + n - n\,\Theta_r^{(k)}(a,f) + n - n\,\Theta(\infty,f)$$

$$n\,\Theta_r(a,f) \leqslant (n+k+1) - \{n\,\Theta(\infty,f) + (k+1)\delta(a,f)\}$$

This proves the theorem.

<u>Theorem 2.9</u> : Let $f_1(z)$, $f_2(z)$ be any two meromorphic functions with $\bar{N}(r, \frac{1}{f_1}) = S(r,f_1)$ and $\bar{N}(r, \frac{1}{f_2}) = S(r,f_2)$ Then for any $a \neq 0, \infty$ $\delta_{1,2}^{(k)}(\infty) + 2\delta_0^{(k)}(\infty) \leqslant 5 - (\Theta_{1,2}(a) + 2\,\Theta_o(a))$

Proof : We know that for any $a \neq 0, \infty$

$$T(r,f) \leq \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-a}\right) + S(r,f)$$

from the above inequality we also have for all non-negative integers k,

$$T(r,f) \leq k\bar{N}(r,f) + N(r,f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-a}\right) + S(r,f)$$

$$= N(r,f^{(k)}) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-a}\right) + S(r,f)$$

Applying above inequality to the two functions f_1 and f_2 and adding we get

$$T(r,f_1) + T(r,f_2) \leq N(r, f_1^{(k)}) + N(r,f_2^{(k)}) + \bar{N}\left(r, \frac{1}{f_1}\right) + \bar{N}\left(r, \frac{1}{f_2}\right) + \bar{N}\left(r, \frac{1}{f_1-a}\right) + \bar{N}\left(r, \frac{1}{f_2-a}\right) + S(r,f_1) + S(r,f_2)$$

$$= N_{1,2}^{(k)}(r, \infty) + 2N_o^{(k)}(r,\infty) + \bar{N}_{1,2}(r,a) + 2N_o(r,a). \quad \ldots (2.20)$$

Dividing by $T(r,f_1) + T(r,f_2)$ and taking limit superior of

(2.20) as $r \to \infty$ we get

$$1 \leq 1 - \delta_{1,2}^{(k)}(\infty) + 2(1 - \delta_0^{(k)}(\infty)) + (1 - \Theta_{1,2}(a))$$

$$+ 2(1 - \Theta_0(a))$$

Equivalently by

$$1 \leq 6 - \delta_{1,2}^{(k)}(\infty) - 2\,\delta_0^{(k)}(\infty) - \Theta_{1,2}(a) - 2\,\Theta_0(a)$$

And so

$$\delta_{1,2}^{(k)}(\infty) + 2\,\delta_0^{(k)}(\infty) \leq 5 - (\Theta_{1,2}(a) + 2\,\Theta_0(a)).$$

We end this Chapter with one more theorem dealing with the common roots of two meromorphic functions, we prove

Theorem 2.10 : Let $f_1(z)$ and $f_2(z)$ be any two meromorphic functions, then for any finite numbers a_1 and a_2 distinct,

$$\Theta_{1,2}(\infty) + 2\,\Theta_0(\infty) \leq 8 - (\Theta_{1,2}(a_1) + \Theta_{1,2}(a_2) -$$

$$- 2(\Theta_0(a_1) + \Theta_0(a_2))$$

Proof : We know that from W.K.Hayman [1,43]

$$\{(q-1) + o(1)\}\quad T(r_n, f) \leq \sum_{\nu=1}^{q} \bar{N}(r_n, a_\nu) + \bar{N}(r_n\,\infty) -$$

$$- N_0\,(r_n\;\frac{1}{f'})$$

Therefore

$$T(r,f) \leq \sum_{\nu=1}^{2} \bar{N}(r,a_\nu) + \bar{N}(r,\infty) - N_o\left(r, \frac{1}{f'}\right)$$

$$\leq \bar{N}(r,a_1) + \bar{N}(r,a_2) + \bar{N}(r,\infty)$$

$$= \bar{N}\left(r, \frac{1}{f-a_1}\right) + N\left(r, \frac{.1}{f-a_2}\right) + \bar{N}(r,f)$$

Applying this inequality for f_1 & f_2 we get

$$T(r,f_1) + T(r,f_2) \leq \bar{N}\left(r, \frac{1}{f_1-a_1}\right) + \bar{N}\left(r, \frac{1}{f_2-a_1}\right)$$

$$+ \bar{N}\left(r, \frac{1}{f_1-a_2}\right) + \bar{N}\left(r, \frac{1}{f_2-a_2}\right) +$$

$$+ \bar{N}(r,f_1) + \bar{N}(r,f_2)$$

$$= \bar{N}_{1,2}(r,a_1) + 2\bar{N}_o(r,a_1) + \bar{N}_{1,2}(r,a_2) +$$

$$+ 2\bar{N}_o(r,a_2) + \bar{N}_{1,2}(r,\infty) + 2\bar{N}_o(r,\infty) \quad \ldots(2.21)$$

Dividing by $T(r,f_1) + T(r,f_2)$ and taking limit superior of (2.21) as $r \to \infty$ we get

$$1 \leq 1 - \Theta_{1,2}(a_1) + 2(1 - \Theta_0(a_1)) + 1 - \Theta_{1,2}(a_2) +$$

$$+ 2(1 - \Theta_0(a_2)) + 1 - \Theta_{1,2}(\infty) + 2(1 - \Theta_0(\infty))$$

which on simplification yields the desired result

$$\Theta_{1,2}(\infty) + 2\,\Theta_0(\infty) \leq 8 - (\Theta_{1,2}(a_1) + \Theta_{1,2}(a_2)) - 2(\Theta_0(a_1) + \Theta_0(a_2)).$$

°0o