CHAPTER - I

Definitions & Results

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Definitions and Results

In this chapter we give some basic definitions and results which we use in Chapter-II and Chapter-III.

§ 1.1. Definitions

Def. 1.1.1 Partially ordered set or poset [6]: Let P be a nonvoid set. Define a relation $\leq$ on P which has following properties for all $a, b, c \in P$

i) $a \leq a$ (reflexivity)

ii) $a \leq b$ and $b \leq a \Rightarrow a = b$ (antisymmetry)

iii) $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitivity)

The relation satisfying above three conditions is called partial ordering relation. And the set equipped with such relation is called partially ordered set or poset.

A poset P is called a chain (or totally ordered set or linearly ordered set) if it satisfy the following condition for all $a, b \in P$

iv) $a \leq b$ or $b \leq a$ (linearity)

Let $H \subseteq P$, $a \in P$, then a is an upper bound of H if $h \leq a$ for all $h \in H$. An upper bound a of H is the least upper bound of H or supremum of H (join) if for any upper bound b of H we have $a \leq b$. We shall write $a = \sup H$ or $a = \vee H$. The concept of lower bound or infimum are similarly defined. The latter is denoted by inf H or $\wedge H$.

Def: 1.1.2 : Zero element and unit element of a poset [6]:

A zero of poset P is an element 0 with $0 \leq x$ for all $x \in P$.
A unit element of a poset P is an element with $x \leq 1$ for all $x \in P$.

Def. 1.1.3 : Lattice as Poset [6] : A poset $\langle L; \leq \rangle$ is a lattice if sup $\{a, b\}$ or $a \vee b$ and inf $\{a,b\}$ or $a \wedge b$ exist for all $a,b \in L$.

Def. 1.1.4 : Lattice as an algebra [6]: An algebra $\langle L; \wedge ,\ \vee \rangle$ is called a lattice if L is nonvoid set, and V are binary operations on L satisfying following properties for all a, b, c $\in L$

i) $a \wedge a = a, \quad a \vee a = a$ (idempotency)

ii) $a \wedge b = b \wedge a, \quad a \vee b = b \vee a$ (commutativity)

iii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

$(a \vee b) \vee c = a \vee (b \vee c)$ (associativity)

iv) $a \wedge (a \vee b) = a$

$a \vee (a \wedge b) = a$ (absorption identities)

<u>Def. 1.1.5</u> : <u>Complete lattice</u> [6] : A lattice L is called complete if $\wedge H$ and $\vee H$ exist for any subset $H \subseteq L$

<u>Def. 1.1.6</u> : <u>Semilattice</u> [6] : A poset is a join semilattice (dually, meet semilattice) if sup $\{a, b\}$ or $a \vee b$ (dually infimum $\{a, b\}$ or $a \wedge b$) exists for any two elements of a poset.

<u>Def. 1.1.7</u> : <u>Distributive lattice</u> [2]: A lattice L is said to be distributive if and only if for all a, b, c $\in$ L the following idemtity will hold.

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

or

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

<u>Def. 1.1.8</u> : <u>O-distributive lattice</u> [10] . Let L be a lattice with O. L is said to be O-distributive if $\{a\}^* = \{x \in L / x \wedge a = 0\}$ is an ideal in L for every a $\in$ L.

i.e. $a \wedge b = 0$, $a \wedge c = 0$ (a, b, c $\in$ L) $\Longrightarrow$ $a \wedge (b \vee c) = 0$

<u>Def. 1.1.9</u> : <u>Modular lattice</u> [6] : A lattice L is called modular if x, y, z $\in$ L and $x \geq z$ implies that

$$(x \wedge y) \vee z = x \wedge (y \vee z)$$

<u>Def. 1.1.10</u> : <u>Semimodular lattice</u> [2] : A lattice L is said to be semimodular if it satisfy one of the

following conditions.

i) If $a \neq b$ and both a and b cover c ($a, b, c \in L$) then $a \vee b$ covers a as well as b

ii) Dually if $a \neq b$ and c covers both a and b, then a and b both cover $a \wedge b$.

Def: 1.1. 11 : Bounded lattice [6] : A lattice L is said to be bounded if it has both 0 and 1.

Def. 1.1.12 : Complement in lattice [6] : Let L be a bounded lattice $a, b \in L$. Then a is called complement of b if $a \wedge b = 0$ and $a \vee b = 1$.

Def. 1.1.13 : Complemented lattice [6] : A complemented lattice is a bounded lattice in which every element has a complement.

Def. 1.1.14 : Pseudocomplement in a lattice [6] : Let L be a lattice with 0. An element a^* is a pseudocomplement of a ($\in L$) if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$

Def. 1.1.15 : Pseudocomplemented lattice [6] : A lattice L with 0 is said to be pseudocomplemented if and only if each element of L has a pseudocomplement.

Def. 1.1.16 : Ideal [6] : Let L be a lattice and let

$I \subseteq L$. I is an ideal, if a, b $\in$ I implies that $a \vee b \in I$ and $a \in I$, $x \in L$, $x \leq a$ imply that $x \in I$

Def. 1.1.17 : 2-ideal [7] . Let L be a finite lattice A nonvoid subset I of L is called 2-ideal if

i) $x \in L$, $x \leq y \in I \implies x \in I$

ii) $x, y \in I$, $x \neq y$ and $t \in L$ such that $t \succ x, y$

$\implies t \in I$.

Def. 1.1.18 : Maximal ideal [6] : Let L be a lattice a proper ideal I of L is called maximal if it is not contained in any other proper ideal of L.

Def. 1.1.19 : Prime ideal [6] : A proper ideal I of L is prime if a, b $\in$ L and $a \wedge b \in I$ imply that $a \in I$ or $b \in I$

Def. 1.1.20 : Principal ideal [6] : Let L be a lattice, $a \in L$ then the intersection of all ideals in L containing a is called principal ideal generated by a. It is denoted by(a] . Equivalently we can define principal ideal as

$$(a] = \{ x \in L / x \leq a \} .$$

The ideal generated by H ($H \subseteq L$) is the intersection of all ideals containing H. It is denoted by (H] .

The concepts of filter, 2-filter, maximal filter, prime filter, principal filter can be defined dually [6]

Let I be the ideal of L, denote

$$I^* = \{x \in L \,/\, x \wedge i = 0, \ \forall\, i \in I\}.$$

Def. 1.1.21 : Annihilator ideal [3] : A ideal J of a lattice L with 0 is called an annihilator ideal if $J = J^{**}$

Def. 1.1.22 : Boolean lattice [6] : A lattice L is called Boolean if it is complemented and distributive.

Def. 1.1.23 : Boolean algebra [6] : A Boolean algebra is a Boolean lattice in which 0, 1 and ' are also considered as operations.

Thus a Boolean algebra is a system $\langle B; \wedge, \vee, ', 0, 1 \rangle$ where $\wedge$ and $\vee$ are binary, ' is unary operation and 0,1 are nullary operations.

Def. 1.1.24 : Semi-ideal in a poset [11] : A non-null subset A of a poset P is called semi-ideal if $x \in A$, $y \in P$ such that $y \leq x$ implies $y \in A$.

Def. 1.1.25 : Ideal in a poset [11] : A non-null subset A of poset P is called an ideal if

i) A is semi-ideal

ii) The supremum or join of any finite number of elements of A whenver it exists, belongs to A.

Def. 1.1.26 : Maximal element [6] : Let P be a poset an element a of P is maximal if $a \leq b$ ($b \in P$) implies that $a = b$.

The minimal element of a poset can be defined dually.

<u>Def. 1.1.27</u> : <u>Maximal ideal in a poset</u> [11] : A maximal ideal of a poset P with 0 is a maximal element of $I\mu$. where $I\mu$ is the set of all ideals of poset P with 0.

<u>Def. 1.1.28</u> : <u>Principal ideal in a poset</u> [11]: The set of all elements x of a poset P such that $x \leq a$, for some fixed $a \in P$, is called principal ideal generated by a. It is denoted by $(a]$.

<u>Def. 1.1.29</u> : <u>Prime ideal in a poset</u> [11] : A proper ideal A of poset P is prime if a, $b \in P$ such that $(a] \cap (b] \subseteq A$ then $a \in A$ or $b \in A$.

The concepts of filter, maximal filter, principal filter, prime filter in a poset can be defined dually.

<u>Def. 1.1.30</u> : <u>Pseudocomplements in a Poset</u> [11] :An element a of a poset P with 0, is said to have pseudo-complement $a^* \in P$ if there exists in P an element a^* such that

i) $(a] \cap (a^*] = (0]$

ii) for $b \in P$, $(a] \cap (b] = (0] \Rightarrow (b] \subseteq (a^*]$

<u>Def. 1.1.31</u> : <u>Pseudocomplemented poset</u> [11]: A poset P with O is said to be pseudocomplemented if every one of its elements has a pseudocomplement.

<u>Def.1.1.32</u> : <u>Ascending chain condition</u> [6]: A poset P is said to satisfy Ascending chain condition if any increasing chain terminates. That is if $x_i \in P$, $i = 0,1,2 \ldots$ .... and $x_0 \leq x_1 \leq \ldots\ldots \leq x_n \leq \ldots$. then for some m we have $x_m = x_{m+1} = \ldots$.

The concept of Descending chain condition can be defined dually.

<u>Def. 1.1.33</u> : <u>Disjunction poset</u> [12]: A poset P with O is called disjunction poset if a, b $\in$ P and $a \neq b$ imply that there exist c $\in$ P such that exactly one of ideals $(a] \cap (c]$, $(b] \cap (c]$ is zero.

<u>Def. 1.1.34</u> : <u>Dense element in poset</u> [11] : An element 'a' in a poset P with O is said to be dense if $a^* = 0$ where a^* is pseudocomplement of $a \in P$.

<u>Def. 1.1.35</u> : $a \succ b$ [6]: Let P be a poset a, b $\in$ P we say that a covers be ($a \succ b$) if $a > b$ and for no x, $a > x > b$.

§ 1.2 Results

Result 1.2.1 [11] : The set I_μ of all ideals of a poset P with 0 is a complete lattice under set inclusion as ordering relation.

Result 1.2.2 [11] : In a poset P a finite join $a_1 \vee a_2 \vee \ldots \vee a_n$ exists if and only if $(a_1] \underline{\vee} (a_2] \underline{\vee} \ldots \underline{\vee} (a_n]$ is a principal ideal (here $(a_1] \underline{\vee} (a_2] \underline{\vee} \ldots \underline{\vee} (a_n]$ is ideal generated by $(a_1] \cup (a_2] \cup \ldots \cup (a_n]$). Also whenever $a_1 \vee a_2 \vee \ldots \vee a_n$ exists we have

$$(a_1 \vee a_2 \vee \ldots \vee a_n] = (a_1] \underline{\vee} (a_2] \underline{\vee} \ldots \underline{\vee} (a_n]$$

Result 1.2.3 [11] : In a poset P with 0 the pseudocomplement a^* of an element a exists if and only if $(a^*]$ is a principal ideal. Further whenever a^* exists $(a]^* = (a^*]$

Result 1.2.4 [9] : Every pseudocomplemented lattice is 0-distributive.

Result 1.2.5 [9] : Every distributive lattice is 0-distributive.

Result 1.2.6 [8] : Every distributive semilattice is 0-distributive.

Result 1.2.7 [6] : In a poset satisfying ascending chain condition every ideal is principal ideal.

Result 1.2.8 [9] : A lattice L with 0 is 0-distributive if and only if the lattice of all ideals is pseudocomplemented.

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