CHAPTER - I.

PRELIMINARIES

1.1 FUZZY SETS.

Definition (1.1.1) :

Let X be any set and L be a lattice with 0 and 1. An element $A \in L^X$ is called a fuzzy set.

Remarks (1.1.2) :

For $L = \{0, 1,\}$ A becomes characteristic function from X to $\{0, 1\}$. In this sense fuzzy set is a generalization of a classical subset of a set.

Definition (1:1.3) :

Let A and B be two fuzzy sets. Then the function $A \cup B$ defined by $(A \cup B)(x) = A(x) \vee B(x)$ is called the union of two fuzzy sets A and B.

Definition (1.1.4) :

Let A and B be two fuzzy sets. Then the function $A \cap B$ defined by $A \cap B(x) = A(x) \wedge B(x)$ is called the intersection of two fuzzy sets A and B.

Definition (1.1.5) :

Let A and B be any two fuzzy sets. We say that A is a fuzzy subset of B and denote $A \subseteq B$ if $A(x) \leq B(x)$ for all $x \in X$

Remark (1.1.6) :

'$\subseteq$' is an order relation in L^X.

Definition (1.1.7) :

Two fuzzy sets A and B are said to be equal if $A(x) = B(x)$ for all $x \in X$.

Definition (1.1.8) :

Let A be a fuzzy set and $t \in L$. The crisp set,

$$A_t = \left\{ x \in x \;/\; A(x) \geq t \right\}$$

is called t - cut of A or t - level subset of A.

Definition (1.1.9) :

If A is a fuzzy set in X and f is a function from X to Y, then fuzzy set B in Y defined by

$$\left.\begin{aligned} B(y) &= \bigvee_{x \in f^{-1}(y)} A(x) \\ &= 0 \text{ if } f^{-1}(y) = \emptyset \end{aligned}\right\} \text{ FOR all } y \in Y.$$

B is called image of A under f and it is denoted by f(A).

Definition (1.1.10) :

Let f be a function from a set X to a set Y and B be a fuzzy set in Y. Then the fuzzy set A in X defined by,

$A(x) = B\left\{ f(x) \right\} = (B o f)(x)$, for all $x \in X$, is called preimage of B under f. It is denoted by $f^{-1}(B)$.

1.2 FUZZY SUBGROUPS

Definition (1.2.1) :

Let . be a binary operation in X. A fuzzy set A is closed under . if,

$$A(x.y) \geqslant A(x) \wedge A(y) \quad \text{for all } x,y \in X.$$

Definition (1.2.2) :

Let (X, .) be a group. A fuzzy set A is called a fuzzy subgroup of X if,

(G1) $A(x.y) \geqslant A(x) \wedge A(y)$ for all $x,y \in X$

(G2) $A(X^{-1}) \geqslant A(x)$ for all $x \in X$ where $x.x^{-1} = x^{-1}.x = e$, identify of X.

Proposition (1.2.3) :

If A is a fuzzy subgroup of X then $A(x^{-1}) = A(x)$ and $A(e) \geqslant A(x)$ for all $x \in X$, where e is the identify of X.

Proposition (1.2.4) :

A fuzzy set A is a fuzzy subgroup of X if and only if, $A(x.y^{-1}) \geqslant A(x) \wedge A(y)$.

Remark (1.2.5) :

If A is a subgroup of X then characteristic function of A i.e. $\chi_A : X \to L$ defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$, is a fuzzy subgroup of X.
Conversely any characteristic function which satisfies (G1) and (G2) gives a subgroup of X.

Proposition (1.2.6) :

Intersection of any family of fuzzy subgroups of X is a fuzzy subgroup of A.

Proposition (1.2.7) :

Let A be a fuzzy subgroup of X. Then level subsets, A_t; $t \in L$ and $t \leqslant A(e)$ is a crisp subgroup of X where e is the identify of X.

Corollary (1.2.8) :

Let A be a fuzzy subgroup of X. A crisp subset $\{ x \in X \,/\, A(x) = A(e) \}$ is a subgroup of X.

Proposition (1.2.9) :

Let A be a fuzzy subset of a group X. Such that A_t is a subgroup of X for each $t \in L$, $t \leqslant A(e)$, then A is a fuzzy subgroup of X.

Definition (1.2.10) :

Let X be a group and A be a fuzzy subgroup of X. Then subgroups A_t , $t \in L$, $t \leqslant A(e)$ are called level subgroups of A.

Proposition (1.2.11):

If A is a fuzzy subgroup of X, then $A(x.y^{-1}) = A(e)$ for $x, y \in X \Longrightarrow A(x) = A(y)$. Here e is the identify of X.

Proof :

$$A(x) = A(x.y^{-1}.y) \geqslant A(x.y^{-1}) \wedge A(y) = A(e) \wedge A(y)$$

$= \quad A(y) = A(y.x^{-1}.x) \geqslant \quad A(y.x^{-1}) \wedge A(x)$

$= \quad A(x.y^{-1}) \wedge A(x) \quad = \quad A(e) \wedge A(x) \quad = \quad A(x).$

Hence $A(x) = A(y)$.

Corollary (1.2.12) :

If A is a fuzzy subgroup of X, then A is constant on each cosets of $A_{A(e)}$

where, $A_{A(e)} = \left\{ x \in X \,/\, A(x) = A(e) \right\}$

Proposition (1.2.13) :

Let X be a group and A be a fuzzy subgroup of X Two level subgroups, A_{t_1}, A_{t_2} (with $t_1 < t_2$) of A are equal if and only if there is no $x \in X$ such that

$t_1 \leq A(x) < t_2$

Proposition (1.2.14) :

Any subgroup H of a group X can be realized as a level subgroup of some fuzzy subgroup of X .

Proposition (1.2.15) :

A homomorphic image or preimage of a fuzzy subgroup is a fuzzy subgroup.

Proof :

Let $f: X \rightarrow Y$ be a homomorphism of a group X into a group Y, and let $A: X \rightarrow I$ be a fuzzy subgroup of X, where I is a closed unit interval $[0,1]$

Define, $B : Y \longrightarrow I$ by,

$$B(y) = \bigvee_{x \in f^{-1}(y)} A(x)$$

$$= 0 \text{ if } f^{-1}(y) = \emptyset$$

Let $y_1 \in Y$ and $y_2 \in Y$.

If $f^{-1}(y_1 . y_2^{-1}) = \emptyset$, then $f^{-1}(y_1) = \emptyset$ or $f^{-1}(y_2) \neq \emptyset$

Hence, $B(y_1 . y_2^{-1}) \geqslant B(y_1) \wedge B(y_2)$.

If $f^{-1}(y_1 . y_2^{-1}) \neq \emptyset$, then either $f^{-1}(y_1) \neq \emptyset$ and $f^{-1}(y_2) \neq \emptyset$ or $f^{-1}(y_1) = \emptyset$ and $f^{-1}(y_2) = \emptyset$.

<u>Case I</u> : $f^{-1}(y_1) \neq \emptyset$ and $f^{-1}(y_2) \neq \emptyset$.

Then,

$$B(y_1 . y_2^{-1}) = \bigvee_{x \in f^{-1}(y_1 . y_2^{-1})} A(x)$$

$$\geqslant \bigvee_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} A(x_1 . x_2^{-1})$$

$$\geqslant \bigvee_{\substack{x_1 \in f^{-1}(y_1) \\ x_2 \in f^{-1}(y_2)}} (A(x_1) \wedge A(x_2))$$

$$= (\bigvee_{x_1 \in f^{-1}(y_1)} A(x_1)) \wedge (\bigvee_{x_2 \in f^{-1}(y_2)} A(x_2))$$ (Since, I being finite distributive & upper continuous, is infinite distributive)[23]

$$= B(y_1) \wedge B(y_2)$$

Thus $B(y_1 . y_2^{-1}) \geqslant B(y_1) \wedge B(y_2)$

<u>Case II</u> :

$$f^{-1}(y_1) = \emptyset, \quad f^{-1}(y_2) = \emptyset$$

Then,

$$B(y_1) = 0 \text{ and } B(y_2) = 0$$

Hence $B(y_1 . y_2^{-1}) \geqslant B(y_1) \wedge B(y_2)$

Thus, homomorphic image of a fuzzy subgroup is fuzzy subgroup.

Next, if B is a fuzzy subgroup of Y then define,

$$A : X \longrightarrow I \text{ by}$$

$$A(x) = B(f(x))$$

Let $x_1 \in X$, $x_2 \in X$. Consider,

$$
\begin{aligned}
A(x_1 . x_2^{-1}) &= B(f(x_1 . x_2^{-1})) \\
&= B(f(x_1) . (f(x_2))^{-1}) \\
&\geq B(f(x_1) \wedge B(f(x_2)) \\
&= A(x_1) \wedge A(x_2)
\end{aligned}
$$

Thus $A(x_1 . x_2^{-1}) \geq A(x_1) \wedge A(x_2)$. Hence A is fuzzy subgroup of X .

Remark (1.2.16) :

In order to prove that homomorphic image of a fuzzy subgroup is a fuzzy subgroup, A. Rosenfeld has assumed that the fuzzy subgroup satisfies sup property [17] . But as shown above we do not need this property to prove the result.