CHAPTER TWO

GENERALIZED HYPERGEOMETRIC POLYNOMIALS

2.1. In this chapter we establish a generating function, differential equation and a mixed recurrence relation for the generalized hypergeometric polynomials (1.3.1). We also mention at the end of each section a few interesting particular cases, some of which are beleived to be new.

2.2. Generating function : We consider

$$\exp\left[\frac{(\delta-1)}{x}t\right]{}_pF_{q+\lambda}\left[\begin{matrix}(a_p);\\ \Delta(\lambda,\alpha),\ (b_q);\end{matrix}\ \frac{ut^{\delta}x^{\{\mu+(\delta-1)\delta\}}}{(-\delta)^{\delta}}\right]$$

$$=\sum_{n=0}^{\infty}\frac{\left[x^{(\delta-1)}t\right]^{n}}{n!}\sum_{K=0}^{\infty}\frac{[(a_p)]_K\,u^k\,t^{\delta k}\,x^{\{\mu k+(\delta-1)\delta k\}}}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_k[(b_q)]_k\,K!\,(-\delta)^{\delta k}}\,.$$

With the help of (1.5.13) the right hand side of the above expression becomes

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/\delta]}\frac{[(a_p)]_k\,u^k\,t^n\,x^{\{\mu k+(\delta-1)\delta k+(\delta-1)(n-\delta k)\}}}{(n-\delta k)!\prod_{j=0}^{(\lambda-1)}\left(\frac{\alpha+j}{\lambda}\right)_k[(b_q)]_k\,K!\,(-\delta)^{\delta k}}\,.$$

On using (1.5.4) it can be written as

$$\sum_{n=0}^{\infty}\sum_{K=0}^{[n/\delta]}\frac{\prod_{j=0}^{\delta-1}\left(\frac{-n+j}{\delta}\right)_k[(a_p)]_k\,u^k\,t^n\,x^{\{\mu k+(\delta-1)n\}}}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_k[(b_q)]_k\,K!\,n!}\,.$$

Using definition (1.3.1) we obtain the relation

$$\exp\left[\frac{(\delta-1)}{x}t\right]{}_pF_q\left[\begin{matrix}(a_p);\\ \Delta(\lambda,\alpha),\ (b_q);\end{matrix}\ \frac{ut^{\delta}x^{\{\mu+(\delta-1)\delta\}}}{(-\delta)^{\delta}}\right]=\sum_{n=0}^{\infty}F_n(x)\frac{t^n}{n!}\,; \qquad (2.2.1)$$

Particular cases : On choosing the parameters suitably as in § 1.3 the result (2.2.1) will yield number of results as particular cases. We mention a few of them here.

(i) For the generalized Rice's polynomials (1.2.17):

$$e^{t}\ {}_2F_2\left[\begin{matrix} n+a+b+1, \rho; \\ 1+b, \sigma; \end{matrix} -Xt\right] = \sum_{n=0}^{\infty} \frac{H_n^{(a,b)}(\rho,\sigma;\ X)\ t^n}{(1+a)_n} \qquad (2.2.2)$$

(ii) For the generalized Sister Celine's polynomials (1.2.18):

$$e^{t}\ {}_pF_{q+1}\left[\begin{matrix} n+a+b+1,\ a_2,\dots,a_p ; \\ 1+a,\ \tfrac{1}{2},\ b_2,\dots,b_q; \end{matrix} -Xt\right] = \sum_{n=0}^{\infty} \frac{f_n^{(a,b)}\left(\begin{matrix} a_2,\dots,a_p; \\ b_2,\dots,b_q; \end{matrix}\ X\right)\ t^n}{(1+a)_n} \qquad (2.2.3)$$

(iii) For the generalized Bessel polynomials (1.2.19):

$$e^{t}\ {}_1F_2\left[\begin{matrix} 2\nu+n ; \\ \nu+\tfrac{1}{2},\ 1+b; \end{matrix} -Xt\right] = \sum_{n=0}^{\infty} y_n(x)\ \frac{t^n}{n!}\ . \qquad (2.2.4)$$

(iv) For the Bedient polynomials Rn $(\beta,\gamma;X)$ and Gn $(\alpha,\beta;X)$ (1.2.20) and (1.2.21) :

$$e^{xt}\ {}_1F_2\left[\begin{matrix} \gamma-\beta; \\ \gamma,\ 1-\beta-n; \end{matrix} \frac{t^2}{4}\right] = \sum_{n=0}^{\infty} \frac{Rn\ (\beta,\gamma;X)\ t^n}{(\beta)_n\ 2^n} \qquad (2.2.5)$$

and

$$e^{xt}\ {}_1F_2\left[\begin{matrix} 1-\alpha-\beta-n; \\ 1-\alpha-n,\ 1-\beta-n; \end{matrix} \frac{t^2}{4}\right] = \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n\ G_n\ (\alpha,\beta;\ X)\ t^n}{(\alpha)_n\ 2^n\ (\beta)_n}\ . \qquad (2.2.6)$$

(v) For the Lommel polynomials $R_{n,\nu}\left(\frac{1}{x}\right)$ (1.2.22):

$$e^{xt}\; {}_0F_3\left[\begin{matrix} -\,; \\ \nu, -n, 1-\nu-n; \end{matrix}\; -\frac{t^2}{4}\right] = \sum_{n=0}^{\infty} \frac{R_{n,\nu}\left(\frac{1}{x}\right) t^n}{(\nu)_n\, 2^n\, n!} \cdot \quad (2.2.7)$$

(vi) For the Toscano polynomials $S_n(x)$ (1.2.23) :

$$e^{t}\; {}_pF_{q+1}\left[\begin{matrix} (a_p)\;; \\ (a+1), (b_q)\;; \end{matrix}\; -xt\right] = \sum_{n=0}^{\infty} \frac{(a)_n\, S_n(x)\, t^n}{(2a)_n} . \quad (2.2.8)$$

(vii) For the Shah's polynomials (1.3.16):

$$\exp\left[x^{(\delta-1)}\, t\right] \; {}_pF_q\left[\begin{matrix} (a_p)\,; \\ (b_q)\,; \end{matrix}\; \frac{ut\, x^{\delta\{\mu+(\delta-1)\}}}{(-\delta)^{\delta}}\right] = \sum_{n=0}^{\infty} F_n(x)\, \frac{t^n}{n!} \cdot \quad (2.2.9)$$

2.3. <u>Differential equation</u>: Let the generalized hypergeometric polynomial be

$$W = z^{(\delta-1)n}\; {}_{p+\delta}F_{q+\lambda}\left[\begin{matrix} \Delta(\delta,-n), \; (a_p)\,; \\ \Delta(\lambda,\alpha)\;, \; (b_q)\,; \end{matrix}\; uz^{\mu}\right]$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-n+i}{\delta}\right)_k \prod_{r=1}^{p} (a_r)_k\, u^k\, z^{\{\mu k + (\delta-1)n\}}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_k \prod_{s=1}^{q} (b_s)_k\, k!} .$$

and the operator $\Theta \equiv z\frac{d}{dz}$ gives $\Theta z^k = kz^k$. Now consider

$$\left\langle \frac{1}{\mu}\left[\Theta - (\delta-1)n\right] \prod_{j=0}^{\lambda-1}\left\{\frac{\alpha+j}{\lambda} + \frac{1}{\mu}\left[\Theta - (\delta-1)n\right] - 1\right\} \times \right.$$

$$\left. \times \prod_{S=1}^{q}\left\{b_s + \frac{1}{\mu}\left[\Theta - (\delta-1)n\right] - 1\right\}\right\rangle W$$

$$= \sum_{K=1}^{\infty} \frac{1}{\mu}\left[\mu k + (\delta-1)n - (\delta-1)n\right] \prod_{j=0}^{\lambda-1}\left\{\frac{\alpha+j}{\lambda} + \frac{1}{\mu}\left[\mu k + (\delta-1)n\right.\right.$$

$$\left.\left. - (\delta-1)n\right] - 1\right\} \prod_{S=1}^{q}\left\{b_s + \frac{1}{\mu}\left[\mu k + (\delta-1)n - (\delta-1)n\right]\right\} \times$$

$$\times \frac{\prod_{i=1}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_k \prod_{r=1}^{p}(a_r)_k\, u^k\, z^{\{\mu k + (\delta-1)n\}}}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_k \prod_{S=1}^{q}(b_s)_k\; K!}$$

$$= \sum_{k=1}^{\infty} \frac{\prod_{i=0}^{\delta-1}\left(\frac{-n+\delta}{\delta}\right)_k \prod_{r=1}^{p}(a_r)_k\, u^k\, z^{\{\mu k + (\delta-1)n\}}}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_{k-1} \prod_{S=1}^{q}(b_s)_{k-1}\; (K-1)!}$$

on replacing K by (K+1) the above expression becomes

$$= \sum_{K=0}^{\infty} \frac{\prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_{k+1} \prod_{r=1}^{p}(a_r)_{k+1}\, u^{k+1}\, z^{\{\mu(K+1)+(\delta-1)n\}}}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_k \prod_{S=1}^{q}(b_s)_k\; K!}$$

$$= \sum_{K=0}^{\infty} uz^{\mu} \cdot \frac{\prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}+k\right) \prod_{r=1}^{p}(a_r+k) \prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_k}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_k \prod_{S=1}^{q}(b_s)_k \quad K!} \times$$

$$\times \prod_{r=1}^{p}(a_r)_k \; u^k \; z^{\{\mu k+(\delta-1)n\}}$$

$$= \sum_{k=0}^{\infty} uz^{\mu} \prod_{i=0}^{\delta-1}\left\{\frac{-n+i}{\delta}+\frac{1}{\mu}\left[\theta-(\delta-1)n\right]\right\} \times$$

$$\times \prod_{r=1}^{p}\left\{a_r+\frac{1}{\mu}\left[\theta-(\delta-1)n\right]\right\} \times$$

$$\times \frac{\prod_{i=0}^{\delta-1}\left(\frac{-n+i}{\delta}\right)_k \prod_{r=1}^{p}(a_r)_k \; u^k \; z^{\{\mu k+(\delta-1)n\}}}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_k \prod_{S=1}^{q}(b_s)_k \quad K!}$$

$$= \left\langle uz^{\mu} \prod_{i=0}^{\delta-1}\left\{\frac{-n+i}{\delta}+\frac{1}{\mu}\left[\theta-(\delta-1)n\right]\right\} \times\right.$$

$$\left.\times \prod_{r=1}^{p}\left\{a_r+\frac{1}{\mu}\left[\theta-(\delta-1)n\right]\right\}\right\rangle W.$$

Hence we have the result : $W = {}^{F}n\,(z)$ (1.3.1) is the solution of the differential equation

$$\left\langle \frac{1}{\mu}\left[\theta-(\delta-1)n\right]\prod_{j=0}^{\lambda-1}\left\{\frac{\alpha+j}{\lambda}+\frac{1}{\mu}\left[\theta-(\delta-1)n\right]-1\right\}\times\right.$$

$$\times\prod_{s=1}^{q}\left\{b_s+\frac{1}{\mu}\left[\theta-(\delta-1)n\right]-1\right\}$$

$$-\,uz^{\mu}\prod_{i=0}^{\delta-1}\left\{\frac{-n+i}{\delta}+\frac{1}{\mu}\left[\theta-(\delta-1)n\right]\right\}\times$$

$$\left.\times\prod_{r=1}^{p}\left\{a_r+\frac{1}{\mu}\left[\theta-(\delta-1)n\right]\right\}\right\rangle W=0 \quad (2.3.1)$$

valid for all finite Z and λ, δ, n, α nonnegative integers.

Special cases . On specializing the parameters in view of § 1.3, the result (2.3.1) will yield several special cases. However we mention here a few of the results.

(i) For the generalized Rice's polynomials (1.2.17) :

$$\left[\theta\,(\theta+a)\,(\theta+\sigma-1)+Z\,(n-\theta)\,(n+a+b+1+\theta)\,(\varrho+\theta)\right]H_n^{(a,b)}(\varrho,\sigma;z)$$

$$=0. \quad (2.3.2)$$

(ii) For the generalized Sister Celine's polynomials (1.2.18) :

$$\left[\theta\,(a+\theta)\prod_{s=1}^{q}\left\{b_s+\theta-1\right\}-Z(\theta-n)\prod_{r=1}^{p}(a_r+\theta)\right]W=0. \quad (2.3.3)$$

(iii) For the generalized Bessel polynomials (1.2.19) :

$$\left[\theta\,(\nu+\theta-\tfrac{1}{2})\,(b+\theta)-Z\,(\theta-n)\,(2\nu+n+\theta)\right]W=0. \quad (2.3.4)$$

(iv) For the Bedient polynomials $R_n(\beta,\gamma;z)$, $G_n(\alpha,\beta;z)$ (1.2.20) and (1.2.21) :

$$\left[(\theta-n)(2\gamma-\theta+n+1)(2\beta+n+\theta) + \frac{1}{z^2}\,\theta(1-\theta)(2\gamma-2\beta-\theta+n)\right] R_n(\beta,\gamma;z) = 0 \quad (2.3.5)$$

and

$$\left[(\theta-n)(\theta+2\alpha+n)(\theta+2\beta+n) - \frac{1}{z^2}\,\theta(\theta-1)(\theta+n+2\beta+2\alpha-2)\right] \times G_n(\alpha,\beta;z) = 0. \quad (2.3.6)$$

(v) For the Lommel polynomials $R_{n,\nu}(\frac{1}{z})$ (1.2.22) :

$$\left[(\theta-n)(2\nu-2+n-\theta)(\theta+n+1)(2\nu+n+\theta) + \frac{4}{z^2}\,\theta(1-\theta)\right] R_{n,\nu}\left(\frac{1}{z}\right) = 0. \quad (2.3.7)$$

(vi) For the Toscano polynomials $S_n(z)$ (1.2.23) :

$$\left[\theta(\theta+a+n-1)\prod_{j=1}^{q}(\theta+b_j-1) - z(\theta-n)\prod_{i=1}^{p}(\theta+a_i)\right] * S_n(z) = 0, \quad (2.3.8)$$

(vii) For the Shah's polynomials (1.3.16) :

$$\left\langle \frac{1}{\mu}\left[\theta-(\delta-1)n\right]\prod_{S=1}^{q}\left\{b_s+\frac{1}{\mu}\left[\theta-(\delta-1)n\right]-1\right\}\right.$$

$$- uz^{\mu} \prod_{i=0}^{\delta-1} \left\{ \frac{-n+i}{\delta} + \frac{1}{\mu} \left[\theta - (\delta-1)n \right] \right\} \times 1$$

$$\times \prod_{r=1}^{p} \left\{ a_r + \frac{1}{\mu} \left[\theta - (\delta-1)n \right] \right\} \Big\rangle W = 0 \qquad (2.3.9)$$

δ , n nonnegative integers.

2.4. <u>Recurrence relation</u>. We consider

$$W = \exp\left[X^{(\delta-1)} t \right] \sum_{k=0}^{\infty} \frac{[(a_p)]_k \; u^k \; t^{\delta k} \; X^{\{\mu k + (\delta-1)\delta k\}}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_k \; [(b_q)]_k \; k!}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma_n(X) \; t^n}{n!}, \qquad (2.4.1)$$

Differentiating (2.4.1) partially with respect to X we get

$$\frac{\partial W}{\partial X} = t(\delta-1) \; X^{(\delta-2)} W + \exp\left[x^{(\delta-1)} t \right] \left[\mu + (\delta-1)\delta \right] \times$$

$$\times \sum_{k=1}^{\infty} \frac{[(a_p)]_k \; u^k \; t^k \; X^{\{[\mu + (\delta-1)\delta]K - 1\}}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_k \; [(b_q)]_k \; (K-1)!}$$

$$= t(\delta-1) \; x^{(\delta-2)} \; W + \frac{[\mu + (\delta-1)\delta]}{X} \exp\left[x^{(\delta-1)} t \right].$$

$$\times \sum_{k=1}^{\infty} \frac{[(a_p)]_k \; u^k \; t^{\delta k} \; x^{\{\mu + (\delta-1)\delta\} K}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha + j}{\lambda}\right)_k [(b_q)]_k \; (K-1)!} \qquad (2.4.2)$$

Now differentiating (2.4.1) partially with respect to t we get

$$\frac{\partial W}{\partial t} = x^{(\delta-1)} W + \exp\left[x^{(\delta-1)} t\right] \sum_{k=1}^{\infty} \frac{[(a_p)]_k \; u^k \; \delta \; t^{\delta k-1}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha + j}{\lambda}\right)_k [(b_q)]_k} \times$$

$$\times \quad \frac{x^{\{\mu + (\delta-1)\delta\} k}}{(k-1)!}$$

$$= x^{(\delta-1)} W + \frac{\delta}{t} \exp\left[x^{(\delta-1)} t\right]$$

$$\sum_{k=1}^{\infty} \frac{[(a_p)]_k \; u^k \; t^{\delta k} \; x^{\{\mu + (\delta-1)\delta\} k}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha - j}{\lambda}\right)_k [(b_q)]_k \; (K-1)!} \qquad (2.4.3)$$

Multiplying (2.4.2) by $\frac{\delta}{t}$ and (2.4.3) by $\left[\mu + \frac{(\delta-1)\delta}{x}\right]$ and subtracting we get

$$\frac{\delta}{t}\frac{\partial W}{\partial x} - \frac{[\mu + (\delta-1)\delta]}{x} \frac{\partial W}{\partial t} = \delta(\delta-1) \; x^{\delta-2} \; W$$

$$- x^{(\delta - 2)} \left[\mu + (\delta - 1)\delta\right] W . \qquad (2.4.4.)$$

Again from (2.4.1) we have

$$\frac{\partial W}{\partial x} = \sum_{n=0}^{\infty} F_n'(x) \frac{t^n}{n!} \; ; \quad \frac{\partial W}{\partial t} = \sum_{n=1}^{\infty} \frac{F_n(x)\; t^{(n-1)}}{(n-1)!} .$$

Substituting these in (2.4.4) we get

$$\delta \sum_{n=0}^{\infty} F_n'(x) \frac{t^{n-1}}{n!} - \left[\frac{\mu + (\delta - 1)\delta}{x}\right] \sum_{n=1}^{\infty} \frac{F_n(x)\; t^{(n-1)}}{(n-1)!}$$

$$= x^{(\delta-2)} \left[\delta(\delta - 1) - \mu - (\delta - 1)\delta\right] \sum_{n=0}^{\infty} \frac{F_n(x)\; t^n}{n!} .$$

Comparing the coefficients of $\frac{t^n}{n!}$ from both the sides we obtain

$$\frac{\delta}{n+1} F_{n+1}'(x) - \left[\frac{\mu + (\delta - 1)\delta}{x}\right] F_{n+1}(x) = -\mu x^{(\delta-2)} F_n(x)$$

which on simplification can be put in the form

$$x\delta F_{n+1}'(x) - \left[\mu + (\delta - 1)\delta\right] (n + 1) F_{n+1}(x)$$

$$+ \mu (n + 1) x^{(\delta-1)} F_n(x) = 0. \qquad (2.4.5)$$

Particular cases. With the suitable selection of the parameters in view of §1.3, the result (2.4.5) will yield several special cases, however we quote here some interesting cases.

(i) For the generalized Rice's polynomials (1.2.17):

$$x\, D\, H_{n+1}^{(a,b)}(\rho, \sigma ; x) - (n+1) H_{n+1}^{(a,b)}(\rho, \sigma ; x)$$

$$+ (1+a+n) H_n^{(a,b)}(\rho, \sigma ; x) = 0 \qquad (2.4.6)$$

(ii) For the generalized Sister Celines polynomials (1.2.18):

$$x\, D\, f_{n+1}^{(a,b)}\left(\begin{matrix} a_2, \ldots, a_p \\ b_2, \ldots, b_q ; \end{matrix}\; x\right) - (n+1)\, f_{n+1}^{(a,b)}\left(\begin{matrix} a_2, \ldots, a_p ; \\ b_2, \ldots, b_q ; \end{matrix}\; x\right)$$

$$+ (1+a+n)\, f_n^{(a,b)}\left(\begin{matrix} a_2, \ldots, a_p \\ b_2, \ldots, b_q ; \end{matrix}\; x\right) = 0, \qquad (2.4.7)$$

(iii) For the generalized Bessel polynomials (1.2.19):

$$x\, D\, y_{n+1}(x) - (n+1)\, y_{n+1}(x) + (n+1)\, y_n(x) = 0. \quad (2.4.8)$$

(iv) For the Bedient polynomials (1.2.20) & (1.2.21):

$$D\, R_{n+1}(\beta, \gamma ; x) - 2(\beta+n)\, R_n(\beta, \gamma ; x) = 0 \quad (2.4.9)$$

and

$$(\alpha+\beta+n)\, D G_{n+1}(\alpha, \beta ; x) - 2(\alpha+n)(\beta+n)\, G_n(\alpha, \beta ; x) = 0.$$

$$(2.4.10)$$

(v) For the Lommel polynomials (1.2.22):

$$D R_{n+1,\nu}\left(\frac{1}{x}\right) - 2 \quad (1+n)\ (\nu+n) \quad R_{n,\nu}\left(\frac{1}{x}\right) = 0. \qquad (2.4.11)$$

(vi) For the Toscano polynomials (1.2.23) :

$$(a+n)\ xDS_{n+1}(x) - (1+n)\ (a+n)\ S_{n+1}(x)$$

$$-(2a+n)\ S_n(x) = 0. \qquad (2.4.12)$$

(vii) For the Shah's polynomials (1.3.16) :

$$x\delta\ F'_{n+1}(x) - [\mu + (\delta-1)\delta]\ (n+1)\ F_{n+1}(x)$$

$$+ \mu\ (n+1)\ x^{\delta-1}\ F_n(x) = 0. \qquad (2.4.13)$$

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