CHAPTER THREE

FINITE INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC POLYNOMIALS AND THE TCHE-BICHEFF POLYNOMIALS OF THE FIRST KIND.

3.1. In this chapter we evaluate two finite integrals involving generalized hypergeometric polynomials (1.3.1) and the Tchebicheff polynomials of first kind with the help of these results we obtain expansion formulae for the generalized hypergometric polynomials . A few of the interesting particular cases are also given at the end of each section.

3.2. In this section we evaluate the two finite integrals :

(A)

$$\int_0^1 x^{\beta+(\delta-1)m}\,(1-x)^{-\frac{1}{2}}\,T_n(2x-1)\;{}_{p+\delta}F_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_p)\,; \\ \Delta(\lambda,\alpha), & (b_q)\,;\end{matrix}\; ux^{\mu}\right]dx$$

$$= \frac{n!\,\pi^{\frac{1}{2}}\,\Gamma(\beta+\delta m-m+1)\,\Gamma(\beta+\delta m-m+\frac{3}{2})}{(\frac{1}{2})_n\,\Gamma(\beta+\delta m-m+n+\frac{3}{2})\,\Gamma(\beta+\delta m-m-n+\frac{3}{2})}\,\times$$

$$\times\;{}_{\delta+p+2\mu}F_{\lambda+q+2\mu}\left[\begin{matrix}\Delta(\delta,-m),\,\Delta(\mu,\beta+\delta m-m+1), \\ \Delta(\mu,\beta+\delta m-m+\frac{3}{2}),\,(a_p)\,; \\ \Delta(\lambda,\alpha),\,\Delta(\mu,\beta+\delta m-m+n+\frac{3}{2}), \\ \Delta(\mu,\beta+\delta m-m-n+\frac{3}{2}),\,(b_q)\,;\end{matrix}\; u\right] \qquad (3.2.1)$$

where $R_e\ (\beta+\delta m - m) > 0,\ \mu > 0.$

(B)
$$\int_0^1 x^{\beta+(\delta-1)m}\,(1-x)^{-\frac{1}{2}}\, T_n(2x-1)\ \times$$

$$\times\ {}_{\delta+p}F_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_p); \\ \Delta(\lambda,\alpha), & (b_q);\end{matrix}\ ux^{-\mu'}\right] dx$$

$$= \frac{n!\,\pi^{\frac{1}{2}}\,\Gamma(\beta+\delta m - m + 1)\ \Gamma(\beta+\delta m - m + \frac{3}{2})}{(\frac{1}{2})_n\ \Gamma(\beta+\delta m - m + n + \frac{3}{2})\ \Gamma(\beta+\delta m - m - n + \frac{3}{2})}$$

$${}_{\delta+p+2\mu'}F_{\lambda+q+2\mu'}\left[\begin{matrix}\Delta(\delta,-m),\ \Delta(\mu',\ -\beta-\delta m+m-n-\frac{1}{2}), \\ \Delta(\mu',-\beta-\delta m+m+n-\frac{1}{2}),\ (a_p); \\ \Delta(\lambda,\alpha),\ \Delta(\mu',-\beta-\delta m+m), \\ \Delta(\mu',-\beta-\delta m+m-\frac{1}{2}),\ (b_q);\end{matrix}\ u\right]$$

where $R_e\ (\beta+\delta m - m) > 0,\ \mu > 0.$ (3.2.2.)

In order to evaluate (3.2.1) consider

$$I_1 = \int_0^1 x^{\{\beta+(\delta-1)m\}}\,(1-x)^{-\frac{1}{2}}\ T_n(2x-1)\ {}_{p+\delta}F_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_p); \\ \Delta(\lambda,\alpha), & (b_q);\end{matrix}\ ux^{\mu}\right] dx$$

$$= \int_0^1 x^{\{\beta+(\delta-1)m\}} (1-x)^{-\frac{1}{2}} T_n(2x-1) \times$$

$$\times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r x^{\mu r}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!} dx$$

Interchanging the order of summation and integration we get

$$I_1 = \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!} \int_0^1 x^{\beta+(\delta-1)m+\mu r} \times$$

$$\times (1-x)^{-\frac{1}{2}} T_n(2x-1)\, dx$$

(using (1.5.1))

$$= \sum_{r=0}^{\infty} \frac{n!\, \pi^{\frac{1}{2}} \Gamma(\beta+\delta m-m+\mu r+1)\, \Gamma(\beta+\delta m-m+\mu r+\frac{3}{2})}{(\frac{1}{2})_n \Gamma(\beta+\delta m-m+n+\frac{3}{2}+\mu r)\, \Gamma(\beta+\delta m-m-n+\frac{3}{2}+\mu r)}$$

$$= \sum_{r=0}^{\infty} \frac{n!\, \pi^{\frac{1}{2}}}{(\frac{1}{2})_n} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r (\beta+\delta m-m+1)_{\mu r}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!\, (\beta+\delta m-m+n+\frac{3}{2})_{\mu r}}$$

$$\times \frac{\Gamma(\beta+\delta m-m+1)\, (\beta+\delta m-m+\frac{3}{2})_{\mu r} \Gamma(\beta+\delta m-m+\frac{3}{2})}{\Gamma(\beta+\delta m-m+n+\frac{3}{2})\, (\beta+\delta m-m-n+\frac{3}{2})_{\mu r} \Gamma(\beta+\delta m-m-n+\frac{3}{2})}$$

(using (1.5.6))

$$= \frac{n!\,\pi^{\frac{1}{2}}\,\Gamma(\beta+\delta m-m+1)\,\Gamma(\beta+\delta m-m+\frac{3}{2})}{(\frac{1}{2})_n\,\Gamma(\beta+\delta m-m+n+\frac{3}{2})\,\Gamma(\beta+\delta m-m-n+\frac{3}{2})} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1}\left(\frac{-m+i}{\delta}\right)_r [(a_p)]_r\, \mu^{\mu r} \prod_{i=0}^{\mu-1}\left(\frac{\beta+\delta m-m+1+i}{\mu}\right)_r u^r}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_r [(b_q)]_r\, r!\, \mu^{\mu r}\prod_{j=0}^{\mu-1}\left(\frac{\beta+\delta m-m+n+\frac{3}{2}+j}{\mu}\right)_r} \times$$

$$\times \frac{\mu^{\mu r}\prod_{j=0}^{\mu-1}\left(\frac{\beta+\delta m-m+\frac{3}{2}+j}{\mu}\right)_r}{\mu^{\mu r}\prod_{i=0}^{\mu-1}\left(\frac{\beta+\delta m-m-n+\frac{3}{2}+i}{\mu}\right)_r}$$

$$= \frac{n!\,\pi^{\frac{1}{2}}\,\Gamma(\beta+\delta m-m+1)\,\Gamma(\beta+\delta m-m+\frac{3}{2})}{(\frac{1}{2})_n\,\Gamma(\beta+\delta m-m+n+\frac{3}{2})\,\Gamma(\beta+\delta m-m-n+\frac{3}{2})} \times$$

$$\times\ {}_{\delta+p+2\mu}F_{\lambda+q+2\mu}\left[\begin{matrix}\Delta(\delta,-m),\ \Delta(\mu,\beta+\delta m-m+1);\\ \Delta(\mu,\beta+\delta m-m+\frac{3}{2}),\ (a_p);\\ \Delta(\lambda,\alpha),\ \Delta(\ ,\beta+\delta m-m+n+\frac{3}{2}),\\ \Delta(\mu,\beta+\delta m-m-n+\frac{3}{2}),\ (b_q);\end{matrix}\ u\right].$$

In order to evaluate (3.2.2) we consider

$$I_2 = \int_0^1 x^{\beta+(\delta-1)m}\,(1-x)^{-\frac{1}{2}}\,T_n(2x-1)\ {}_{\delta+p}F_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m),\ (a_p);\\ \Delta(\lambda,\alpha),\ (b_q);\end{matrix}\ ux^{-\mu}\right] dx$$

$$= \int_0^1 x^{\beta+(\delta-1)m} (1-x)^{-\frac{1}{2}} \,\Gamma_n(2x-1) \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r x^{-\mu' r}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!} \, dx$$

Interchanging summation and integration operations we have

$$I_2 = \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!} \int_0^1 x^{\beta+(\delta-1)m-\mu' r} \times$$

$$\times (1-x)^{-\frac{1}{2}} \,\Gamma_n(2x-1) \quad dx$$

(using (1.5.1))

$$= \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r \, n! \, \pi^{\frac{1}{2}} \,\Gamma(\beta+\delta m-m-\mu' r+1)}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r! \, (\tfrac{1}{2})_n \,\Gamma(\beta+\delta m-m-\mu' r+n+\frac{3}{2})} \times$$

$$\times \frac{\Gamma(\beta+\delta m-m-\mu' r+\frac{3}{2})}{\Gamma(\beta+\delta m-m-\mu' r-n+\frac{3}{2})}$$

$$= \frac{n!\,\pi^{\frac{1}{2}}}{(\frac{1}{2})_n} \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r \,(\beta+\delta m-m+1)_{-\mu' r}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r! \,(\beta+\delta m-m+n+\frac{3}{2})_{-\mu' r}} \times$$

$$\frac{\Gamma(\beta+\delta m-m+1)\,(\beta+\delta m-m+\tfrac{3}{2})_{-\mu' r}\,\Gamma(\beta+\delta m-m+\tfrac{3}{2})}{\Gamma(\beta+\delta m-m+n+\tfrac{3}{2})\,(\beta+\delta m-m-n+\tfrac{3}{2})_{-\mu' r}\,\Gamma(\beta+\delta m-m+\tfrac{3}{2})}$$

(using (1.5.7))

$$= \frac{n!\,\pi^{\frac{1}{2}}\,\Gamma(\beta+\delta m-m+1)\,\Gamma(\beta+\delta m-m+\tfrac{3}{2})}{(\tfrac{1}{2})_n\,\Gamma(\beta+\delta m-m+n+\tfrac{3}{2})\,\Gamma(\beta+\delta m-m-n+\tfrac{3}{2})}\,\times$$

$$\times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1}\left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r (-1)^{\mu' r}(\mu')^{\mu' r}\prod_{j=0}^{\mu'-1}\left(\frac{-\beta-\delta m+m-n-\frac{1}{2}+j}{\mu'}\right)_r}{\prod_{j=0}^{\lambda-1}\left(\frac{\lambda+j}{\lambda}\right)_r \left[(b_q)\right]_r\, r!\,(-1)^{\mu' r}\qquad(\mu')^{\mu' r}}\,\times$$

$$\times \frac{(-1)^{\mu' r}\,(\mu')^{\mu' r}\prod_{i=0}^{(\mu'-1)}\left(\frac{-\beta-\delta m+m+n-\frac{1}{2}+i}{\mu'}\right)_r\, u^r}{\prod_{i=0}^{\mu'-1}\left(\frac{-\beta-\delta m+m+i}{\mu'}\right)_r (-1)^{\mu' r}(\mu')^{\mu' r}\prod_{j=0}^{(\mu'-1)}\left(\frac{-\beta-\delta m+m-\frac{1}{2}+j}{\mu'}\right)_r}$$

$$= \frac{n!\,\pi^{\frac{1}{2}}\,\Gamma(\beta+\delta m-m+1)\,\Gamma(\beta+\delta m-m+\tfrac{3}{2})}{(\tfrac{1}{2})_n\,\Gamma(\beta+\delta m-m+n+\tfrac{3}{2})\,\Gamma(\beta-\delta m-m-n+\tfrac{3}{2})}\,\times$$

$$_{\delta+p+2\mu'}F_{\lambda+q+2\mu'}\left[\begin{matrix}\Delta(\delta,-m),\ \Delta(\mu',\ -\beta-\delta m+m-n-\tfrac{1}{2}), \\ \Delta(\mu',-\beta-\delta m+m+n-\tfrac{1}{2}),\ (a_p); \\ \Delta(\lambda,\alpha),\ \Delta(\mu',\ -\beta-\delta m+m), \\ \Delta(\mu',\ -\beta-\delta m+m-\tfrac{1}{2}),\ (b_q);\end{matrix}\ u\right].$$

Particular cases : On choosing the parameters suitably as in §1.3, the result (3.2.1) will yield number of results as particular cases. We mention as few of them here.

(i) For the generalized Rice's polynomials (1.2.17) :

$$\int_0^1 x^{\beta}(1-x)^{-\frac{1}{2}}\, T_n(2x-1)\, H_m^{(a,b)}(\rho,\sigma;x)\,dx$$

$$= \frac{(1+a)_m\ n!\ \pi^{\frac{1}{2}}\ \Gamma(\beta+1)\ \Gamma(\beta+\frac{3}{2})}{m!\ (\frac{1}{2})_n\ \Gamma(\beta+n+\frac{3}{2})\ \Gamma(\beta-n+\frac{3}{2})}$$

$$_5F_4\left[\begin{matrix}-m,\ m+a+b+1,\ \beta+1,\ \beta+\frac{3}{2},\ \rho; \\ 1+a,\ \beta+n+\frac{3}{2},\ \beta-n+\frac{3}{2},\ \sigma;\end{matrix}\ 1\right]$$

$$R_e(\beta)>0. \qquad (3.2.3)$$

(ii) For the generalized Sister Celine's polynomials (1.2.18) :

$$\int_0^1 x^{\beta}(1-x)^{-\frac{1}{2}}\, T_n(2x-1)\ f_m^{(a,b)}\left(\begin{matrix}a_2,\cdots,a_p \\ b_2,\cdots,b_q;\end{matrix}\ x\right)dx$$

$$= \frac{(1+a)_m \; n! \, \pi^{\frac{1}{2}} \; \Gamma(\beta+1) \; \Gamma(\beta+\frac{3}{2})}{m! \; (\frac{1}{2})_n \; \Gamma(\beta+n+\frac{3}{2}) \; \Gamma(\beta-n+\frac{3}{2})} \times$$

$$\times \; {}_{p+3}F_{q+3}\left[\begin{matrix} -m, \; m+a+b+1, \; \beta+1, \; \beta+\frac{3}{2}, \; a_2, \ldots, a_p; \\ 1+a, \; \frac{1}{2}, \; \beta+n+\frac{3}{2}, \; \beta-n+\frac{3}{2}, \; b_2, \ldots, b_q; \end{matrix} \; 1\right];$$

$$R_e(\beta) > 0. \qquad (3.2.4)$$

(iii) For the generalized Bessel polynomials (1.2.19) :

$$\int_0^1 x^{\beta} (1-x)^{-\frac{1}{2}} \; T_n(2x-1) \; y_m(x) \; dx$$

$$= \frac{n! \, \pi^{\frac{1}{2}} \; \Gamma(\beta+1) \; \Gamma(\beta+\frac{3}{2})}{(\frac{1}{2})_n \; \Gamma(\beta+n+\frac{3}{2}) \; \Gamma(\beta-n+\frac{3}{2})} \times$$

$$\times \; {}_4F_4\left[\begin{matrix} -m, \; m+2\nu, \; \beta+1, \; \beta+\frac{3}{2}; \\ \nu+\frac{1}{2}, \; 1+b, \; \beta+n+\frac{3}{2}, \; \beta-n+\frac{3}{2}; \end{matrix} \; 1\right]$$

$$R_e(\beta) > 0. \qquad (3.2.5)$$

(iv) For the Toscano polynomials (1.2.23) :

$$\int_0^1 x^{\beta} (1-x)^{-\frac{1}{2}} \; T_n(2x-1) \; S_m(x) \; dx$$

$$= \frac{(2a)_m \; n! \; \pi^{\frac{1}{2}} \; \Gamma(\beta+1) \; \Gamma(\beta+\frac{3}{2})}{m! \; (a)_m \; (\frac{1}{2})_n \; \Gamma(\beta+n+\frac{3}{2}) \; \Gamma(\beta-n+\frac{3}{2})} \times$$

$$\times \; {}_{p+3}F_{q+3}\left[\begin{matrix} -m, \; \beta+1, \; \beta+\frac{3}{2}, \; (a_p) \; ; \\ a+m, \; \beta+n+\frac{3}{2}, \; \beta-n+\frac{3}{2}, \; (b_q) \; ; \end{matrix} \; 1 \right]$$

$$R_e \; (\beta) > 0 . \qquad (3.2.6)$$

The result (3.2.2) will also yield number of particular cases.We quote here a few of them.

(v) For the Bedient polynomials $R_n \; (\beta \; ; \gamma \; ; x)$ and $G_n \; (\alpha, \beta \; ; x)$ (1.2.20) and (1.2.21) :

$$\int_0^1 x^{\beta} \; (1-x)^{-\frac{1}{2}} \; T_n \; (2x-1) \; R_m \; (\beta, \gamma \; ; \; x) \; dx$$

$$= {}_7F_6\left[\begin{matrix} \Delta(2,-m), \; \Delta(2,-\beta-m+n-\frac{1}{2}), \\ \Delta(2,-\beta-m+n-\frac{1}{2}), \; \gamma-\beta \; ; \\ \gamma, \; 1-\beta-m, \; \Delta(2,-\beta-m), \; \Delta(2,-\beta-m-\frac{1}{2}); \end{matrix} \; 1 \right] \times$$

$$\times \frac{(\beta)_m \; 2^m \; n! \; \pi^{\frac{1}{2}} \; \Gamma(\beta+m+1) \; \Gamma(\beta+m+\frac{3}{2})}{m! \; (\frac{1}{2})_n \; \Gamma(\beta+m+n+\frac{3}{2}) \; \Gamma(\beta+m-n+\frac{3}{2})} \qquad (3.2.7)$$

and

$$\int_0^1 x^{\beta} \; (1-x)^{-\frac{1}{2}} \; T_n \; (2x-1) \; G_m \; (\alpha, \beta \; ; x) \; dx$$

$$= \frac{(\alpha)_m \; (\beta)_m \; 2^m \; n! \; \pi^{\frac{1}{2}} \; \Gamma(\beta+m+1) \; \Gamma(\beta+m+\frac{3}{2})}{m! \; (\alpha+\beta)_m \; (\frac{1}{2})_n \; \Gamma(\beta+m+n+\frac{3}{2}) \; \Gamma(\beta+m-n+\frac{3}{2})} \times$$

$$
{}_{7}F_{6}\left[\begin{matrix}\Delta(2,-m),\ \Delta(2,-\beta-m-n-\tfrac{1}{2}),\\ \Delta(2,-\beta-m+n-\tfrac{1}{2}),\ 1-\alpha-\beta-m;\\ 1-\alpha-m,\ 1-\beta-m,\ \Delta(2,-\beta-m),\ \Delta(2,-\beta-m-\tfrac{1}{2});\end{matrix}\ 1\right],
$$

with $\mu' = 2$, $R_e(\beta+m) > 0$. (3.2.8)

(vi) For the Lommel polynomials (1.2.22) :

$$
\int_0^1 x^{\beta}(1-x)^{-\frac{1}{2}}\, T_n(2x-1)\, R_{m,\nu}\left(\frac{1}{x}\right) dx
$$

$$
= \frac{(\nu)_m\, 2^m\, n!\, \pi^{\frac{1}{2}}\, \Gamma(\beta+m+1)\, \Gamma(\beta+m+\frac{3}{2})}{(\frac{1}{2})_n\, \Gamma(\beta+m+n+\frac{3}{2})\, \Gamma(\beta+m-n+\frac{3}{2})}\ \times
$$

$$
\times\ {}_{6}F_{7}\left[\begin{matrix}\Delta(2,-m),\ \Delta(2,-\beta-m-n-\tfrac{1}{2}),\ \Delta(2,-\beta+n-m-\tfrac{1}{2});\\ \nu,\ -m,\ 1-\nu-m,\ \Delta(2,-\beta-m),\ \Delta(2,-\beta-m-\tfrac{1}{2})\end{matrix}\ -1\right],
$$

$R_e(\beta+m) > 0$. (3.2.9)

3.3. Expansion formulae : In this section we establish two expansion formulae

(A) $$x^{\{\beta+(\delta-1)m+\frac{1}{2}\}}\ {}_{p+\delta}F_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m);\ (a_p);\\ \Delta(\lambda,\alpha);\ (b_q);\end{matrix}\ ux^{\mu}\right]$$

$$
= \sum_{r=0}^{\infty} \frac{r!\,\Gamma(\beta+\delta m-m+1)\,\Gamma(\beta+\delta m-m+\frac{3}{2})}{(\frac{1}{2})_r\,\pi^{\frac{1}{2}}\,\Gamma(\beta+\delta m-m+r+\frac{3}{2})\,\Gamma(\beta-\delta m-m-r+\frac{3}{2})}\ \times
$$

$$
{}_{p+\delta+2\mu}F_{\lambda+q+2\mu}\left[\begin{matrix}\Delta(\delta,-m),\Delta(\mu,\beta+\delta m-m+1),\\ \Delta(\mu,\beta+\delta m-m+\frac{3}{2}),\quad (a_p)\ ;\\ \\ \Delta(\lambda,\alpha),\Delta(\mu,\beta+\delta m-m+r+\frac{3}{2}),\\ \Delta(\mu,\beta+\delta m-m-r+\frac{3}{2}),\quad (b_q)\ ;\end{matrix}\ u\right]\times
$$

$$
\times\, T_r(2x-1),
$$

$R_e\ (\beta+\delta m-m)>0,\ r\neq 0$, integer $\mu>0$ (3.3.1)

and

(B) $$x^{\{\beta+(\delta-1)m+\frac{1}{2}\}}\ {}_{\delta+p}F_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_p);\\ \\ \Delta(\lambda,\alpha), & (b_q);\end{matrix}\ ux^{-\mu'}\right]$$

$$
=\sum_{r=0}^{\infty}\frac{r!\,2\,\Gamma(\beta+\delta m-m+1)\,\Gamma(\beta+\delta m-m+\frac{3}{2})}{\pi^{\frac{1}{2}}\,(\frac{1}{2})_r\,\Gamma(\beta+\delta m-m+r+\frac{3}{2})\,\Gamma(\beta+\delta m-m-r+\frac{3}{2})}
$$

$$
{}_{\delta+p+2\mu'}F_{\lambda+q+2\mu'}\left[\begin{matrix}\Delta(\delta,-m),\Delta(\mu',-\beta-\delta m+m-r-\frac{1}{2}),\\ \Delta(\mu',-\beta-\delta m+m+r-\frac{1}{2}),(a_p);\\ \\ \Delta(\lambda,\alpha),\Delta(\mu',-\beta-\delta m+m),\\ \Delta(\mu',-\beta-\delta m+m-\frac{1}{2}),\quad (b_q)\ ;\end{matrix}\ u\right] T_r(2x-1),
$$

$R_e\ (\beta+\delta m-m)>0,\ r\neq 0,$ (3.3.2)

In order to establish (3.3.1), we suppose that

$$x^{\{\beta+(\delta-1)m+\frac{1}{2}\}}\ {}_{\delta+p}F_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_p)\ ;\\ \\ \Delta(\lambda,\alpha), & (b_q)\ ;\end{matrix}\ ux^{\mu}\right]$$

(37)

$$= \sum_{r=0} A_r \, T_r(2x-1) \tag{3.3.3}$$

Multiply both the sides of (3.3.3) by $x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ $T_n(2x-1)$ and integrate with respect to X over the interval (0,1) to get

$$\int_0^1 x^{\{\beta+(\delta-1)m\}}(1-x)^{-\frac{1}{2}} \, T_n(2x-1) \; \Gamma_{\delta+p}^{\;q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_p); \\ \Delta(\lambda,\alpha), & (b_q);\end{matrix} \; ux^{\mu}\right] dx$$

$$= \int_0^1 \sum_{r=0}^{\infty} A_r \, T_r(2x-1) \, x^{\frac{1}{2}} (1-x)^{-\frac{1}{2}} \; Tn(2x-1) \, dx$$

$$= \int_0^1 A_n \, x^{\frac{1}{2}} (1-x)^{-\frac{1}{2}} \left[T_n(2x-1)\right]^2 dx$$

$$= A_n \frac{\pi}{2}$$

with the aid of (1.5.2) . On using (3.2.1) we get

$$A_n = \frac{n! \; 2 \; \Gamma(\beta+\delta m-m+1) \; \Gamma(\beta+\delta m-m+\frac{3}{2})}{\pi^{\frac{1}{2}} (\frac{1}{2})_n \; \Gamma(\beta+\delta m-m+n+\frac{3}{2}) \; \Gamma(\beta+\delta m-m-n+\frac{3}{2})} \times$$

$$\Gamma_{\delta+p+2\mu}^{\;\lambda+q+2\mu}\left[\begin{matrix}\Delta(\delta,-m), \Delta(\mu, \beta+\delta m-m+1), \\ \Delta(\mu,\beta+\delta m-m+\frac{3}{2}), \; (a_p); \\ \Delta(\lambda,\alpha), \Delta(\mu,\beta+\delta m-m+n+\frac{3}{2}), \\ \Delta(\mu,\beta+\delta m-m-n+\frac{3}{2}), \; (b_q);\end{matrix} \; u\right].$$

Substituting this value of A_n in (3.3.3) we get

$$x^{\{\beta+(\delta-1)m+\frac{1}{2}\}} \; \Gamma_{\delta+p}^{\;q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_p); \\ \Delta(\lambda,\alpha), & (b_q);\end{matrix} \; ux^{\mu}\right]$$

$$= \sum_{r=0}^{\infty} \frac{r!\; 2\; \Gamma(\beta+\delta m - m + 1)\; \Gamma(\beta+\delta m - m + \frac{3}{2})}{(\frac{1}{2})_r\; \pi^{\frac{1}{2}}\; \Gamma(\beta+\delta m - m + r + \frac{3}{2})\; \Gamma(\beta+\delta m - m - r + \frac{3}{2})}$$

$${}_{\delta+p+2\mu}F_{q+\lambda+2\mu}\left[\begin{matrix}\Delta(\delta,-m),\ \Delta(\mu,\beta+\delta m - m + 1), \\ \Delta(\mu,\beta+\delta m - m + \frac{3}{2}),\ (a_p)\ ; \\ \\ \Delta(\lambda,\alpha),\ \Delta(\mu,\beta+\delta m - m + r + \frac{3}{2}), \\ \Delta(\mu,\beta+\delta m - m - r + \frac{3}{2}),\ (b_q)\ ;\end{matrix}\ u\right] \times$$

$$\times\ T_r\ (2x - 1)\ ;$$

which establishes (3.3.1).

Now to establish (3.3.2), let

$$x^{\{\beta + (\delta - 1)m + \frac{1}{2}\}}\ {}_{\delta+p}F_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_p)\ ; \\ \Delta(\lambda,\alpha), & (b_q)\ ;\end{matrix}\ ux^{-\mu'}\right]$$

$$= \sum_{r=0}^{\infty} A_r\ T_r\ (2x - 1)\ . \qquad (3.3.4)$$

Multiply both the sides of (3.3.4) by $x^{-\frac{1}{2}}\ (1 - x)^{\frac{1}{2}}$ $T_n\ (2x - 1)$ and integrate with respect to x over the interval (0,1) to get

$$= \int_0^1 \sum_{r=0}^{\infty} A_r\ x^{-\frac{1}{2}}\ (1-x)^{-\frac{1}{2}}\ T_r\ (2x - 1)\ T_n\ (2x - 1)\ dx$$

$$= A_n \int_0^1 x^{-\frac{1}{2}}\ (1 - x)^{-\frac{1}{2}} \left[T_n\ (2x - 1)\right]^2\ dx$$

$$= A_n\ (\frac{\pi}{2})$$

with the aid of (1.5.2). With the help of integral (3.2.2) we get

$$A_n = \frac{n!\; 2\; \Gamma(\beta+\delta m - m + 1)\; \Gamma(\beta+\delta m - m + \frac{3}{2})}{\pi^{\frac{1}{2}}\; (\frac{1}{2})_n\; \Gamma(\beta+\delta m - m + n + \frac{3}{2})\; \Gamma(\beta+\delta m - m - n + \frac{3}{2})}$$

$$F^{\delta+p+2\mu'}_{\lambda+q+2\mu'}\left[\begin{matrix}\Delta(\delta, -m),\ \Delta(\mu', -\beta-\delta m + m - n - \frac{1}{2}), \\ \Delta(\mu', -\beta-\delta m + n - \frac{1}{2}),\quad (a_p)\ ; \\ \\ \Delta(\lambda,\alpha),\ \Delta(\mu', -\beta-\delta m + m), \\ \Delta(\mu', -\beta-\delta m + m - \frac{1}{2}),\quad (b_q)\ ;\end{matrix}\ u\right].$$

Substituing this value of A_n in (3.3.4) we have

$$\{x\}^{\beta+(\delta-1)m+\frac{1}{2}}\; F^{\delta+p}_{q+\lambda}\left[\begin{matrix}\Delta(\delta, -m)\ ;\quad (a_p)\ ; \\ \\ \Delta(\lambda,\alpha)\ ,\quad (b_q)\ ;\end{matrix}\ ux^{-\mu'}\right]$$

$$= \sum_{r=0}^{\infty} \frac{r!\; 2\; \Gamma(\beta+\delta m - m + 1)\; \Gamma(\beta+\delta m - m + \frac{3}{2})}{(\frac{1}{2})_r\; \pi^{\frac{1}{2}}\; \Gamma(\beta+\delta m - m + r + \frac{3}{2})\; \Gamma(\beta+\delta m - m - r + \frac{3}{2})}$$

$$F^{p+\delta+2\mu'}_{\lambda+q+2\mu'}\left[\begin{matrix}\Delta(\delta, -m),\ \Delta(\mu', -\beta-\delta m + m - r - \frac{1}{2}), \\ \Delta(\mu', -\beta-\delta m + m + r - \frac{1}{2}),\ (a_p)\ ; \\ \\ \Delta(\lambda,\alpha),\ \Delta(\mu', -\beta-\delta m + m)\ , \\ \Delta(\mu', -\beta-\delta m + m - \frac{1}{2}),\quad (b_q)\ ;\end{matrix}\ u\right] \times$$

$$\times\ T_r\ (2x - 1)$$

which establishes (3.3.2).

Particular cases : On specializing the parameters in view of §1.3, the result (3.3.1) will yield several special cases. We mention here a few of these results.

(i) For the generalized Rice's polynomials (1.2.17):

$$x^{\{\beta+\frac{1}{2}\}} H_m^{(a,b)}(\varrho, \sigma ; x) = \frac{(1+a)_m}{m!} \sum_{r=0}^{\infty} \frac{r!\, 2\, \Gamma(\beta+1)\, \Gamma(\beta+\frac{3}{2})}{(\frac{1}{2})_r\, \pi^{\frac{1}{2}}\, \Gamma(\beta+r+\frac{3}{2})}$$

$$\cdot \frac{T_r(2x-1)}{\Gamma(\beta-r+\frac{3}{2})} \; {}_{p+3}F_{q+3}\left[\begin{matrix} -m, \beta+1, \beta+\frac{3}{2}, 1+a+b+m, a_2, \cdots, a_p; \\ 1+a, \frac{1}{2}, \beta+r+\frac{3}{2}, \beta-r+\frac{3}{2}, b_2, \cdots, b_q; \end{matrix} 1\right]. \quad (3.3.5)$$

$R_e(\beta) > 0,\ r \neq 0.$

(ii) For the generalized Sister Celine's polynomials (1.2.18) :

$$x^{\beta+\frac{1}{2}} f_m^{(a,b)}\left(\begin{matrix} a_2, \cdots, a_p; \\ b_2, \cdots, b_q; \end{matrix}\ x\right) = \frac{(1+a)_m}{m!} \sum_{r=0}^{\infty} \frac{r!\, 2\, \Gamma(\beta+1)}{(\frac{1}{2})_r\, \pi^{\frac{1}{2}}}$$

$$\frac{\Gamma(\beta+\frac{3}{2})}{\Gamma(\beta+r+\frac{3}{2})\, \Gamma(\beta-r+\frac{3}{2})} \; {}_{p+3}F_{q+3}\left[\begin{matrix} -m, \beta+1, \beta+\frac{3}{2}, \\ 1+a, \frac{1}{2}, \beta+r+\frac{3}{2}, \end{matrix}\right.$$

$$\left.\begin{matrix} 1+a+b+m, a_2, \cdots, a_p; \\ \beta-r+\frac{3}{2}, \quad b_2, \cdots, b_q; \end{matrix}\ 1\right] T_r(2x-1),$$

$R_e(\beta) > 0,\ r \neq 0.$ (3.3.6)

(iii) For the generalized Bessel polynomials (1.2.19) :

$$x^{\beta+\frac{1}{2}} \mathcal{Y}_m(x) = \sum_{r=0}^{\infty} \frac{r!\, 2\, \Gamma(\beta+1)\, \Gamma(\beta+\frac{3}{2})}{(\frac{1}{2})_r\, \pi^{\frac{1}{2}}\, \Gamma(\beta+r+\frac{3}{2})\, \Gamma(\beta-r+\frac{3}{2})}$$

$${}_4F_4\left[\begin{matrix} -m, \beta+1, \beta+\frac{3}{2}, 2\nu+m; \\ \nu+\frac{1}{2}, 1+b, \beta+r+\frac{3}{2}, \beta-r+\frac{3}{2}; \end{matrix} 1\right] T_r(2x-1),$$

$R_e(\beta) > 0, \quad r \neq 0.$ (3.3.7)

(iv) For the Toscano polynomials (1.2.23) :

$$x^{\beta+\frac{1}{2}} S_m(x) = \frac{(2a)_m}{m!\,(a)_m} \sum_{r=0}^{\infty} \frac{r!\, 2\, \Gamma(\beta+1)\, \Gamma(\beta+\frac{3}{2})}{(\frac{1}{2})_r\, \pi^{\frac{1}{2}}\, \Gamma(\beta+r+\frac{3}{2})\, \Gamma(\beta-r+\frac{3}{2})}$$

$${}_{p+3}F_{q+3}\left[\begin{matrix} -m, \beta+1, \beta+\frac{3}{2}, (a_p); \\ a+m, \beta+r+\frac{3}{2}, \beta-r+\frac{3}{2}, (b_q); \end{matrix} 1\right] T_r(2x-1),$$

$R_e(\beta) > 0, \quad r \neq 0.$ (3.3.8)

Now the result (3.3.2) gives rise to various special cases when we choose the parametêrs as in § 1.3 we quote here a few of these.

(v) For the Bedient polynomials $R_m(\beta, \gamma; x)$ and $G_m(\alpha, \beta; x)$ (1.2.20) and (1.2.21) :

with $\mu' = 2$

$$x^{\beta+\frac{1}{2}} R_m(\beta,\nu,x) = \frac{(\beta)_m 2^{m+1}}{m!} \sum_{r=0}^{\infty} \frac{r!\,\Gamma(\beta+m+1)}{\pi^{\frac{1}{2}}(\frac{1}{2})_r} \times$$

$$\times \frac{\Gamma(\beta+m+\frac{3}{2})}{\Gamma(\beta+m+r+\frac{3}{2})\,\Gamma(\beta+m-r+\frac{3}{2})} \times$$

$$F^{7}_{6}\left[\begin{matrix}\Delta(2,-m),\ \Delta(2,-\beta-m-r-\frac{1}{2}), \\ \Delta(2,-\beta-m+r-\frac{1}{2}), \ldots, \quad (\nu-\beta); \\ \nu,\ 1-\beta-m,\ (2,-\beta-m),\ (2,-\beta-m-\frac{1}{2});\end{matrix}\ 1\right] T_r(2x-1)$$

$R_e\ (\beta+m) > 0, \quad r \neq 0$ (3.3.9)

and

$$x^{\beta+\frac{1}{2}} G_m(\alpha,\beta;x) = \frac{(\alpha)_m(\beta)_m}{(\alpha+\beta)_m\, m!} \sum_{r=0}^{\infty} \frac{r!\,2\,\Gamma(\beta+m+1)\,\Gamma(\beta+m+\frac{3}{2})}{\pi^{\frac{1}{2}}(\frac{1}{2})_r\,\Gamma(\beta+m+r+\frac{3}{2})} \times$$

$$\times \frac{1}{\Gamma(\beta+m-r+\frac{3}{2})} \times$$

$$\times F^{7}_{6}\left[\begin{matrix}\Delta(2,-m),\ \Delta(2,-\beta-m-r-\frac{1}{2}), \\ \Delta(2,-\beta-m+r-\frac{1}{2}),\ 1-\alpha-\beta-m; \\ 1-\alpha-m,\ 1-\beta-m,\ \Delta(2,-\beta-m), \\ \Delta(2,-\beta-m-\frac{1}{2});\end{matrix}\ 1\right] T_r(2x-1),$$

$R_e\ (\beta+m) > 0, \quad r \neq 0.$ (3.3.10)

(vi) For the Lommel polynomials $R_{n,\nu}\left(\frac{1}{x}\right)$ (1.2.22) :

$$x^{\beta+\frac{1}{2}}\, R_{m,\nu}\left(\frac{1}{x}\right) = (\nu)_m\, 2^{m+1} \sum_{r=0}^{\infty} \frac{r!\, \Gamma(\beta+m+1)}{\pi^{\frac{1}{2}}\,(\tfrac{1}{2})_r\, \Gamma(\beta+m+r+\frac{3}{2})}$$

$$\frac{\Gamma(\beta+m+\frac{3}{2})}{\Gamma(\beta+m-r+\frac{3}{2})}$$

$${}_6F_7\left[\begin{matrix}\Delta(2,-m),\ \Delta(2,-\beta-m-r-\tfrac{1}{2}), \\ \Delta(2,-\beta-m+r-\tfrac{1}{2});\\ \nu,\ -m,\ 1-\nu-m,\ \Delta(2,-\beta-m),\\ \Delta(2,-\beta-m-\tfrac{1}{2});\end{matrix}\ -1\right] T_r(2x-1),$$

$R_e\,(\beta+m) > 0,\quad r \neq 0.$ (3.3.11)

~~~ x ~~~
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