CHAPTER FOUR.

INFINITE INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC POLYNOMIALS AND THE HERMITE POLYNOMIALS.

4.1. Hermite even and Hermite Odd polynomial solutions of self adjoint differential equation

$$D\left[\exp(-x^{2k})\, Dy\right] + x^{2k-2} \exp(-x^{2k})\, y = 0,$$

where $D \equiv \frac{d}{dx}$ and $k = 1,2,3,----$

were studied by Thakare and Karande [31]. These polynomials are orthogenal with respect to weight function $x^{2k-2} \exp(-x^{2k})$ over the interval $(-\infty, \infty)$. They established the Rodrigue's formulae (1.5.9) and (1.5.10).

In this chapter we evaluate infinite integrals involing generalized hypergeometric polynomials (1.3.1) and the Hermite even polynomials $H_{2pk}(x;k)$ or Hermite Odd polynomials $H_{2pk+1}^{(x;k)}$. Using these integrals we obtain the expansion formulae for the generalized hypergeometric polynomials in terms of the Orthogonal polynomials $H_{2pk}(x;k)$ (or $H_{2pk+1}^{(x;k)}$). A few of the interesting special cases are also given at the end of each section.

4.2. In this section we evaluate the two infinite integrals for even m and $\mu = -2ck$.

(i) $$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k})\, x^{2pk+2sk}\, H_{2pk}^{(x;k)}\, x^{(\delta-1)m}$$

$$_{p'+\delta}F_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_p); \\ \Delta(\lambda,\alpha), & (b_q);\end{matrix}\; ux^{-2ck}\right] dx$$

$$= \frac{(-2\beta)\ (1+p)_p\ \Gamma(s+p+m\beta-\delta m\beta+1)\ \Gamma(p+s+m\beta-\delta m\beta+\beta+1)}{(1+\beta)_p\ \Gamma(s+m\beta-\delta m\beta+1)}$$

$$\bar{H}^{p'+\delta+c}_{q+\lambda+2c}\left[\left.\begin{matrix}\Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta),\ (a_{p'});\\ \Delta(\lambda,\alpha),\ \Delta(c,-s-p-m\beta+\delta m\beta),\\ \Delta(c,-s-p-m\beta+\delta m\beta-\beta),\ (b_q);\end{matrix}\right| \frac{u}{(-c)^c}\right],$$

$R_e\ (p+s+\beta+1) > 0$ and $\beta = -\frac{1}{2k}$, (4.2.1)

related to the even Hermite polynomials and

(ii) $$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k})\ x^{2pk+2sk+1}\ H_{2pk+1}(x;k)\ x^{(\delta-1)m}$$

$$\bar{H}^{p'+\delta}_{q+\lambda}\left[\left.\begin{matrix}\Delta(\delta,-m), & (a_{p'});\\ \Delta(\lambda,\alpha), & (b_q);\end{matrix}\right| ux^{-2ck}\right] dx$$

$$= \frac{(-\beta)\ 2^{2p+2}\ (\frac{3}{2})_p\ \Gamma(s+m\beta-\delta m\beta+p+1)\ \Gamma(s+m\beta-\delta m\beta+p-\beta+1)}{(1-\beta)_p\ \Gamma(s+m\beta-\delta m\beta+1)}$$

$$\bar{H}^{\delta+p'+c}_{\lambda+q+2c}\left[\left.\begin{matrix}\Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta),\ (a_{p'});\\ \Delta(\lambda,\alpha),\ \Delta(c,-s-p-m\beta+\delta m\beta),\\ \Delta(c,-s-p-m\beta+\delta m\beta+\beta),\ (b_q);\end{matrix}\right| \frac{u}{(-c)^c}\right],$$

$R_e\ (p+s+m\beta-\delta m\beta-\beta) > 0$ and $\beta = \frac{-1}{2k}$, (4.2.2)

related to the Odd Hermite polynomials.

In order to evaluate (4.2.1) consider

$$I_1 = \int_{-\infty}^{\infty} \exp(-x^{2k})\, x^{2pk+2sk}\, H_{2pk}(x;k)\, x^{(\delta-1)m}\, x^{2k-2}$$

$$p'+\delta \Big\lceil q+\lambda \left[\begin{matrix} \Delta(\delta,-m), & (a_p); \\ \Delta(\lambda,\alpha), & (b_q); \end{matrix} \; ux^{-2kc} \right] dx .$$

The integrand being even function of x we have

$$I_1 = 2 \int_0^{\infty} x^{2k-2} \exp(-x^{2k})\, x^{2pk+2sk}\, H_{2pk}(x;k)\, x^{(\delta-1)m} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r\, x^{-2crk}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r \cdot r!} dx .$$

Interchanging order of summation and integration

$$I_1 = 2 \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!} \times$$

$$\times \int_0^{\infty} x^{2k-2} \exp(-x^{2k})\, x^{2pk+2sk}\, H_{2pk}(x;k)\, x^{(\delta-1)m-2crk}\, dx$$

(on putting $z = x^{2k}$ and $-\beta = \frac{1}{2k}$)

(7)

$$=(-2\beta)\sum_{r=0}^{\infty}\frac{\prod_{i=0}^{\delta-1}\left(\frac{-m+i}{\delta}\right)_r\left[(a_{p'})\right]_r\ u^r}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_r\left[(b_q)\right]_r\ r!}\int_0^{\infty} z^{p+\beta+s+(1-\delta)m\beta-cr}\ \times$$

$$\exp(-z)\ H_{2pk}(z)\ dz.$$

With the help of result (1.5.9) we get

$$I_1=(-2\beta)\sum_{r=0}^{\infty}\frac{\prod_{i=0}^{\delta-1}\left(\frac{-m+i}{\delta}\right)_r\left[(a_{p'})\right]_r\ u^r\ (-1)^p\ (1+p)_p}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_r\left[(b_q)\right]_r\ r!\ (1+\beta)_p}$$

$$\int_0^{\infty} z^{\beta+p+s+(1-\delta)m\beta-cr}\ e^{-z}\ z^{-\beta}\ e^{z}\ \frac{d^p}{dz^p}\left[e^{-z}\ z^{p+\beta}\right] dz$$

$$=\frac{(-1)^p\ (1+p)_p\ (-2\beta)}{(1+\beta)_p}\sum_{r=0}^{\infty}\frac{\prod_{i=0}^{\delta-1}\left(\frac{-m+i}{\delta}\right)_r\left[(a_{p'})\right]_r\ u^r}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_r\left[(b_q)\right]_r\ r!}\ \times$$

$$\times\int_0^{\infty} z^{p+s+(1-\delta)m\beta-cr}\ \frac{d^p}{dz^p}\left[e^{-z}\ z^{p+\beta}\right] dx$$

Integrating by parts p times we have

$$I_1=\frac{(-1)^{p+1}\ (1+p)_p\ 2\beta}{(1+\beta)_p}\sum_{r=0}^{\infty}\frac{\prod_{i=0}^{\delta-1}\left(\frac{-m+i}{\delta}\right)_r\left[(a_{p'})\right]_r\ u^r}{\prod_{j=0}^{\lambda-1}\left(\frac{\alpha+j}{\lambda}\right)_r\left[(b_q)\right]_r\ r!}\ \times$$

$$\times (-1)^p \, (s+m\beta-\delta m\beta-cr+1)_p \int_0^\infty z^{s+(1-\delta)m\beta-cr} \, e^{-z} \, z^{p+\beta} \, dz$$

$$= \frac{(-2\beta)\,(1+p)_p}{(1+\beta)_p} \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_{p'})\right]_r \, u^r}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r \, r!} \times$$

$$(s+m\beta-\delta m\beta-cr+1)_p \int_0^\infty e^{-z} \, z^{\{\beta+p+s+m\beta-\delta m\beta-cr\}} \, dz$$

(by the definition of the gamma function)

$$= \frac{(-2\beta)\,(1+p)_p}{(1+\beta)_p} \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_{p'})\right]_r \, u^r}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r \, r!} \times$$

$$\times \frac{\Gamma(s+p+m\beta-\delta m\beta-cr+1)\,\Gamma(p+s+m\beta-\delta m\beta-cr+\beta+1)}{\Gamma(s+m\beta-\delta m\beta-cr+1)}$$

$$= \frac{(-2\beta)\,(1+p)_p\,\Gamma(s+p+m\beta-\delta m\beta+1)\,\Gamma(p+s+m\beta-\delta m\beta+\beta+1)}{(1+\beta)_p\,\Gamma(s+m\beta-\delta m\beta+1)} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_{p'})\right]_r \, u^r \, (s+p+m\beta-\delta m\beta+1)_{-cr}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r \, r! \, (s+m\beta-\delta m\beta+1)_{-cr}}$$

$$\times \, (p+s+m\beta-\delta m\beta+\beta+1)_{-cr} \, .$$

Now using (1.5.7)

$$I_1 = \frac{(-2\beta)\ (1+p)_p\ \Gamma(s+p+m\beta-\delta m\beta+1)\ \Gamma(p+s+m\beta-\delta m\beta+\beta+1)}{(1+\beta)_p\ \Gamma(s+m\beta-\delta m\beta+1)} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_{p'})\right]_r u^r (-1)^{cr} \prod_{j=0}^{c-1} \left(\frac{j-s-m\beta+\delta m\beta}{c}\right)_r}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r\ r!\ \prod_{i=0}^{c-1} \left(\frac{-s-p-m\beta+\delta m\beta+i}{c}\right)_r (c)^{cr}}$$

$$\frac{1}{\prod_{j=0}^{c-1} \left(\frac{-s-p-m\beta+\delta m\beta-\beta+j}{c}\right)_r} \times$$

$$= \frac{(-2\beta)\ (1+p)_p\ \Gamma(s+p+m\beta-\delta m\beta+1)\ \Gamma(p+s+m\beta-\delta m\beta+\beta+1)}{(1+\beta)_p\ \Gamma(s+m\beta-\delta m\beta+1)}$$

$$F_{\lambda+q+2c}^{\delta+p'+c}\left[\begin{matrix} \Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta),\ (a_{p'}); \\ \Delta(\lambda,\alpha),\ \Delta(c,-s-p-m\beta+\delta m\beta), \\ \Delta(c,-s-p-m\beta+\delta m\beta-\beta),\quad (b_q); \end{matrix}\ \frac{u}{(-c)^c}\right]$$

$R_e\ (p+s+\beta+1) > 0$. This establishes (4.2.1).

Now inorder to evaluate (4.2.2) consider

$$I_2 = \int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k})\ x^{2pk+2sk+1}\ H_{2pk+1}(x;k)\ x^{(\delta-1)m} \times$$

$$F_{q+\lambda}^{p'+\delta}\left[\begin{matrix} \Delta(\delta,-m), & (a_{p'}); \\ \Delta(\lambda,\alpha), & (b_q); \end{matrix}\ ux^{-2ck}\right] dx$$

(integrand again being an even function of x)

$$= 2\int_{0}^{\infty} x^{2k-2} \exp(-x^{2k})\ x^{2pk+2sk+1}\ H_{2pk+1}(x;k)\ x^{(\delta-1)m} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r x^{-2ckr}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!} dx$$

(interchanging the order of summation & integration)

$$= 2 \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!} \times$$

$$\times \int_0^{\infty} x^{\{2k-1+2pk-2sk+(\delta-1)m-2ckr\}} \times$$

$$\exp(-x^{2k}) \; H_{2pk+1}(x;k) \; dx$$

(putting $z = x^{2k}$ and $-\beta = \frac{1}{2k}$)

$$= (-2\beta) \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!} \int_0^{\infty} z^{p+s+m\beta-\delta m\beta-cr} \times$$

$$\times \exp(-z) \; H_{2pk+1}(z) \; dz$$

(using (1.5.10))

$$= (-2\beta) \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!} \int_0^{\infty} z^{p+s+m\beta-\delta m\beta-cr} e^{-z} \times$$

$$\times \frac{(-1)^{p} \; 2^{2p+1} \; (\frac{3}{2})_p \; e^{z} \; d^{p} \left[e^{-z} \; x^{p-\beta} \right]}{(1-\beta)_p \; dz^{p}} \, dz$$

(integrating by parts p times)

$$= \frac{(-1)^{p+1} \; 2^{2p+2} \; (\frac{3}{2})_p}{(1-\beta)_p} \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} (\frac{-m+i}{\delta})_r \left[(a_p)\right]_r \; u^r}{\prod_{j=0}^{\lambda-1} (\frac{\alpha+j}{\lambda})_r \left[(b_q)\right]_r \; r!} \times$$

$$\times (-1)^{p} \; (s+m\beta-\delta m\beta-cr+1)_p \int_0^{\infty} z^{s+m\beta-\delta m\beta-cr} \; e^{-z} \; z^{p-\beta} \, dz$$

(by the definition of the gamma function)

$$= \frac{(-\beta) \; 2^{2p+2} (\frac{3}{2})_p}{(1-\beta)_p} \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} (\frac{-m+i}{\delta})_r \left[(a_p)\right]_r \; u^r}{\prod_{j=0}^{\lambda-1} (\frac{\alpha+j}{\lambda})_r \left[(b_q)\right]_r \; r!} \times$$

$$\times (s+m\beta-\delta m\beta-cr+1)_p \; \Gamma(s+m\beta-\delta m\beta-cr+p-\beta+1)$$

$$= \frac{(-\beta) \; 2^{2p+2} \; (\frac{3}{2})_p}{(1-\beta)_p} \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} (\frac{-m+i}{\delta})_r \left[(a_p)\right]_r \; u^r}{\prod_{j=0}^{\lambda-1} (\frac{\alpha+j}{\lambda})_r \left[(b_q)\right]_r \; r!} \times$$

$$\times \frac{\Gamma(s+m\beta-\delta m\beta-cr+p+1) \; \Gamma(s+m\beta-\delta m\beta-cr+p-\beta+1)}{\Gamma(s+m\beta-\delta m\beta-cr+1)}$$

$$= \frac{(-\beta)\, 2^{2p+2} \left(\frac{3}{2}\right)_p \Gamma(s+m\beta-\delta m\beta+p+1)\, \Gamma(s+m\beta-\delta m\beta+p-\beta+1)}{(1-\beta)_p\, \Gamma(s+m\beta-\delta m\beta+1)} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{(\delta-1)} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r u^r\, \Gamma(s+m\beta-\delta m\beta+p+1)_{-cr}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r r!\, \Gamma(s+m\beta-\delta m\beta+1)_{-cr}}$$

$$(s+m\beta-\delta m\beta+p-\beta+1)_{-cr}$$

(using (1.5.7))

$$= \frac{(-\beta)\, 2^{2p+2} \left(\frac{3}{2}\right)_p \Gamma(s+m\beta-\delta m\beta+p+1)\, \Gamma(s+m\beta-\delta m\beta+p-\beta+1)}{(1-\beta)_p\, \Gamma(s+m\beta-\delta m\beta+1)} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-m+i}{\delta}\right)_r \left[(a_p)\right]_r \prod_{i=0}^{c-1} \left(\frac{\delta m\beta-m\beta-s+i}{c}\right)_r}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_r \left[(b_q)\right]_r \prod_{j=0}^{c-1} \left(\frac{\delta m\beta-m\beta-s-p+j}{c}\right)_r} \times$$

$$\times \frac{\left[\frac{u}{(-c)^c}\right]^r}{\prod_{j=0}^{c-1} \left(\frac{\delta m\beta-m\beta+\beta-p-s+j}{c}\right)_r r!}$$

$$= \frac{(-\beta)\, 2^{2p+2} \left(\frac{3}{2}\right)_p \Gamma(s+m\beta-\delta m\beta+p+1)\, \Gamma(s+m\beta-\delta m\beta+p-\beta+1)}{(1-\beta)_p\, \Gamma(s+m\beta-\delta m\beta+1)} \times$$

$$\times \; {}_{\delta+p'+c}F_{\lambda+q+2c}\left[\begin{matrix}\Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta),\ (a_p); \\ \Delta(\lambda,\alpha),\ \Delta(c,-s-m\beta+\delta m\beta-p), \\ \Delta(c,-s-p-m\beta+\delta m\beta+\beta),\ (b_q);\end{matrix}\ \frac{u}{(-c)^c}\right].$$

This establishes (4.2.2).

Particular cases : Selecting parameters as in §1.3 with $\mu=-2ck$ and even n the result (4.2.1) gives number of special cases We mention here a few of them.

(i) For the generalized Rice's polynomials (1.2.17):

$$\int_{-\infty}^{\infty} x^{2k-2}\ \exp(-x^{2k})\ x^{2pk+2sk}\ H_{2pk}(x;k)\ H_m(a,b)\ (\rho,\sigma;\ x^{-2k})\ dx$$

$$= \frac{(-2\beta)\ (1+p)_p\ \Gamma(s+p+1)\ \Gamma(s+p+\beta+1)\ (1+a)_m}{(1+\beta)_p\ \Gamma(s+1)\ m!} \times$$

$${}_4F_4\left[\begin{matrix}-m,\ -s,\ m+a+b+1,\ \rho; \\ 1+a,\ -s-p,\ -s-p-\beta,\ \sigma;\end{matrix}\ -1\right]$$

with $c=1$, $R_e(p+s+\beta)>-1$, $-\beta=\frac{1}{2k}$. (4.2.3)

(ii) For the generalized Sister Celine's polynomials (1.2.18) :

$$\int_{-\infty}^{\infty} x^{2k-2}\ \exp(-x^{2k})\ x^{2pk+2sk}\ H_{2pk}(x;k)\ f_m^{(a,b)}\left(\begin{matrix}a_2,\dots,a_{p'}; \\ b_2,\dots,b_q;\end{matrix}\ x^{-2k}\right)dx$$

$$= \frac{(-2\beta)\ (1+p)_p\ \Gamma(s+p+1)\ \Gamma(s+p+\beta+1)\ (1+a)_m}{(1+\beta)_p\ \Gamma(s+1)\ m!} \times$$

$$\times \; {}_{p'+2}F_{q+3}\left[\begin{matrix} -m, & -s, & m+a+b+1, & a_2, --- , & a_{p'}; \\ 1+a, & -s-p, & -s-p-\beta, & \tfrac{1}{2}, b_2, ---, & b_q; \end{matrix} \; - 1\right]$$

with $c = 1$, $R_e \; (p+s+\beta) > -1$, $\beta = -\frac{1}{2k}$. (4.2.4)

(iii) For the generalized Bessel polynomials (1.2.19) :

$$\int_{-\infty}^{\infty} x^{2k-2} \; \exp(-x^{2k}) \; x^{2pk+2sk} \; H_{2pk}^{(x;k)} \; y_m \; (x^{-2k}) \; dx$$

$$= \frac{(-2\beta) \; (1+p)_p \; \Gamma(s+p+1) \; \Gamma(s+p+\beta+1)}{(1+\beta)_p \; \Gamma(s+1)}$$

$${}_3F_4\left[\begin{matrix} -m, & -s, & 2\nu+m; & \\ 1+b, & \nu+\tfrac{1}{2}, & -s-p, & -s-p-\beta; \end{matrix} \; - 1\right]$$

with $c = 1$, $R_e \; (p + s + \beta) > -1, -\beta = \frac{1}{2k}$. (4.2.5)

(iv) For the Bedient polynomials $R_m(\beta, \gamma; x)$ (1.2.20):

With $c = 1$, $a_1 = d-b$, $b_1 = 1-b-m$, $\alpha = d$

$$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k}) \; x^{2pk+2sk+m-2mk} \; R_m(b, d; x^k) \; dx$$

$$= \frac{(-2\beta) \; (1+p)_p \; \Gamma(s+p-m\beta+1) \; \Gamma(p+s-m\beta \; +\beta+1) \; 2^m \; (b)_m}{(1+\beta)_p \; \Gamma(s-m\beta+1) \; m!} \times$$

$$\times \; {}_4F_4 \left[\begin{matrix} \Delta(2,-m), \; d-b, \; -s+m\beta \; ; \\ d, \; 1-b-m, \; -s-p+m\beta, \; -s-p+m\beta-\beta \; ; \end{matrix} \; -1 \right],$$

$R_e\,(p+s+\beta-m\beta+1) > 0, \; -\beta = \frac{1}{2k}$ and $c = 1$. (4.2.6)

(v) For the Lommel polynomials (1.2.22) :

$$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k}) \; x^{2pk+2sk+m-mk} \; H_{2pk}^{(x;k)} \; R_{m,\nu}\left(\frac{1}{x^k}\right) dx$$

$$= \frac{(\nu)_m \; 2^m \; (-2\beta) \; (1+p)_p \; \Gamma(s+p-m\beta+1) \; \Gamma(s+p-m\beta+\beta+1)}{(1+\beta)_p \; \Gamma(s-m\beta+1)}$$

$${}_3F_5 \left[\begin{matrix} \Delta(2,-m), \; -s+m\beta \; ; \\ \nu, \; -s-p+m\beta, \; -s-p+m\beta-\beta, \; -m, \; 1-\nu-\beta \; ; \end{matrix} \; 1 \right]$$

with $c=1$, $R_e\,(p+s-m\beta+\beta+1) > 0$, $\beta = -\frac{1}{2k}$, (4.2.7)

(vi) For the Toscano polynomials (1.2.23) :

$$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k}) \; x^{2pk+2sk} \; H_{2pk}^{(x;k)} \; S_n(x^{-2k}) \; dx$$

$$= \frac{(-2\beta) \; (1+p)_p \; (a)_{2m} \; \Gamma(s+p+1) \; \Gamma(s+p+\beta+1)}{(1+\beta)_p \; m! \; (a)_m \; \Gamma(s+1)} \times$$

$${}_{p'+2}F_{q+3} \left[\begin{matrix} -m, \; -s, \; (a_{p'}) \; ; \\ a+m, \; -s-p, \; -s-p-\beta, \; (b_q) \; ; \end{matrix} \; -1 \right],$$

with $c = 1$, $R_e\,(p+s+\beta+1) > 0$, $\beta = -\frac{1}{2k}$. (4.2.8)

(vii) For the Shah's polynomials (1.3.16) :

$$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k})\; x^{2pk+2sk}\; H_{2pk}^{(x;k)}\; x^{(\delta-1)m}$$

$$\bar{\Gamma}^{p'+\delta}_{q}\left[\begin{matrix}\Delta(\delta,-m), & (a_{p'}) ; \\ (b_q) ; & \end{matrix}\; ux^{-2ck}\right] dx$$

$$= \frac{(-2\beta)\;(1+p)_p\;\Gamma(s+p+m\beta-\delta m\beta+1)\;\Gamma(s+p+m\beta-\delta m\beta+\beta+1)}{(1+\beta)_p\;\Gamma(s+m\beta-\delta m\beta+1)}$$

$$\bar{\Gamma}^{p'+\delta+c}_{q+2c}\left[\begin{matrix}\Delta(\delta,-m),\ \Delta(c,\ -s-m\beta+\delta m\beta),\ (a_{p'}) ; \\ \Delta(c,-s-p-m\beta+\delta m\beta),\ \Delta(c,-s-p-m\beta+\delta m\beta-\beta), \\ (b_q) ;\end{matrix}\; \frac{u}{(-c)^c}\right],$$

$R_e\,(p+s+m\beta-\delta m\beta+\beta) > -1$, $\beta = -\frac{1}{2k}$. (4.2.9)

Similarly particular cases related to the Odd Hermite polynomials can be obtained from the result (4.2.2).

4.3. <u>Expansion formulae</u> : In this section we establishes two expansion formulae precisely for even m and $\mu = -2ck$

(A) $$x^{2pk+2sk+(\delta-1)m}\, H^{p'+\delta}_{q+\lambda}\left[ux^{-2ck}\ \middle|\ \begin{matrix}\Delta(\delta,-m), & (a_p,); \\ \Delta(\lambda,\alpha), & (b\ \ ); \end{matrix}\right]$$

$$= \sum_{r=0}^{\infty} \frac{\Gamma(s+r+m\beta-\delta m\beta+1)\ \Gamma(s+r+m\beta-\delta m\beta+\beta+1)}{\Gamma(1+\beta)\ \Gamma(s+m\beta-\delta m\beta+1)\,(2r)!}\times$$

$$H^{p'+\delta+c}_{q+\lambda+2c}\left[\frac{u}{(-c)^c}\ \middle|\ \begin{matrix}\Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta), & (a_p,) ; \\ \Delta(\lambda,\alpha),\ \Delta(c,-s-r-m\beta+\delta m\beta), & \\ \Delta(c,-s-r-m\beta+\delta m\beta-\beta), & (b_q) ; \end{matrix}\right]\times$$

$H_{2kr}(x;k)$,

$R_e\,(s+m\beta-\delta m\beta) > -1$, $R_e\,(\beta) > 0$ and $\beta = \frac{1}{2k}$. (4.3.1)

(B) $$x^{2pk+2sk+(\delta-1)m+1}\, H^{p'+\delta}_{q+\lambda}\left[ux^{-2ck}\ \middle|\ \begin{matrix}\Delta(\delta,-m), & (a_p,) ; \\ \Delta(\lambda,\alpha), & (b_q) ; \end{matrix}\right]$$

$$= \sum_{r=0}^{\infty} \frac{2^{(2r-1)}\ (\frac{3}{2})_r\ \Gamma(s+r+m\beta-\delta m\beta+1)\ \Gamma(s+r+m\beta-\delta m\beta+\beta+1)}{(1+r)_{r+1}\ (2r+1)!\ \Gamma(1-\beta)\ \Gamma(s+m\beta-\delta m\beta+1)}\times$$

$$\times\ H^{\delta+p'+c}_{\lambda+q+2c}\left[\frac{u}{(-c)^c}\ \middle|\ \begin{matrix}\Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta), & (a_p,) ; & ; \\ \Delta(\lambda,\alpha),\ \Delta(c,-s-r-m\beta+\delta m\beta), & & , \\ \Delta(c,-s-r-m\beta+\delta m\beta+\beta), & (b_q) ; & \end{matrix}\right]\times$$

$\times\ H_{2kr+1}(x;k)$, (4.3.2)

$R_e\,(s+m\beta-\delta m\beta-\beta) > -1$ and $\beta = -\frac{1}{2k}$.

In order to establish (4.3.1) we suppose

$$x^{2pk+2sk+(\delta-1)m} \; \bar{\Gamma}_{p'+\delta}\overline{\Gamma}_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_{p'}) & ; \\ \Delta(\lambda,\alpha), & (b_q) & ;\end{matrix} \; ux^{-2ck}\right]$$

$$= \sum_{r=0}^{\infty} g_{2kr} \; H_{2kr}(x;k) .$$

Multiply both sides by $x^{2k-2}\exp(-x^{2k})\,H_{2pk}^{(x;k)}$ and integrate both the sides with respect to x over the interval $(-\infty,\infty)$

$$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k})\, H_{2pk}^{(x;k)}\; x^{\{2pk+2sk+(\delta-1)m\}} \times$$

$$\times\; \bar{\Gamma}_{p'+\delta}\overline{\Gamma}_{q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_{p'}) & ; \\ \Delta(\lambda,\alpha), & (b_q) & ;\end{matrix} \; ux^{-2ck}\right] dx$$

$$= \int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k})\, H_{2pk}^{(x;k)} \sum_{r=0}^{\infty} g_{2kr}\, H_{2kr}^{(x;k)}\; dx$$

(using orthogonality property)

$$= g_{2pk}\int_{-\infty}^{\infty} x^{2k-2}\exp(-x^{2k})\left[H_{2pk}^{(x;k)}\right]^2 dx$$

(using (1.5.11))

$$= g_{2pk}\;\frac{(-2\beta)\left[(2p)!\right]^2\Gamma(1+\beta)}{p!\;(1+\beta)_p} .$$

Using (4.2.1), we get

$$g_{2pk} = \frac{p!\,(1+p)_p\,\Gamma(s+p+m\beta-\delta m\beta+1)\,\Gamma(s+p+m\beta-\delta m\beta+\beta+1)}{\left[(2p)!\right]^2\Gamma(1+\beta)\,\Gamma(s+m\beta-\delta m\beta+1)} \times$$

$$\times\; H_{p'+\delta+c,\; q+\lambda+2c}\left[\frac{u}{(-c)^{c}}\;\middle|\;\begin{matrix}\Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta),\ (a_{p'});\\[6pt] \Delta(\lambda,\alpha),\ \Delta(c,-s-p-m\beta+\delta m\beta),\\ \Delta(c,-s-p-m\beta+\delta m\beta-\beta),\ (b_q);\end{matrix}\right]$$

substituting this value of g_{2pk} in (4.3.3) we get

$$x^{2pk+2sk+(\delta-1)m}\; H_{p'+\delta,\; q+\lambda}\left[ux^{-2ck}\;\middle|\;\begin{matrix}\Delta(\delta,-m),\ (a_{p'});\\ \Delta(\lambda,\alpha),\ (b_q);\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{r!\,(1+r)_r\,\Gamma(s+r+m\beta-\delta m\beta+1)\,\Gamma(s+r+m\beta-\delta m\beta+\beta+1)}{[(2r)!]^{2}\,\Gamma(1+\beta)\,\Gamma(s+m\beta-\delta m\beta+1)}$$

$$H_{p'+\delta+c,\; q+\lambda+2c}\left[\frac{u}{(-c)^{c}}\;\middle|\;\begin{matrix}\Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta),\ (a_{p'});\\[6pt] \Delta(\lambda,\alpha),\ \Delta(c,-s-r-m\beta+\delta m\beta-\beta),\\ \Delta(c,-s-r-m\beta+\delta m\beta-\beta),\ (b_q);\end{matrix}\right]\times$$

$$\times\; H_{2kr}(x;k)$$

since $r!\,(1+r)_r = (2r)!$

$$=\sum_{r=0}^{\infty}\frac{\Gamma(s+r+m\beta-\delta m\beta+1)\,\Gamma(s+r+m\beta-\delta m\beta+\beta+1)}{(2r)!\,\Gamma(1+\beta)\,\Gamma(s+m\beta-\delta m\beta+1)}$$

$$H_{\delta+p'+c,\; \lambda+q+2c}\left[\frac{u}{(-c)^{c}}\;\middle|\;\begin{matrix}\Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta),\ (a_{p'});\\[6pt] \Delta(\lambda,\alpha),\ \Delta(c,-s-r-m\beta+\delta m\beta),\\ \Delta(c,-s-r-m\beta+\delta m\beta-\beta),\ (b_q);\end{matrix}\right]\times$$

$\times H_{2kr}(x;k)$,

$R_e\ (s + m\beta - \delta m\beta) > -1, \quad R_e\ (\beta) > 0.$

Now inorder to establish the relation (4.3.2) we consider

$$x^{\{2pk+2sk+(\delta-1)m+1\}}\ \overline{\underline{\ \ }}\,{}^{p'+\delta}_{\ \ q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_{p'}) ; \\ \Delta(\lambda,\alpha), & (b_q) ;\end{matrix}\quad ux^{-2ck}\right]$$

$$= \sum_{r=0}^{\infty} g_{2kr+1}\ H_{2kr+1}^{(x;k)} \qquad (4.3.4)$$

Multiply both sides by $x^{2k-2}\ \exp(-x^{2k})\ H_{2pk+1}^{(x;k)}$ and integrate with respect to x over the interval $(-\infty,\infty)$

$$\int_{-\infty}^{\infty} x^{2k-2}\ \exp(-x^{2k})\ H_{2pk+1}^{(x;k)}\ x^{\{2pk+2sk+(\delta-1)m+1\}} \times$$

$$\overline{\underline{\ \ }}\,{}^{p'+\delta}_{\ \ q+\lambda}\left[\begin{matrix}\Delta(\delta,-m), & (a_{p'}) ; \\ \Delta(\lambda,\alpha), & (b_q) ;\end{matrix}\quad ux^{-2ck}\right] dx$$

$$= g_{2pk+1}\int_{-\infty}^{\infty} x^{2k-2}\ \exp(-x^{2k})\ H_{2pk+1}^{(x;k)} \sum_{r=0}^{\infty} g_{2kr+1}\ H_{2kr+1}^{(x;k)}\ dx$$

(using (1.5.12))

$$= g_{2pk+1}\ \frac{(-8\beta)\ \ (1+p)_{p+1}\ (2p+1)!\ \Gamma(1-\beta)}{(1-\beta)_p}\ .$$

Using the integral (4.2.2) we get

$$g_{2pk+1} = \frac{2^{(2p-1)} \, (\tfrac{3}{2})_p \, \Gamma(s + m\beta - \delta m\beta + p + 1)}{(1+p)_{p+1} \, (2p+1)! \, \Gamma(1-\beta) \, \Gamma(s+m\beta - \delta m\beta + 1)}$$

$$F^{\,p'+\delta+c}_{\;q+\lambda+2c}\left[\begin{matrix}\Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta),\ (a_{p'});\\[1em] \Delta(\lambda,\alpha),\ \Delta(c,-s-r-m\beta+\delta m\beta),\\ \Delta(c,-s-r-m\beta-\delta m\beta+\beta),\quad (b_q);\end{matrix}\ \frac{u}{(-c)^c}\right]$$

substituting this value of g_{2pk+1} in (4.3.4) we get (4.3.2),

Particular cases: By suitable choice of the parameters as in § 1.3, (4.3.1) and (4.3.2) will give rise to number of particular cases with $\mu = -2ck$, m even number and p replaced by p' .However we quote here a few interesting particular cases of the relation (4.3.1).

(i) For the generalized Rice's polynomials (1.2.17):

$$x^{2pk+2sk} \, H_m^{(a,b)}(\rho, \sigma; x^{-2k}) = \frac{(1+a)_m}{m! \, \Gamma(1+\beta) \, \Gamma(s+1)} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(s+r+1) \, \Gamma(s+r+\beta+1)}{(2r)!} \, {}_4F_4\left[\begin{matrix}-m,\ m+a+b+1,\ -s,\ \rho;\\ 1+a,\ -s-r,\ -s-r-\beta,\ \sigma;\end{matrix}\ -1\right] \times$$

$$\times \, H_{2kr}(x;k)$$

with $c = 1$, $R_e(s + r + \beta + 1) > 0$. (4.3.5)

(ii) For the generalized Sister Celine's polynomials (1.2.18) :

$$x^{2pk+2sk} \, f_m^{(a,b)}\left(\begin{matrix} a_2, & a_3, ---, & a_p, ; \\ b_2, & -----, b_q ; \end{matrix} \; x^{-2k}\right)$$

$$= \sum_{r=0}^{\infty} \frac{\Gamma(s+r+1)\;\Gamma(s+r+\beta+1)}{(2r)!\;\Gamma(1+\beta)\;\Gamma(s+1)} \times$$

$$\times f_m^{(a,b)}\left(\begin{matrix} a_2, ---, & a_p, -s; \\ b_2, ----, & bq, -s-r, -s-r-\beta ; \end{matrix} \; -1\right) H_{2kr}^{(x;k)}$$

with $c = 1$, $R_e\ (s + r + \beta + 1) > 0$. (4.3.6)

(iii) For the generalized Bessel polynomials (1.2.19) :

$$x^{2pk+2sk} \, y_n\,(x^{-2k}) = \sum_{r=0}^{\infty} \frac{\Gamma(s+r+1)\;\Gamma(s+r+\beta+1)}{\Gamma(\beta+1)\;\Gamma(s+1)\;(2r)!} \times$$

$${}_3F_4\left[\begin{matrix} -m, -s, \gamma-\beta ; \\ \gamma, 1+b, -s-r, -s-r-\beta ; \end{matrix} \; -1\right] H_{2kr}^{(x;k)}$$

with $c = 1$, $R_e\ (s + r + \beta + 1) > 0$. (4.3.7)

(iv) For the Bedient polynomials $R_n\ (\beta, \gamma, x)$ (1.2.20) with $c = 1$, $a_1 = d-b$, $b_1 = 1 - b-m$, $\alpha = d$ and $p' = 1$ we get

$$x^{\{2pk+2sk+m-mk\}} \, R_m(b, d; x^k) =$$

$$= \frac{(b)_m \; 2^m}{m!} \sum_{r=0}^{\infty} \frac{\Gamma(s+r-m\beta+1)\;\Gamma(s+r-m\beta+\beta+1)}{\Gamma(1+\beta)\;\Gamma(s-m\beta+1)\;(2r)!} \times$$

$$\times\; {}_4F_4\left[\begin{matrix}\Delta(2,-m), & -s+m\beta, & d-b; \\ d, & 1-b-m, & -s-r+m, & -s-r+m\beta-\beta, & (b_q);\end{matrix}\; -1\right] \times$$

$$\times\; H_{2kr}(x;k) \qquad (4.3.8)$$

$R_e\,(s+r+\beta-m\beta+1) > 0,\quad c = 1.$

(v) For the Lommel polynomials (1.2.22):

$$x^{\{2pk+2sk+\nu m-m\alpha\}}\; R_{m,\nu}\left(\frac{1}{x^k}\right)$$

$$= \frac{(\nu)_m \; 2^m}{1} \sum_{r=0}^{\infty} \frac{\Gamma(s+r-m\beta+1)\;\Gamma(s+r-m\beta+\beta+1)}{\Gamma(1+\beta)\;\Gamma(s-m\beta+1)\;(2r)!} \times$$

$$\times\; {}_3F_5\left[\begin{matrix}\Delta(2,-m), & -s+m\beta; \\ \nu, & -m, & 1-\nu-m, & -s-r+m\beta, & -s-r+m\beta-\beta;\end{matrix}\; 1\right] \times 1$$

$$\times\; H_{2kr}(x;k),$$

with $c = 1$, $R_e\,(s+r-m\beta+\beta+1) > 0.$ (4.3.9)

(vi) For the Toscano polynomials (1.2.23) :

$$x^{\{2pk+2sk\}}\; S_m(x^{-2k}) = \frac{(2a)_m}{m!\,(a)_m} \sum_{r=0}^{\infty} \frac{\Gamma(s+r+1)\;\Gamma(s+r+\beta+1)}{\Gamma(1+\beta)\;\Gamma(s+1)\;(2r)!} \times$$

$$ {}_{p'+2}F_{q+3}\left[\begin{matrix} -m, & -s, & (a_{p'}); & \\ a+m, & -s-r, & -s-r-\beta, & (b_q); \end{matrix} \; -1\right] H_{2kr}^{(x;k)} $$

with $c = 1$, $R_e\ (s+r+\beta+1) > 0.$ (4.3.10)

(vii) For the Shah's polynomials (1.3.16) :

$$ x^{\{2pk+2sk+(\delta-1)m\}} \; {}_{p'+\delta}F_{q}\left[\begin{matrix} \Delta(\delta,-m), & (a_{p'}) ; \\ & (b_q) ; \end{matrix} \; ux^{-2ck}\right] $$

$$ = \sum_{r=0}^{\infty} \frac{\Gamma(s+r+m\beta-\delta m\beta+1)\ \Gamma(s+r+m\beta-\delta m\beta+\beta+1)}{\Gamma(1+\beta)\ \Gamma(s+m\beta-\delta m\beta+1)\ \ (2r)!} $$

$$ {}_{p'+\delta+c}F_{q+2c}\left[\begin{matrix} \Delta(\delta,-m),\ \Delta(c,-s-m\beta+\delta m\beta),\ (a_{p'}) ; \\ \Delta(c,-s-r-m\beta+\delta m\beta), \\ \Delta(c,-s-r-m\beta+\delta m\beta-\beta),\ (b_q) ; \end{matrix} \; \frac{u}{(-c)^c}\right] \times $$

$$ \times H_{2kr}^{(x;k)}, \qquad (4.3.11) $$

$R_e\ (s + m\beta-\delta m\beta) > -1$, $R_e\ (\beta) > 0$ and $\beta = -\frac{1}{2k}$.

Similarly we can have the particular cases of the relation (4.3.2) related to the Odd Hermite polynomials.

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