-: CHAPTER ONE :-

Introduction

1.1 Fractional calculus

In the study of calculus we encounter the differential operators $\frac{d}{dx}, \frac{d^2}{dx^2}, \ldots, \frac{d^n}{dx^n}$, and one may think whether it is necessary for the order n of differentiation to be always an integer. Why should there not be a $\frac{d^{1/2}}{dx^{1/2}}$ operator or $\frac{d^{\sqrt{2}}}{dx^{\sqrt{2}}}$, or even $\frac{d^{-1}}{dx^{-1}}$? In fact we know that $\frac{d^{-1}}{dx^{-1}}$ is nothing but an indefinate intergral, but fractional orders of differentiation are more mysterious because they have no obvious geometric interpreation along the lines of the customary introduction to derivatives and integrals as slopes and areas. Even great mathematicians like Leibnitz L'Hospital and others tried to give the meaning to $\frac{d^n}{dx^n}$, when n is fraction. L'Hospital wrote a letter to Leibnitz asking about " What if n be 1/2?" In 1695 Leibnitz [1] replied to L'Hospital "It will lead to a paradox, from which one day useful consequences will be drawn." In 1697, Leibnitz [2] discussed walli's infinite product for π and used the notation $d^{1/2}y$ to denote a derivative of order 1/2. In 1819, Lacroix [3] in his book developed a formula for fractional differentiation for the n^{th} derivative of V^m by induction. Then he formally replaced n with the fraction 1/2, and together with the fact that $\Gamma(1/2) = (\pi)^{1/2}$, he obtained

$$(1.1) \qquad \frac{d^{1/2} v}{dv^{1/2}} = \frac{2\ (v)^{1/2}}{(\pi)^{1/2}}$$

The systematic studies seem to have been made in the beginning and middle of the 19^{th} century by Liouville [4], Riemann [5] and Holmgren [6]. Able [7] was probably the first to give an application of fractional calculus. He used derivatives of arbitrary order to solve the tautochrone problem. The integral he worked with

$$(1.2) \qquad \int_0^x (x-t)^{-1/2} f(t)\, dt$$

is precisely of the same form that Riemann used to define fractional operators. The first major study of fractional calculus started with Liouville [4]. He considered $\left(\frac{d^{1/2}}{dx^{1/2}}\right) e^{2x}$, and solved some problems in mechanics and geometry by using fractional operations.

In the present century remarkable contributions have been made to both theory and application of the fractional calculus. Weyl [8], Hardy [9], Hardy and Littlewood [10,11], Kober [12] and Kuttner [13] examined some rather special, but natural properties of fractional operators of functions belonging to Lebesgue and Lipschitz classes. Erdelyi [14,15,16] and Osler [17] have given definations of fractional operators with respect to arbitrary functions and Post [18] used difference quotients to define generalized differentiation for operators f(D), where D denotes differentiation and f is suitably restricted function. Riesz [19] has developed a theory of fractional integration for functions of more than one variable. Erdelyi [20,21] has applied the fractional calculus to integral equations and Higgins [22] has used fractional integral operators to solve differential equations, Prabhakar [23] has studied some integral equations containing hypergeometric functions in two variables by using fractional integration.

1.11 Operators of fractional integration of one variable

Fractional integration is an immediate generalization of repeated integration. If the function $f(x)$ is integrable in any interval say $(0, a)$ where $a > 0$, we define the first integral $F_1(x)$ of $f(x)$ by the formula

$$(1.3) \qquad F_1(x) = \int_0^x f(t)\,dt$$

and the subsequent integrals by the recursion formula

$$(1.4) \qquad F_{r+1}(x) = \int_0^x F_r(t)\,dt, \quad r = 1,2 \ldots\ldots$$

We can prove by induction that for any positive integer n

$$(1.5) \qquad F_{n+1}(x) = \frac{1}{n!} \int_0^x (x-t)^n f(t)\,dt$$

Similarly we define an indefinate integral $F_n^*(x)$ by the formulae

$$(1.6) \qquad F_1^*(x) = -\int_x^\infty f(t)dt, \; F_{r+1}^*(x) = -\int_x^\infty F_r^*(t)dt,$$

$r = 1, 2, \ldots\ldots\ldots$

Again we can prove by induction that for any positive integer n

$$(1.7) \qquad F_{n+1}^*(X) = \frac{1}{n!} \int_X^\infty (t-x)^n f(t)\,dt$$

provided that $f(x)$ is of such a nature that the integral exists. The Riemann – Liouville fractional integral is a generalization of the integral on the right hand side of equation (1.5).

The integral

$$(1.8)\quad R_{\alpha}\left\{f(t);\, x\right\} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

is convergent for a wide class of functions f (t) if $\operatorname{Re}\alpha > 0$. The upper limit of integration x may be real or complex; in the latter case the path of integration is the straight line $t = xs$, $0 \leq s \leq 1$. The integral reduces to the integral (1.5) in the case when $\alpha = n+1$, a positive integer, so that when α is a positive integer the integral (1.8) is a repeated integral. It is called the " Riemann-Liouville fractional integral of order α."

Hardy and Littlewood [24] considered the fractional integral

$$(1.9)\quad f_{\alpha}(x) = \int_{-\infty}^{x} f(t)\, (x-t)^{\alpha-1} dt, \quad 0 < \operatorname{Re}\alpha < 1$$

while Love and Young [25] considered the integral

$$(1.10)\quad f_{\alpha}^{+}(a,x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)\, (x-t)^{\alpha-1} dt,$$

$$a \leq x \leq b, \quad \operatorname{Re}\alpha > 0$$

f(x) being integrable in (a, b).

The Weyl fractional integral is a generalization of the integral on the right hand side of equation (1.7), it is defined by the equation

(1.11) $$W_\alpha \left\{ f(t);x \right\} = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\, dt, \quad \operatorname{Re}\alpha > 0$$

A fractional integral closely related to Weyl's has been introduced by Love and Young [25] who considered the integral

(1.12) $$f_\alpha^- (x,b) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)\, (t-x)^{\alpha-1} dt, \quad \operatorname{Re}\alpha > 0$$

We adopt the convention that

(1.13) $$R_0 = I, \quad W_0 = I$$

where I denotes the identity operator.

The fractional integral operators as defined by Erdelyi [15] are as follows

(1.14) $$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

$$I_x^0 f(x) = f(x)$$

(1.15) $$K_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)\, dt$$

$$K_x^{0} f(x) = f(x)$$

(1.16) $$I_x^{\eta,\alpha} f(x) = x^{-\eta-\alpha}\, I_x^{\alpha}\, x^{\eta} f(x)$$
$$= x^{-\eta-\alpha} \frac{1}{\Gamma(\alpha)} \int_0^{x} (x-t)^{\alpha-1} t^{\eta} f(t)\,dt.$$

(1.17) $$K_x^{\eta,\alpha} f(x) = x^{\eta} K_x^{\alpha}\, x^{-\eta-\alpha} f(x)$$
$$= \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1}\, t^{-\eta-\alpha} f(t)\, dt.$$

Fractional integral operators with respect to x^A may be defined for $A > 0$ by similer formulae by replacing x by x^A. Thus we write

(1.18) $$I_{x^A}^{\eta,\alpha} f(x) = \frac{x^{-A\eta-A\alpha}}{\Gamma(\alpha)} \int_0^{x} (x^A-t^A)^{\alpha-1} t^{A\eta} f(t)\, d(t^A)$$

(1.19) $$K_{x^A}^{\eta,\alpha} f(x) = \frac{x^{A\eta}}{\Gamma(\alpha)} \int_x^{\infty} (t^A-x^A)^{\alpha-1} t^{-A\eta-A\alpha} f(t)\,d(t^A)$$

Here the function $x^{\eta+\alpha} I_x^{\eta,\alpha} f$ is the " Riemann - Liouville" integral of order α of $t^{\eta} f(t)$, While the function $x^{-\eta} K_x^{\eta,\alpha} f$ is the " Weyl- integral of order α.

1.12 Fractional Integration of the functions of two variables.

Mourya [26] has developed fractional integration for the functions of two variables on the line of Kober and Erdelyi and discussed some of their fundamental properties and simple identities.

Two of the fractional integral operators defined by Mourya [26, p 173], are as follows :

(1.20) $$I_x^{\alpha} I_y^{\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x-t)^{\alpha-1} (y-z)^{\beta-1} \times f(t,z)\, dt\, dz,$$

(1.21) $$K_x^{\alpha} K_y^{\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^{\infty} \int_y^{\infty} (t-x)^{\alpha-1} (z-y)^{\beta-1} \times f(t,z)\, dt\, dz,$$

$$K_x^0 K_y^0 f(x,y) = f(x,y),\ I_x^0 I_y^0 f(x,y) = f(x,y)$$

(1.22) $$I_x^{\eta,\alpha} I_y^{\tau,\beta} f(x,y) = x^{-\eta-\alpha} y^{-\tau-\beta} I_x^{\alpha} I_y^{\beta} x^{\eta} y^{\tau} f(x,y)$$

(1.23) $$K_x^{\eta,\alpha} K_y^{\tau,\beta} f(x,y) = x^{\eta} y^{\tau} K_x^{\alpha} K_y^{\beta} x^{-\eta-\alpha} y^{-\tau-\beta} f(x,y)$$

Here the function $x^{\eta+\alpha} y^{\tau+\beta} I_x^{\eta,\alpha} I_y^{\tau,\beta} f$ is " Riemann - Liouville" type double integrals of order η and τ of $t^{\eta} z^{\tau}$ f (t,z) with/* the function $x^{-\eta} y^{-\tau} K_x^{\eta,\alpha} K_y^{\tau,\beta} f$ is Weyl type double integral. The function f(t,z) is a complex valued function in the open Set D. We consider it as a meromorphic function which can be summed up in

*/ origin (0,0), while

terms of power series near the vicinities of its poles. Thus f(t,z) may be entire function of two variables with essential singularities (0,0) pr (∞,∞), or (0,∞) and (∞, 0). We may visualize the function f(t,z) is an analytic function in the connected open set D of complex field C^2, X, and y be its subsets such that all $t \in X$ and $z \in Y$. Now the Lebesgue integral of the function f(t,z) in the unbounded set D is denoted by $\iint_D f(t,z)\,dt\,dz$. If f(t,z) is analytic in the rectangle D [(o,∞),(0,∞)] then

$$\iint_D f(t,z)\,dt\,dz = \int_0^\infty \int_0^\infty f(t,z)\,dt\,dz.$$

Further ∅ and ψ infinitely differentiable functions with $\phi'(x) \geq 0$, $\psi'(y) > 0$, we write

$$(1.24)\qquad I^{\eta,\alpha}_{\phi(x)}\, I^{\tau,\beta}_{\psi(y)}\, f(x,y) = \frac{[\phi(x)]^{-\eta-\alpha}\,[\psi(y)]^{-\tau-\beta}}{\Gamma(\alpha)\;\Gamma(\beta)}\; x$$

$$\int_0^x \int_0^y [\phi(x)-\phi(t)]^{\alpha-1}.\,[\psi(y)-\psi(z)]^{\beta-1}\; x$$

$$[\phi(t)]^{\eta}\,[\psi(z)]^{\tau} f(t,z)\, d\phi(t).\, d\psi(z).$$

$$(1.25)\qquad K^{\eta,\alpha}_{\phi(x)}\; K^{\tau,\beta}_{\psi(y)}\; f(x,y) = \frac{[\phi(x)]^{\eta}\,[\psi(y)]^{\tau}}{\Gamma(\alpha)\;\Gamma(\beta)}\; x$$

$$\int_x^\infty \int_y^\infty [\phi(t)-\phi(x)]^{\alpha-1}\,[\psi(z)-\psi(y)]^{\beta-1}\; x$$

$$\left[\phi(t)\right]^{-\eta-\alpha} \times \left[\psi(z)\right]^{-\tau-\beta} f(t,z)\, d\phi(t) d\psi(z).$$

1.13 Applications of fractional calculus

Many problems in the physical sciences can be expressed and solved succinctly by the use of the fractional calculus. Fractional calculus can be categorized as applicable mathematics. The properties and theory of these fractional operators are proper objects of study in their own right. Scientists and applied Mathematicians in the last decade, found the fractional calculus useful in various fields. Within mathematics, the subject makes contact with a very large segment of classical analysis and provides a unifying theme for great many known, and some new, results. Applications outside mathematics include such otherwise unrelated topics as, transmission line theory, chemical analysis of aqueous solutions, design of heat-flux meters, rheology of soils, growth of intergranular grooves at metal surfaces, quantum mechanical calculations, electro-chemistry, general transport problems, diffusion, scattering theory and dissemination of atmospheric pollutants. Virtually no area of classical analysis has been left untouched by the fractional calculus.

1.2 Dual Integral Equations

1.21 Defination :

The pair of equations

(1.26) $$\int_0^\infty G(p)\, f(p) K\,(r,p)\, dp = g(r), \quad 0 < r < 1$$

(1.27) $$\int_0^\infty f(p)\, K(r,p)\, dp = h(r), \quad r > 1.$$

Where G(p), g(r), h(r), K(r,p) are given functions of the variables indicated, and f(p) is to be found are known as " Dual integral equations." They arise in the solution of boundry-value problems in which the condition on one boundry is a "mixed" one and it is usually a simple matter to reduce this type of problem to the solution of such a pair of integral equations.

1.22 A Specific example of reduction of a physical problem to a pair of dual integral equations, as discussed by Titchmarsh [27, P 334]

Let v (r,z) be the potential of flat circular electrified disk of conducting material, its centre being at the origin, and its axis along the z-axis. The potential satisfies the differential equation

(1.28) $$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} = 0$$

This is Laplace's equation in cylindrical coordinates, Let V (u,z) be Hankel transform of v (r,z), that is,

(1.29) $$V(u,z) = \int_0^\infty v(r,z)\, r\, J_0(ru)\, dr$$

Then clearly V (u,z) satisfies the Laplace's equation (1.28)

Since

$$\frac{\partial^2 V}{\partial z^2} = \int_0^\infty r\frac{\partial^2 v}{\partial z^2} J_0(ru)\, du$$

$$= -\int_0^\infty \left(r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r}\right) J_0(ru)\, dr$$

and using integrations by part we have

$$\int_0^\infty r\frac{\partial^2 v}{\partial r^2} J_0(ru)\, dr = -\int_0^\infty \frac{\partial v}{\partial r}\left[J_0(ru) + ru\, J_0'(ru)\right] dr$$

Hence

$$\frac{\partial^2 V}{\partial z^2} = -u \int_0^\infty v \left\{ J_0'(ru) + ru\, J_0''(ru) \right\} dr$$

Using Bessel's equation,

$$J_0'(ru) + ru\, J_0''(ru) = ur\, J_0(ru)$$

Thus

$$\frac{\partial^2 V}{\partial z^2} = u^2 \int_0^\infty V\, J_0(ru)\, r\, dr.$$

$$= u^2 V, \quad \text{using (1.29)}$$

Thus we get $\dfrac{\partial^2 V}{\partial z^2} = u^2 V$

The solution of above differential equation is

$$V = A(u)\, e^{-uz} + B(u)\, e^{uz}$$

Using initial condition, $B(u) = 0$

We have

$$V = A(u)\, e^{-uz}$$

Now using (1.29) and by Hankel theorem

$$v(r,z) = \int_0^\infty u\, A(u)\, e^{-uz}\, J_0(ru)\, du.$$

Taking the radius of the disk to be unity, the boundary conditions are $V = \text{const.}$ ($z=0$, $0 < r < 1$)

$$\frac{\partial v}{\partial z} = 0 \quad (z = 0,\ r > 1)$$

writting $u\, A(u) = f(u)$

Hence $f(u)$ must satisfy

$$\int_0^\infty f(u)\, J_0(ru)\, du = g(r), \quad 0 < r < 1$$

$$\int_0^{\infty} f(u)\, u\, J_0\,(ru) = 0, \quad r > 1$$

In above case g(r) is a constant, and f(u) is to be found.

1.23 Survey

For potential problems with axial symmetry, Tranter, in 1950 [28] has considered Bessel function of zero order as kernels. He considered the dual equations.

(1.30) $$\int_0^{\infty} G(p)\, f\,(p)\, J_0\,(rp)\, dp = g(r), \quad o < r < 1.$$

(1.31) $$\int_0^{\infty} f(p)\, J_0(rp)\, dp = 0, \quad r > 1.$$

Where G(p), g(r) are given functions of the variables indicated and f(p) is to be found. He has considered the solution of above pair in series form as

(1.32) $$f(p) = p^{1-k} \sum_{m=0}^{\infty} a_m\, J_{2m+K}(p)$$

and used the result of Watson [29,P401] so that the equation (1.31) is satisfied by this choice of f(p). By taking the coefficients a_m properly he has proved that the form of f(p) assumed in (1.32) also satisfies the equation (1.30). Thus he obtained the solution of above pair in series form. The solution given is a formal one and the difficult question of convergence is not considered.

Again in 1951, Tranter [30] has considered the dual integral equations as :

$$(1.33) \qquad \int_0^\infty t\, f(t)\, J_\mu(rt)\, dt = g(r),\ 0 < r < 1.$$

$$(1.34) \qquad \int_0^\infty f(t)\, J_\mu(rt)\, dt = F(r),\ r > 1.$$

where g(r), F(r) are prescribed functions of r and f(t) is a function of t to be found. By applying Hankel's inversion [27] theorem to equation (1.34) and using the result of Watson, [29, P 373] he obtained the solution of above pair. But his method is cumbersome.

Mitra [31] in his paper discussed the solution of a class of dual integral equations which appear in the formulation of electristatic and electromagnetic boundry-value problems possessing circular symmetry. He has discussed two classes of equations, out of which one admits a closed form solution. A Fredholm's equation of the second kind is derived for the second class and iterative means of solution are suggested.

Firstly, he considered the dual equations as :

$$(1.35) \qquad \int_0^\infty p^\alpha f(p)\, J_\mu(pr)\, dp = \int_0^\infty H(p)\, J_\mu(pr)\, dp,\ 0 < r < 1.$$

$$(1.36) \qquad \int_0^\infty f(p)\, J_\mu(pr)\, dp = 0, \quad r > 0.$$

Where H(p) is a given function, $\alpha > -1-2\mu$, $\mu \geqslant 0$. His method of finding the soulution is similar to the method given by Tranter.

As a first step, he assumes that f(p) is of the form.

$$(1.37) \qquad f(p) = P^{1-k} \sum_{m=0}^{\infty} C_m\, J_{2m+\mu+k}(p), \quad K > 0$$

and on using Watson [29, P 401], the representation of f(p) in (1.37), automatically makes it satisfy equation (1.36).

Using the known result of the Bessel functions and the result of Wilkins [32], he obtained the solution of the above pair in closed form as

$$(1.38) \qquad f(p) = P^{\mu-\lambda-\alpha/2}\, H(p) - P^{\mu-\lambda} \int_1^\infty r\, J_\lambda(pr)\, dr \times \int_0^\infty H(t)\, t^{-\alpha/2}\, J_\lambda(tr)\, dt.$$

Secondly, he considered the dual equations of the type.

$$(1.39) \qquad \int_0^\infty p^\alpha [1 + T(p)]\, f(p)\, J_\mu(pr)\, dp = \int_0^\infty H(p) \times J_\mu(pr)\, dp, \quad 0 < r\ 1.$$

(1.40) $$\int_0^\infty f(p)\, J_\mu\,(pr)\;\; dp = 0 \qquad r > 1.$$

Where T(p) is a known function and T(p) tends to 0 or to a constant for large positive values of P. Here also he considered the solution of (1.39) and (1.40) in series from as before. Putting the value of f(p) from (1.37) in (1.39) and using Wilkin's result [32] as before, and finally using Hankels formula [28] one can arrive at the Fredholm's equation of second kind.

The chief advantages of this method over that of Tranter's are the following :

a) The solution is obtained in a closed form rather than a series form given by Tranter.

b) By this method he obtained a Fredholm's equation of second kind for the dual equations of class second, where as Tranter's method involves the solution of an infinite set of equations for this case.

In 1958, Noble [33] has considered the pair.

(1.41) $$\int_0^\infty t^{-2\alpha}\, A(t)\; J_\mu(xt)\; dt = F(x),\; 0 \leqslant x \leqslant 1.$$

(1.42) $$\int_0^\infty A(t)\; J_\mu(xt)\; dt = G(x),\; x > 1.$$

This pair with general value of α in the range $-2 < \alpha < 2$ was considered by Noble, who reduced the problem to that of solving an integral equation, by the use of operator of fractional integration. His analysis involves considerable manipulation and cannot be regarded as elementry. In 1960, Sneddon [34] has given an elementry method of finding the solution of the pair(1.41) and (1.42), by using the operators of fractional integration. In 1961, Copson [35] has given a simple and elegent solution of the pair(1.41) and (1.42) by a method which is a generalization of an elementry method suggested by Sneddon.

In 1961, Williams [36] has considered the dual integral equations as,

$$\int_0^\infty y^\alpha f(y)\, J_\mu(xy)\, dy = G(x), \quad 0 \leqslant x \leqslant 1. \tag{1.43}$$

$$\int_0^\infty f(y)\, J_\mu(xy)\, dy = F(x), \quad x > 1. \tag{1.44}$$

He obtained the solution of above pair by a formal application of Mellin - transform. The manipulation here is formally more simple because much of it can be absorbed in the calculation of Mellin - transform.

In 1962, Burlack [37] considered a pair of dual integral equations occuring in diffraction theory as

(1.45) $$\int_0^\infty u^{-\nu-\mu}(u^2-k^2)\ \psi(u)\ J_\mu(xu)\ du = f(x), 0 \leq x \leq 1.$$

(1.46) $$\int_0^\infty \psi(u)\ J_\nu(xu)\ du = g(x),\ x > 1.$$

He obtained the solution of above pair by using Laplace -transform.

In 1962, Erdelyi and Sneddon [38] obtained the solution of pair (1.43) and (1.44) by using fractional integral operators.

In 1964, Buschman [39] considered a pair of dual integral equations as,

(1.47) $$\int_0^\infty y^\alpha\ J_\mu(xy)\ f(y)\ dy = g(x),\ 0 < x < 1.$$

(1.48) $$\int_0^\infty y^\beta\ J_\nu(xy)\ f(y)\ dy = h(x),\ x > 1.$$

He has defined fractional integral operators as,

(1.49) $$I^{\eta,\alpha,A}(x) = \frac{A}{\Gamma(\alpha)}\ (x^A-1)^{\alpha-1}\ x^{-A\eta-A\alpha}\ U(x-1).$$

(1.50) $$K^{\eta,\alpha,A}(x) = \frac{A}{\Gamma(\alpha)}\ (1-x^A)^{\alpha-1}\ x^{A\eta}\ U(1-x).$$

Where U(x) is the Heaviside unit step function, and proved that these integral operators with respect to x^A can be written in the form of convolution, (1.98) and also he has identified these operators with the elements of algebra. He obtained the Mellin - transform of these operators, and finally using these results in the convolution theorem(1.111) He has reduced the above pair to a single integral equation as

$$(1.51) \quad \int_0^\infty y^k J_\lambda (xy)\, f(y)\, dy = F(x),\ 0 < x < \infty$$

Where k, λ are related to α, β, μ, ν and F(x) to g(x) and h(x).

A systematic treatment of this subject is given by Fox [40].

In 1965, he has considered the most general case in which a dual integral equations contain H - functions as Kernels. These H - functions contain almost all special functions as perticular cases. He obtained the solution by inspection.

He considered the pair,

$$(1.52) \quad \int_0^\infty H\left(ux \,\middle|\, \begin{matrix} \alpha_i, a_i \\ \beta_i, a_i \end{matrix} : n\right) f(u)\, du = g(x),\ 0 < x < 1.$$

$$(1.53)\quad \int_0^\infty \ddot{H}\left(ux \,\middle|\, \begin{matrix} \lambda_i, a_i \\ \beta_i, a_i \end{matrix} : n\right) f(u)\, du = h(x), \quad x \geq 1.$$

Where g(x) and h(x) are given and f(x) is to be determined, and

$H\left(X \,\middle|\, \begin{matrix} \alpha_i, a_i \\ \beta_i, a_i \end{matrix} : n\right)$ is a H -function of order n, defined by

$$(1.54)\quad H\left(X \,\middle|\, \begin{matrix} \alpha_i, a_i \\ \beta_i, a_i \end{matrix} : n\right) = H\left(X \,\middle|\, \begin{matrix} \alpha_1, a_1 \; \ldots \; (\alpha_n, a_n) \\ ; \\ \beta_1, a_1 \; \ldots \; (\beta_n, a_n) \end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_C \prod_{i=1}^{n}\left[\frac{\Gamma(\alpha_i + sa_i)}{\Gamma(\beta_i - sa_i)}\right] X^{-s}\, ds.$$

Observe that the constants a_i, $i = 1,2,\ldots n$ are the same for both (1.52) and (1.53).

He has used the Parseval theorem, which states that :

If the Mellin - transform of f(u) is denoted by M[f(u)], That is M [f(u)] = F (s) and if M [P(u)] = P(s) then

$$(1.55)\quad \int_0^\infty P(u)\, f(u)\, du = \frac{1}{2\pi i}\int_C P(s)\, F(1-s)\, ds.$$

Where the contour C is some straight line.

Using the defination of Mellin-transform he obtained,

(1.56) $M[f(ux)] = x^{-s} M[f(u)]$.

Where f(ux) is a function with u and x as a parameter. From (1.55) and (1.56), he obtained :

(1.57) $\int_0^\infty P(ux)\, f(u)du = \frac{1}{2\pi i}\int_C P(s)\, X^{-s} F(1-s)\, ds.$

This form of Parseval theorem has been used by Fox.

From (1.54)

(1.58) $M\left[H\left(u\begin{pmatrix}\alpha_i, a_i \\ \beta_i, a_i\end{pmatrix} : n\right)\right] = \prod_{i=1}^{n} \frac{\Gamma(\alpha_i + sa_i)}{\Gamma(\beta_i - sa_i)}$

Using parseval theorem and (1.58) to (1.52) and (1.53), he obtained the following equations,

(1.59) $\frac{1}{2\pi i}\int_C \prod_{i=1}^{n}\left\{\frac{\Gamma(\alpha_i + sa_i)}{\Gamma(\beta_i - sa_i)}\right\} x^{-s} F(1-s)\, ds = g(x),$

$0 < x < 1.$

(1.60) $\frac{1}{2\pi i}\int_C \prod_{i=1}^{n}\left\{\frac{\Gamma(\lambda_i + sa_i)}{\Gamma(\mu_i - sa_i)}\right\} x^{-s} F(1-s)\, ds = h(x),\ X > 1.$

Then he has defined two operators of fractional integration as :

(1.61) $$I\left[\gamma, \epsilon : m : W(x)\right] = \frac{m}{\Gamma(\gamma)} x^{-\epsilon - m\gamma + m - 1} \times \int_0^x (x^m - v^m)^{\gamma-1} v^{\epsilon} W(v)\, dv.$$

$$R\left[\gamma, \epsilon : m : W(x)\right] = \frac{m}{\Gamma(\gamma)} x^{\epsilon} \int_x^{\infty} (v^m - x^m)^{\gamma-1} \times v^{-\epsilon - m\gamma + m - 1} W(v)\, dv.$$

Using these operators step by step on (1.59) and (1.60), he obtained a single equation with a common Kernel as

(1.63) $$\frac{1}{2\pi i}\int_C \left\{ \prod_{i=1}^{m} \frac{\Gamma(\alpha_i + s a_i)}{\Gamma(\mu_i + s a_i)} \right\} x^{-s} F(1-s)\, ds = K(x)$$

Where

(1.64) $$K(x) = I_1\left[I_2 \ldots I_n\left[(x)\right] \ldots\right], \quad 0 < x < 1.$$

$$= R_1\left[R_2 \ldots R_n\left[h(x)\right] \ldots\right], \quad x > 1.$$

To solve (1.63) for $f(x) = M^{-1}\left[F(s)\right]$, he has used the generalized Fourier- transform which consists of reciprocity

(1.65) $$\phi(x) = \int_0^{\infty} P(ux)\, f(u)\, du.$$

(1.66) $$f(x) = \int_0^{\infty} q(ux)\, \phi(u)\, du.$$

and the functional equation

(1.67) $P(s)\, Q(1-s) = 1$ Where $M\left[P(u)\right] = P(s)$

and $M\left[q(u)\right] = Q(s)$.

Using, Parseval theorem to right hand side of (1.65) and (1.66), and using (1.67) he obtained :

(1.68) $$\phi(x) = \frac{1}{2\pi i} \int_C P(s) \cdot X^{-s} F(1-s)\, ds.$$

(1.69) $$f(x) = \frac{1}{2\pi i} \int_C \frac{1}{P(1-s)} X^{-s} \Phi(1-s)\, ds.$$

Where $M[\phi(u)] = \Phi(s)$

Hence if P(s) and $\phi(x)$ are known in (1.68) one can solve for $f(x) = M^{-1}[F(s)]$, by means of equation (1.69)

By applying this idea to (1.63) he obtained the solution as, :

(1.70) $$f(x) = \frac{1}{2\pi i} \int_C \prod_{i=1}^{n} \left\{ \frac{\Gamma(\mu_i - a_i + s a_i)}{\Gamma(\alpha_i + a_i - s a_i)} \right\} X^{-s} K(1-s)\, ds.$$

Where $M[K(x)] = K(s)$.

Again using Parseval theorem, (1.57) one can transform the integral of (1.70) so that the equation takes the form

(1.71) $$f(x) = \int_0^{\infty} H\left(ux \,\middle|\, \begin{matrix} \mu_i - a_i, a_i \\ \alpha_i + a_i, a_i \end{matrix} : n\right) K(u)\, du.$$

Where K(x) is given by (1.64)

In 1967, Kesarwani [41] has considered the dual integral equations with Meijers G functions as Kernels:

$$(1.72) \quad \int_0^{\infty} G^{m,n}_{p,q}\left((xy)^A \middle| \begin{matrix} a_1, \ldots a_p \\ b_1 \ldots b_q \end{matrix}\right) f(y)\, dy = g(x), 0 < x < 1.$$

$$(1.73) \quad \int_0^{\infty} G^{m,n}_{p,q}\left((xy)^A \middle| \begin{matrix} c_1, \ldots c_p \\ d_1, \ldots d_q \end{matrix}\right) f(y)\, dy = h(x), X > 1.$$

Using the method of Buschman [39] he has shown that the above equations can be reduced into two others having the same Kernel. The problem of solving a single integral equation has been discussed by Kesarwani in a series of earlier paper [42] In 1967, Saxena [43] has also discussed the formal solution of certain dual integral equations involving H - functions. He has shown that by applications of fractional integration operators that the given integral equations can be reduced into two others with a common Kernel and the problem then reduces to that of solving one integral equation. In the first case the Kernels of transformed equations involve the H-function, as a symmetrical Fourier Kernel given earlier by Fox [44] and the solution is then immediate. The second case deals with the solution of another pair of integral equations which are more general than one given by Fox [40] in which the common Kernel comes out to be generalized Fourier Kernel studied by Fox [40] and solution can obtained by following his method. In the first he considered a pair

$$(1.74) \quad \int_0^{\infty} H^{q,p+m}_{2p+m,2q+m}(xu)\, f(u)\, du = g(x), \quad 0 < x < 1.$$

$$(1.75) \quad \int_0^{\infty} H^{q+n,p}_{2p+n,2q+n}(xu)\, f(u) du = h(x), \quad X > 1.$$

Where g(x), h(x) are given and f(x) is to be found. Using the same technique as given by Fox [40] he obtained the solution of above pair. In the second case, he considered a pair

$$(1.76) \quad \int_0^{\infty} H^{p,m}_{m,2p+m}\left(xu \,\middle|\, \begin{matrix} (1-\alpha_k, \Delta_k) \\ (\gamma_i, a_i), \ (1-\delta_i, a_i), (1-\beta_k, \Delta_k) \end{matrix}\right) f(u)du$$

$$= g(x), \quad 0 < x < 1.$$

$$(1.77) \quad \int_0^{\infty} H^{p+n,0}_{n,2p+n}\left(xu \,\middle|\, \begin{matrix} (\mu_1, \xi_1) \\ \lambda_1, \xi_L \ ) \ (\nu_i, a_i \ ), (1-p_i, a_i) \end{matrix}\right) f(u)du$$

$$= \phi(x), \quad x > 1.$$

Where g(x) ϕ (x) are given and f(x) is to be determined.

In 1969, Saxena [45] has obtained a formal solution of equations (1.43) and (1.44) by using the technique of Mellin transform. Instead of Bessel's function as Kernel, he has used Watson's Kernel [46]

In 1970, Mourya [26] has developed fractional integral operators for the function of two variables, on the line of Erdelyi and Kober [16], and discussed some of their fundamental properties and some identities. The algebra of these operators have been developed by Koranne [47] and used in the solution of certain dual integral equations of function of two variables. He has used Agarwals [48] function as Kernels.

In 1970, Dwivedi [49] and in 1974, Saxena and Kumbhat [50] have used fractional integral operators and the Mellin-transform theory to solve the d al integral equations with Kernels as H-functions.

In 1974, Pathak [51] has given a formal solution Of following pair of dual integral equations by a method based on multiplying factor a nd Wiener-Hop f techniques as illustrated by Noble [52] for the Bessel function dual integral equations.

$$(1.78)\quad \int_0^{\infty} H^{m,n}_{p,q}\left(xy \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right) f(y)\, dy = u(x),\ 0 < x < 1.$$

$$(1.79)\quad \int_0^{\infty} H^{M,N}_{P,Q}\left(xy \,\middle|\, \begin{matrix}(c_P,\gamma_P)\\(d_Q,\delta_Q)\end{matrix}\right) f(y)\, dy = v(x),\ x > 1.$$

where u(x), v(x) are given functions and f(x) is to determined.

1.3 The H-Functions :

1.31 The H-Function of one variable

Fox[44,53] introduced a general function which is well-known as Fox's H-function or the H-Function. This function is defined and represented by means of the Mellin-Barnes type of contour integral. A very general class of Barnes integral was first introduced by Dixon and Ferror [54]. Baraaksma [55] has studied this function in detail with reference to asymptotic expansion and analytic continuation.

The H-function is defined and represented in the following manner [56].

$$(1.80)\quad H^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix}(a_p,c_p)\\(b_q,\beta_q)\end{matrix}\right) = \frac{1}{2\pi i}\int_L \theta(s)\, x^s\, ds$$

where $i = (-1)^{1/2}$, x is not zero and is a complex number, and

(1.81) $X^s = \exp [s \text{ Log } |X| + i \arg X]$

In which Log |X| represents the natural logarithm of |X| and arg X is not necessarily the principal value. An empty product is interpreted as unity. Also

(1.82) $$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{P} \Gamma(a_j - \alpha_j s)}$$

Where m, n, P, q, are nonnega-tive integers satisfying $0 \leq n \leq P$, $1 \leq m \leq q$, α_j (j = 1,2 ...P) and β_j (j = 1,2,...q) are assumed to be possitive quantities. Also a_j (j = 1,2 ...P) and b_j (j = 1,2 ....q) are complex number such that none of the points.

(1.83) $$s = \frac{(b_h + \lambda)}{\beta_h}, \quad h = 1,2 \ldots m, \lambda = 0,1 \ldots$$

Which are the poles of $\Gamma(b_h - \beta_h s)$ h = 1,2...m and the points

(1.84) $$s = \frac{a_i - \eta - 1}{\alpha_i} \qquad i = 1,2, \ldots n, \eta = 0,1 \ldots$$

Which are the poles of $\Gamma(1 - a_i + \alpha_i s)$ coincide with one another.

that is

(1.85) $\alpha_i (b_h + \lambda \neq \beta_h(a_i - \eta - 1)$

For $\lambda, \eta = 0, 1, \ldots,$ $h = 1, 2, \ldots m;$ $i = 1, 2 \ldots n.$

Further, the contour L runs from $-i\infty$ to $+i\infty$ such that the poles of $\Gamma(b_h - \beta_h s)$, $h = 1,2,..m$ lie to the right of L and the poles of $\Gamma(1-a_i + \alpha_i s)$, $i =1, 2...n$ lie to the left of L, such a contour is possible on account of (1.85). These assumptions will be adhered to throughout the present work.

We state the following useful properties of the H.-function.

(1.86) $$x^k H^{m,n}_{p,q}\left(x \middle| \begin{matrix} (a_p , \alpha_p) \\ (b_q , \beta_q) \end{matrix} \right) = H^{m,n}_{p,q}\left(x \middle| \begin{matrix} (a_p+k\alpha_p , \alpha_p) \\ (b_q + K\beta_q , \beta_q) \end{matrix} \right)$$

If one of the (a_i, α_i), $i =1,2,...n$ is equal to one of the (b_j , β_j) $j = m+1,...q$.

[or one of the pairs (a_i , α_i), $i = n+1,...p$, is equal to one of the (b_j, β_j) $j = 1,....m$]

then H-function reduces to one of the lower order i.e. P, q and n(or m) decrease by unity.

We give below one such reduction formulas

$$(1.87)\quad H_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}(a_1,\alpha_1)\;\ldots\ldots(a_{p-1},\alpha_{p-1}),\,(a_p\,,\,\alpha_p)\\(b_1,\beta_1)\;,\ldots\ldots(b_{q-1},\,\beta_{q-1}),\;(a_1,\alpha_1)\end{matrix}\right)$$

$$= H_{p-1,q-1}^{m,n-1}\left(x\,\middle|\,\begin{matrix}(a_2\,,\,\alpha_2),\ldots\ldots(a_p\,,\,\alpha_p)\\(b_1\,,\,\beta_1),\ldots\ldots(b_{q-1},\,\beta_{q-1})\end{matrix}\right)$$

provided $n \geqslant 1$ and $q > m$.

When $\alpha_i = \beta_j$ ($i=1, \ldots P$, $j = 1, \ldots q$) then H-function reduces to the well known Meijer's G-function.

$$(1.88)\quad H_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}(a_p\,,\,\alpha_p)\\(b_q\,,\,\beta_q)\end{matrix}\right) = \frac{1}{C}\; G_{p,q}^{m,n}\left(x^{1/c}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right)$$

1,32 <u>The H-function of two variables</u>

We shall define and represent the H-function of two variables [57,P117] using the following notation [58,P266].

(1.89) $H\,(x,y)$

$$= H_{p_1,q_1:p_2,q_2;p_3,q_3}^{0,n_1:m_2,n_2;m_3,n_3}\left(\begin{matrix}x\\y\end{matrix}\,\middle|\,\begin{matrix}(a_{p_1},\alpha_{p_1},A_{p_1})\;:\;(c_{p_2},\gamma_{p_2});(e_{p_3},E_{p_3})\\(b_{q_1},\beta_{q_1},B_{q_1})\;:\;(d_{q_2},\delta_{q_2});(f_{q_3},F_{q_3})\end{matrix}\right)$$

$$= \frac{-1}{4\pi^2}\int_{L_1}\int_{L_2} \phi\,(s,t)\,\theta_2\,(s)\,\theta_3(t)\,x^s\,y^t\;ds\,dt.$$

Where

$$(1.90)\quad \phi(s,t) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\,s + A_j t)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)\;\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)}$$

$$(1.91)\quad \theta_2(s) = \frac{\prod_{j=1}^{n_2} \Gamma(1-c_j+\gamma_j s) \prod_{j=1}^{m_2} \Gamma(d_j-\delta_j s)}{\prod_{j=n_2+1}^{p_2} \Gamma(c_j-\gamma_j s) \prod_{j=m_2+1}^{q_2} \Gamma(1-d_j+\delta_j s)}$$

$$(1.92)\quad \theta_3(t) = \frac{\prod_{j=1}^{n_3} \Gamma(1-e_j+E_j t) \prod_{j=1}^{m_3} \Gamma(f_j-F_j t)}{\prod_{j=n_3+1}^{p_3} \Gamma(e_j-E_j t) \prod_{j=m_3+1}^{q_3} \Gamma(1-f_j+F_j t)}$$

X and y are not equal to zero, and empty product is interpreted as,unity, p_i, q_i, n_j and m_j are non-negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$ ($j=1,2$; $i = 1,2,3$). Also all the A's, α's, B's , β's, γ's, δ's E's and F's are assumed to be positive quantities.

The contour L_1 is in the s -plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j s)$ ($j=1,\ldots m_2$) lie to the right, and the poles of $\Gamma(1-c_j+\gamma_j s)$ ($j=1,2,\ldots n_2$), $\Gamma(1-a_j+\alpha_j s+A_j t)$ ($j=1,\ldots n_1$) to the left of the countour. The contour L_2 is in the t-plane and runs from $-i\infty$ to $+i\infty$, with loops, to ensure that the poles of $\Gamma(f_j-F_j t)$ ($j=1,\ldots n_3$) lie to the right, and the poles of $\Gamma(1-e_j+E_j t)$ ($j=1,\ldots n_3$) and $\Gamma(1-a_j+\alpha_j s+A_j t)$ ($j=1,2..n_1$)

to the left of the contour.

Following the result of Braaksma [55, P278] it can be shown that the function defined by (1.89) is an analytic function of X and y if

$$(1.93)\quad R = \sum_{j=1}^{p_1} \alpha_j + \sum_{j=1}^{p_2} \gamma_j - \sum_{j=1}^{q_1} \beta_j - \sum_{j=1}^{q_2} \delta_j < 0.$$

$$(1.94)\quad S = \sum_{j=1}^{p_1} A_j + \sum_{j=1}^{p_3} E_j - \sum_{j=1}^{q_1} B_j - \sum_{j=1}^{q_3} F_j < 0$$

Buschman [59] has given the following conditions for the convergence of the double Mellin - Barnes integral representing the extended H-function of two variables :

* (1.95)

$$(1.96)\quad V = - \sum_{j=n_1+1}^{p_1} A_j - \sum_{j=1}^{q_1} B_j + \sum_{j=1}^{m_3} F_j - \sum_{j=m_3+1}^{q_3} F_j + \sum_{j=1}^{n_3} E_j - \sum_{j=n_3+1}^{p_3} E_j > 0$$

(1.97) $|\arg X| < 1/2\, U\pi$, $|\arg y| < 1/2\, V\pi$

We state the following useful property of the H -function of two variables.

If one of the (c_i, γ_i) $(i=1,2,\ldots n_2)$ is equal to the one of the (d_i, δ_i) $(i= m_2+1,\ldots q_2)$ then the H- function reduces to one of the lower order, and similar other results. We give one of such reduction formulas:

$$*(1.95)\quad U = - \sum_{j=n_1+1}^{p_1} \alpha_j - \sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{m_2} \delta_j - \sum_{j=m_2+1}^{q_2} \delta_j + \sum_{j=1}^{n_2} \gamma_j - \sum_{j=n_2+1}^{p_2} \gamma_j > 0 .$$

(1.98)

$$H^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left(\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});\ (e_{p_3},E_{p_3})\\(b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2-1},\delta_{q_2-1}),(c_1,\gamma_1);(f_{q_3},F_{q_3})\end{matrix}\right)$$

$$= H^{0,n_1:m_2,n_2-1;m_3,n_3}_{p_1,q_1:p_2-1,q_2-1;p_3,q_3}\left(\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_{p_1},\alpha_{p_1},A_{p_1}):(c_2,\gamma_2),\ldots(c_{p_2},\gamma_{p_2});(e_{p_3},E_{p_3})\\(b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2-1},\delta_{q_2-1});(f_{q_3},F_{q_3})\end{matrix}\right)$$

1.4 Mellin -Convolution :

1.41 The Mellin -Convolution of one variable

We know from Titchmarsh [27,p 59] that if

$f \in L\ (0,\infty)$, $g \in L\ (0,\infty)$ then $(f * g)\ (x) \in L'\ (0,\infty)$, where

(1.99) $$(f * g)\ (x) = \int_0^\infty u^{-1} f(x/u)\ g(u)\ du.$$

Hence the set $L'\ (0,\infty)$ of complex-valued functions froms an algebra over the field of complex numbers with the usual defination of addition and scalar multiplication and the convolution (1.99) as the defination of product.

we can show that the convolution (1.99) as the defination of product is commutative.

Now $(f * g)\ (x) = \int_0^\infty u^{-1} f(x/u)\ g\ (u)\ du.$

by puting $u = x/t$

we get $(f * g)\ (x) = (g * f)\ (x)$.

Simple calculations show that the algebra is also associative.

BUSCHMAN [39] pointed out that if we defined

(1.100) $I^{\eta,\alpha,A}(x) = A/\Gamma(\alpha)\ (x^A-1)^{\alpha-1}\ x^{-A\eta-A\alpha}\ U\ (x-1)$

Where U(x) is the Heaviside unit step function,

$$U(x) = \begin{cases} 0 & , \text{ for } x \leq 0 \\ 1 & , \text{ for } x > 0 \end{cases}$$

The fractional integral operator (1.18) can be written in the form of convolution (1.99).

(1.101) $I_{x^A}^{\eta,\alpha} f(x) = (I^{\eta,\alpha,A} * f)\ (x)$

Since, using (1.99) we have

$$(I^{\eta,\alpha,A} * f)\ (x)$$

$$= \int_0^\infty u^{-1} f(x/u)\ I^{\eta,\alpha,A}\ (u)\ du.$$

$$= \frac{A}{\Gamma(\alpha)} \int_0^\infty u^{-1} f(x/u)\ (u^A-1)^{\alpha-1}\ u^{-A\eta-A\alpha}\ U\ (u-1)\ dx$$

$$\text{Where } U(u-1) = \begin{cases} 0 & \text{for } u-1 \leq 0 \\ 1 & \text{for } u-1 > 0 \end{cases}$$

$$= \frac{A}{\Gamma(\alpha)} \int_1^\infty u^{-1} f(x/u)\ (u^A-1)^{\alpha-1}\ u^{-A\eta-A\alpha}\ du.$$

By putting t = x/u,

$$= \frac{x^{-A\eta-A\alpha}}{\Gamma(\alpha)} \int_0^\infty (x^A-t^A)^{\alpha-1}\ t^{A\eta}\ f(t)\ d\ (t^A)$$

$= I_{x^A}^{\eta,\alpha} f(x)$, Also $I^{\eta,\alpha,A}\ (x) \in L'\ (0,\infty)$ for Re $\alpha > 0$,

$\mathrm{Re}\,\eta > 1/A - 1$

Similarily (1.19) can be written in the form

(1.102) $K_{x^A}^{\eta,\alpha} f(x) = (K^{\eta,\alpha,A} * f)(x)$ if

(1.103) we define

$$K^{\eta,\alpha,A}(x) = A/\Gamma(\alpha) \quad (1-x^A)^{\alpha-1} \quad x^{A\eta}\, U(1-x)$$

Which belongs to L' $(0,\infty)$ for

$\mathrm{Re}\,\alpha \geq 0$, $\mathrm{Re}\,\eta > -1/A$

Thus we can identify the fractional integral operators I's and K's with the elements of algebra and hence we conclude that they associate and commute.

A direct computation in order to varify the commutativity of $[I^{\eta,\alpha}\ I^{\eta_1,\alpha_1}]$ can also be carried out. However

$I_{x^A}^{\eta,\alpha}$ and $I_{x^B}^{\eta_1,\alpha_1}$ do not commute unless A = B.

1.42 <u>The Mellin -Convolution of two variables</u>

From Koranne [47], if f,g $\in$ D[$(0,\infty)$, $(0,\infty)$] then $(f ** g) \in D[(0,\infty),\ (0,\infty)]$ Where

(1.104)
$$(f ** g)\ (xy) = \int_0^\infty \int_0^\infty u^{-1} v^{-1}\ f(x/u, y/v)\ g(u,v)\ du, dv.$$

Hence the set of complex - valued functions belonging to D [$(0,\infty)$, $(0,\infty)$] from an algebra over the field of complex numbers with usual defination of addition and scalar multiplication and (1.104) as one defination of the product.

Simple calculations show that this algebra is associative and commutative. We note that fractional integral (1.22) can be weitten in the form of a convolution

(1.105)

$$I_x^{\eta,\alpha} \; I_y^{\tau,\beta} f(x,y) = \int_0^\infty \int_0^\infty t^{-1} z^{-1} \left[1/\Gamma(\alpha) \, (x/t)^{-\eta-\alpha} \right.$$

$$\left. (x/t-1)^{\alpha-1} \, U(x/t-1) \right] \left[1/\Gamma(\beta) \, (y/z)^{-\tau-\beta} (y/z-1)^{\beta-1} U(y/z-1) \right] X$$

$f(t,z)dt\,dz.$

$$=(I^{\eta,\alpha} \; I^{\tau,\beta} * * \; f)\,(x,y)$$

Where we define a function

$$(1.106) \quad I^{\eta,\alpha} \; I^{\tau,\beta} \; (x,y)$$

$$= \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)} \; x^{-\eta-\alpha} \, y^{-\tau-\beta} \, (x-1)^{\alpha-1} \, (y-1)^{\beta-1} \; U(x-1)U(y-1),$$

In which U denotes unit step function.

Also $I^{\eta,\alpha} \; I^{\tau,\beta} \in D\,[(0,\infty),(0,\infty)]$ for

$(\eta,\alpha) > 0$, $(\tau,\beta) > 0$, So that we can identify these fractional integral operators with elements of algebra.

Similarly

$$K_x^{\eta,\alpha} K_y^{\tau,\beta} f(x,y) = (K^{\eta,\alpha} K^{\tau,\beta} ** f)(x,y)$$

Here we can define the function

(1.107) $K^{\eta,\alpha} K^{\tau,\beta}(x,y)$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\eta}(1-x)^{\alpha-1}U(1-x)y^{\tau}(1-y)^{\beta-1}U(1-y)$$

Which belongs to D [(0,∞), (0,∞)] for $(\alpha,\beta) > 0$, $(\eta,\tau) > -1$

The equation (1.24) can be expressed in the form

(1.108)

$$I_{\phi(x)}^{\eta,\alpha} I_{\psi(y)}^{\tau,\beta} f(x,y)$$

$$= \int_0^\infty \int_0^\infty t^{-1} z^{-1} \left[\frac{1}{\Gamma(\alpha)} \left(\frac{\phi(x)}{\phi(t)} - 1\right)^{\alpha-1} \left(\frac{\phi(x)}{\phi(t)}\right)^{-\eta-\alpha} \times \right.$$

$$\left. U\left(\frac{\phi(x)}{\phi(t)} - 1\right)\right] \left[\frac{1}{\Gamma(\beta)} \left(\frac{\psi(y)}{\psi(z)} - 1\right)^{\beta-1} \left(\frac{\psi(y)}{\psi(z)}\right)^{-\tau-\beta} \times\right.$$

$$\left. U\left(\frac{\psi(y)}{\psi(z)} - 1\right)\right] f(t,z) \left(t \frac{\phi'(t)}{\phi(t)}\right) \left(z \frac{\psi'(z)}{\psi(z)}\right) dt\, dz.$$

This can be written as the convolution product if,

$$\frac{\phi(x)}{\phi(t)} = \phi(x/t), \quad \frac{\psi(y)}{\psi(z)} = \psi(y/z) \quad \text{and}$$

$$x \frac{\phi'(x)}{\phi(x)} = A, \quad y \frac{\psi'(y)}{\psi(y)} = B.$$

Hence if $\phi(x) = c\, x^{A}$, $\psi(y) = c\, y^{B}$: then follows

$$(1.109) \qquad I_{x}^{\eta,\alpha}{}_{A}\; I_{y}^{\tau,\beta}{}_{B}\; f(x,y) = (\, I^{\eta,\alpha,A}\; I^{\tau,\beta,B} ** f)(x,y)$$

When $(A,B) > 0$

We define the function

$$(1.110) \qquad I^{\eta,\alpha,A}\; I^{\tau,\beta,B}\; (x,y)$$

$$= \frac{A \quad B}{\Gamma(\alpha)\Gamma(\beta)} (x^{A}-1)^{\alpha-1} (y^{B}-1)^{\beta-1} x^{-A\eta-A\alpha} \times y^{-B\tau-B\beta} \times$$

$$U(x-1).\ U(y-1).$$

Since the operators correspond to functions of algebra for $(\alpha,\eta) > 0$, $(\tau,\beta) > 0$, they commute and associate independently of the choice for A and B. Similar arguments can also be applied to $K_{\phi(x)}^{\eta,\alpha}$ $K_{\psi(y)}^{\tau,\beta}$

1.5 . The Mellin Transform :

1.51 The Mellin transform of operators of one variable:

Let us denote F(s), the Mellin transform of f(x) by M [f(x)], that is

$$M\,[f(x)] = F(s) = \int_{0}^{\infty} f(x)\; x^{s-1} dx$$

and regard $s = \sigma + i\tau$ as complex variable. Under certain conditions [28], f(x), the inverse Mellin transform of F(s) may be represented as an integral.

$$M^{-1}[F(s)] = f(x) = 1/2\pi i \int_{C-i\infty}^{C+i\infty} F(s)\, X^{-s}\, ds.$$

Associated with these transforms is the following convolution theorem [27, th 44, P 60] If $s = C + i\tau$, $X^{C} f(x)$ and $X^{C} g(x) \in L'(0,\infty)$.

Then

(1.111) $F(s)G(s) = M[(f * g)(x)]$ and
$X^{C}(f * g)(x) \in L'(0,\infty)$.

It has been proved by BUSCHMAN [40] that

(1.112) $M[I^{\eta,\alpha,A}(x)] = \dfrac{\Gamma(1+\eta-s/A)}{\Gamma(1+\eta+\alpha-s/A)}$ Re $\alpha > 0$, Re $s < A(\text{Re}\,\eta+1)$

(1.113) $M[K^{\eta,\alpha,A}(x)] = \dfrac{\Gamma(\eta+s/A)}{\Gamma(\eta+\alpha+s/A)}$, Re $\alpha > 0$, Re $s > -A\,\text{Re}\,\eta$

1.52 <u>The Mellin transform of operators of two variables:</u>

Let us denote $F(s,t)$, the mellin- transform of $f(x,y)$ by $M[f(x,y)]$, that is

$$M[f,(x,y)] \equiv F(s,t) = \int_0^\infty \int_0^\infty f(x,y)\, X^{s-1}\, y^{t-1}\, dx\, dy$$

and regard $s = \sigma_1 + i\tau_1$ $t = \sigma_2 + i\tau_2$, as complex variables. Under the conditions [26, 47] $f(x,y)$, the inverse Mellin transform of $F(s,t)$ may be represented as an integral

$$M^{-1}[F(s,t)] = f(x,y) = \frac{1}{(2\pi i)^2} \int_{C_1-i\infty}^{C_1+i\infty} \int_{C_2-i\infty}^{C_2+i\infty} F(s,t) x^{-s} y^{-t} \, ds \, dt$$

We have associated with this transform the following convolution theorem.:

If $x^{c_1} y^{c_2} f(x,y)$ and $x^{c_1} y^{c_2} g(x,y) \in D[(0,\infty),(0,\infty)]$ then

(1.114) $F(s,t)\,G(s,t) = M[(f ** g)(x,y)]$ and

$$x^{c_1} y^{c_2} (f ** g)(x,y) \in D[(0,\infty),(0,\infty)]$$

It has been proved by Koranne [47] that

(1.115) $M[\, I^{\eta,\alpha,A}\, I^{\tau,\beta,B}(x,y)\,]$

$$= \frac{\Gamma(1+\eta-s/A)\,\Gamma(1+\tau-t/B)}{\Gamma(1+\eta+\alpha-s/A)\,\Gamma(1+\tau+\beta-t/B)}$$

$Re\,\alpha > 0$, $Re\,\beta > 0$, $Re\,s < A(Re\,\eta + 1)$, $Re\,t < B(Re\,\tau + 1)$

(1.116) $M[\, K^{\eta,\alpha,A}\, K^{\tau,\beta,B}(x,y)\,]$

$$= \frac{\Gamma(\eta+s/A)\,\Gamma(\tau+t/B)}{\Gamma(\eta+\alpha+s/A)\,\Gamma(\tau+\beta+t/B)}$$

$Re\,\alpha > 0$, $Re\,\beta > 0$, $Re\,s > -A\,Re\,\eta$, $Re\,t > -B\,Re\,\tau$.

1.53 <u>The Mellin - transform of the H-function of one variable.</u> :

The Mellin-transform of the H-function follows from the defination of H-function, in the view of the well-known Mellin inversion theorem,

We have:

(1.117) $$M\left[H^{m,n}_{p,q}\left(ax\middle|\begin{matrix}(a_p, \alpha_p)\\(b_q,\beta_q)\end{matrix}\right)\right]$$

$$= a^{-s}\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}$$

Where $a > 0$, $|\arg a| < \frac{a\pi}{2}$, m,n,p,q, are non negative integers satisfying $0 \leq n \leq p, 1 \leq m \leq q$, $\alpha_j, (j=1,2\ldots p)$ and β_j, $(j=1,2 \ldots q)$ are assumed to be a positive quantities. Also a_j and b_j are complex numbers such that

(1.118) $$-\min_{1\leq j\leq m} Re\,(b_j/\beta_j) < Re\,(s) \min_{1\leq j\leq n} Re(1-a_j/\alpha_j)$$

1.54 <u>The Double Mellin Transform of the H-function of Two variables.</u>

The result is a direct consequence of the defination of H [ax,by] function.

(1.119) $$\int_0^{\infty}\int_0^{\infty} x^{s-1}\, y^{t-1}\, H\; ax,by\;\, dx\, dy$$

$$= a^{-s}\, b^{-t}\;\phi(-s,-t),\; \theta_2(-s)\;\theta_3(-t)$$

Where $\phi(s,t)$ $\theta_2(s)$, $\theta_3(t)$ are given by (1.90),(1.91) (1.92) The conditions given by(1.93),(1.94),(1.95),(1.96) (1.97)are assumed to be satisfied, and

$$-\lambda \min_{1 \leq j \leq m_2} Re\left(\frac{d_j}{\delta_j}\right) < Re(s) < \lambda \min_{1 \leq j \leq n_2} Re \frac{1-c_j}{\gamma_j}$$

and

$$-\mu \min_{1 \leq j \leq m_3} Re\left(\frac{f_j}{F_j}\right) < Re(t) < \mu \min_{1 \leq j \leq n_3} Re \frac{1-e_j}{E_j}$$

<u>Motivation of the work done</u> :-

Dual Integral Equations involving many special functions as Kernels have been tackled from time to time by various mathematicians like Tranter, Noble, Buschman, Saxena, Fox, Koranne and others, by using various techniques. This motivated us to study dual integral equations of one and two variables by choosing Kernels in a very general form and using the technique of fractional integral operators. This technique offers the convenience of converting dual integral equations to a single integral equation. Various techniques are available in the literature to solve such single integral equation.