-: CHAPTER TWO :-

Operators of Fractional Integration and Their Applications to Dual Integral Equations

2.1 An Application of Fractional Integral Operators of one variable :

We consider the dual integral equations :

$$(2.1)\quad \int_0^\infty H_{p,q}^{m,n}\left(ax\,\middle|\,\begin{matrix}(a_1,\alpha_1)\ldots.(a_p,\alpha_p)\\(b_1,\beta_1)\ldots\ (b_q,\beta_q)\end{matrix}\right)f(y)\,dy = g(x),\ 0<x<1.$$

$$(2.2)\quad \int_0^\infty H_{p,q}^{m,n}\left(ax\,\middle|\,\begin{matrix}(c_1,\alpha_1)\ldots.(c_p,\alpha_p)\\(d_1,\beta_1)\ldots(d_q,\beta_q)\end{matrix}\right)f(y)\,dy = h(x),\ x>1.$$

Where $H_{p,q}^{m,n}\left[ax\,\middle|\,\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right]$ is the Fox's H-function [1.80]. Observe that α_i $(i=1,2,\ldots p)$, β_j $(j=1,2,\ldots q)$ are the same for both H-function in (2.1) and (2.2).

The importance of the pair (2.1), (2.2) of dual equations is due to the very general, yet simple form of the Kernels which include as particular cases many special functions used as Kernels of dual integral equations in earlier studies.

We use the method of BUSCHMAN [39] to transform (2.1) and (2.2) in to equations with a common Kernel so that the problem is reduced to solving a single integral equation.

Using (1.117) with (1.112) in the convolution theorem (1.111) we obtain

$$M\left[I^{\eta,\alpha,A} \quad * \; H^{m,n}_{p,q}\left(ax\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right)\right]$$

$$= M\left[I^{\eta,\alpha,A}\;(x)\right] \quad M\left[H^{m,n}_{p,q}\left(ax\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right)\right]$$

On using (1.100), the left hand side of above equation becomes

$$M\left[I^{\eta,\alpha}_{x^A}\; H^{m,n}_{p,q}\left(ax\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right)\right]$$

On using (1.112) and (1.117), the right hand side of above equation becomes,

$$\text{R.H.S} = \frac{\Gamma(1+\eta-\frac{s}{A})}{\Gamma(1+\eta+\alpha-\frac{s}{A})} \times \frac{a^{-s}\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}.$$

$$= \frac{\Gamma(1+\eta-\frac{s}{A})\; a^{-s}\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=2}^{n}\Gamma(1-a_j-\alpha_j s)\times\Gamma(1-a_1-\alpha_1 s)}{\Gamma(1+\eta+\alpha-\frac{s}{A})\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}$$

Let $\eta = C_1$, $\alpha = C_1 - a_1$, $A = \frac{1}{\alpha_1}$

$$\text{R.H.S} = \frac{\Gamma(1-C_1-\alpha_1 s)\; \bar{a}^{s} \prod_{j=a}^{m} \Gamma(b_j+\beta_j s) \prod_{j=2}^{n} \Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-\beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j+\alpha_j s)}$$

Thus we obtain

$$M\left[I^{c_1,\,c_1-a_1}_{x^{\frac{1}{\alpha_1}}} \; H^{m,n}_{p,q}\left(ax \left| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right. \right) \right]$$

$$= \frac{\Gamma(1-c_1-\alpha_1 s)\; a^{-s} \prod_{j=1}^{m} \Gamma(b_j+\beta_j s) \prod_{j=2}^{n} \Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-\beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j+\alpha_j s)}.$$

Now taking the inverse Mellin-transform of above equation, we get

$$I^{-c_1,\,c_1-a_1}_{x^{\frac{1}{\alpha_1}}} \; H^{m,n}_{p,q}\left(ax \left| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right. \right)$$

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{m} \Gamma(b_j+\beta_j s) \prod_{j=2}^{n} \Gamma(1-a_j-\alpha_j s) \times \Gamma(1-c_1-\alpha_1 s)\,(ax)^{-s}}{\prod_{j=m+1}^{q} \Gamma(1-b_j-\beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j+\alpha_j s)}\, ds$$

Now, using the defination of H-function (1.80), we get -

$$(2.3) \quad I_{\frac{1}{x^{\alpha_1}}}^{-c_1, c_1-a_1} \; H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_p,\alpha_p) \\ (b_1,\beta_1),(b_2,\beta_2),\ldots,(b_q,\beta_q) \end{matrix} \right.\right)$$

$$= H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (c_1,\alpha_1),(a_2,\alpha_2),\ldots\ldots(a_p,\alpha_p) \\ (b_1,\beta_1),(b_2,\beta_2),\ldots\ldots(b_q,\beta_q) \end{matrix} \right.\right)$$

The effect of above operator, as is clear from above result, is to change the parameter (a_1, α_1) in the H-function.

Again using (1.117) with (1.113) in the convolution theorem(1.111), we have

$$M\left[K^{\eta,\alpha,A} \quad * \quad H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right.\right)\right]$$

$$= M\left[K^{\eta,\alpha,A}(x)\right] M\left[H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right.\right)\right]$$

On using (1.101), the left hand side of above equation becomes,

$$M\left[K_x^{\eta,\alpha}{}_{A} \quad H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right.\right)\right]$$

and using (1.113) and (1.117), the right hand side of above equation becomes

$$\text{R.H.S} = \frac{\Gamma(\eta+\frac{s}{A}) \quad a^{-s} \prod_{j=1}^{m} \Gamma(b_j+\beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j-\alpha_j s)}{\Gamma(\eta+\alpha+\frac{s}{A}) \prod_{j=m+1}^{q} \Gamma(1-b_j-\beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j+\alpha_j s)}$$

$$= \frac{\Gamma(\eta+\frac{s}{A}) \quad a^{-s} \prod_{j=1}^{m} \Gamma(b_j+\beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j-\alpha_j s)}{\Gamma(\eta+\alpha+\frac{s}{A}) \prod_{j=m+1}^{q} \Gamma(1-b_j-\beta_j s) \prod_{j=n+1}^{p-1} \Gamma(a_j+\alpha_j s)\,\Gamma(a_p+\alpha_p s)}$$

Let $\eta = a_p$, $\alpha = C_p - a_p$, $A = \frac{1}{\alpha_p}$.

$$\text{R.H.S} = \frac{a^{-s} \prod_{j=1}^{m} \Gamma(b_j+\beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j-\alpha_j s)}{\Gamma(c_p+\alpha_p s) \prod_{j=m+1}^{q} \Gamma(1-b_j-\beta_j s) \prod_{j=n+1}^{p-1} \Gamma(a_j+\alpha_j s)}$$

Thus we obtain :

$$M\left[\, {}_{x}K^{a_p,\, c_p-a_p}_{1/\alpha_p}\; H^{m,n}_{p,q}\left(ax \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right)\right]$$

$$= \frac{a^{-s}\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\Gamma(c_p+\alpha_p s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p-1}\Gamma(a_j+\alpha_j s)}$$

Now taking the inverse Mellin-transform of above equation we have :

$${}_{x}K^{a_p,\, c_p-a_p}_{1/\alpha_p}\; H^{m,n}_{p,q}\left(ax \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)\quad (ax)^{-s}}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p-1}\Gamma(a_j+\alpha_j s)\times\Gamma(c_p+\alpha_p s)}\, ds$$

Now using the defination of H-function (1.80) we get :

(2.4)
$$K_{x^{1/\alpha_p}}^{a_p, c_p - a_p} \; H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (a_1,\alpha_1),\ldots(a_{p-1},\alpha_{p-1})(a_p,\alpha_p) \\ (b_1,\beta_1),\ldots(b_{q-1},\beta_{q-1})(b_q,\beta_q) \end{matrix} \right.\right)$$

$$= H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (a_1,\alpha_1),\ldots(a_{p-1},\alpha_{p-1}),(c_p,\alpha_p) \\ (b_1,\beta_1),\ldots(b_{q-1},\beta_{q-1}),(b_q,\beta_q) \end{matrix} \right.\right)$$

The effect of above operator, is to change the parameter (a_p, α_p) in the H-function.

Similarly, we obtain :

(2.5)
$$K_{x^{1/\beta_1}}^{d_1, b_1 - d_1} \; H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (a_1,\alpha_1)\ldots(a_p,\alpha_p) \\ (b_1,\beta_1)\ldots(b_q,\beta_q) \end{matrix} \right.\right)$$

$$= H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (a_1,\alpha_1),\ldots(a_p,\alpha_p) \\ (d_1,\beta_1),(b_2,\beta_2),\ldots(b_q,\beta_q) \end{matrix} \right.\right)$$

(2.6)
$$I_{x^{1/\beta_q}}^{-b_q, b_q - d_q} \; H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (a_1,\alpha_1),\ldots(a_p,\alpha_p) \\ (b_1,\beta_1),\ldots(b_q,\beta_q) \end{matrix} \right.\right)$$

$$= H_{p,q}^{m,n}\left(ax \left| \begin{matrix} (a_1,\alpha_1),\ldots(a_p,\alpha_p) \\ (b_1,\beta_1),\ldots(b_{q-1},\beta_{q-1})(d_q,\beta_q) \end{matrix} \right.\right)$$

By applying these operators step by step, we can obtain formulae of the following forms.

$$(2.7)\quad \prod_{i=1}^{n} I_{x^{1/\alpha_i}}^{-c_i,\, c_i-a_i} H_{p,q}^{m,n}\left(ax \,\middle|\, \begin{matrix}(a_1,\alpha_1),\ldots(a_p,\alpha_p)\\ (b_1,\beta_1)\ldots(b_q,\beta_q)\end{matrix}\right)$$

$$= H_{p,q}^{m,n}\left(ax \,\middle|\, \begin{matrix}(c_1,\alpha_1),\ldots(c_n,\alpha_n),(a_{n+1},\alpha_{n+1}),\ldots(a_p,\alpha_p)\\ (b_1,\beta_1)\ \ldots\ldots\ldots((b_q,\beta_q)\end{matrix}\right)$$

$$(2.8)\quad \prod_{i=n+1}^{p} K_{x^{1/\alpha_i}}^{a_i,\, c_i-a_i} H_{p,q}^{m,n}\left(ax \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\ (b_q,\beta_q)\end{matrix}\right)$$

$$= H_{p,q}^{m,n}\left(ax \,\middle|\, \begin{matrix}(a_1,\alpha_1),\ldots(a_n,c_n),(c_{n+1},\alpha_{n+1})\ldots(c_p,\alpha_p)\\ (b_1,\beta_1),\ldots\ldots\ldots\ldots(b_q,\beta_q)\end{matrix}\right)$$

$$(2.9)\quad \prod_{i=1}^{m} K_{x^{1/\beta_i}}^{d_i,\, b_i-d_i} H_{p,q}^{m,n}\left(ax \,\middle|\, \begin{matrix}(a_p,\alpha_q)\\ (b_q,\beta_q)\end{matrix}\right)$$

$$= H_{p,q}^{m,n}\left(ax \,\middle|\, \begin{matrix}(a_1,\alpha_1),\ldots\ldots\ldots(a_p,\alpha_p)\\ (d_1,\beta_1),\ldots(d_m,\beta_m),(b_{m+1},\beta_{m+1})\ldots(b_q,\beta_q)\end{matrix}\right).$$

$$(2.10)\quad \prod_{i=m+1}^{q} I_{x\ 1/\beta_i}^{-b_i,\, b_i-d_i}\ H_{p,q}^{m,n}\left(ax \middle| \begin{matrix}(a_p,\alpha_p)\\ (b_q,\beta_q)\end{matrix}\right)$$

$$= H_{p,q}^{m,n}\left(ax \middle| \begin{matrix}(a_1,\alpha_1),\ \ldots\ldots\ldots(a_p,\alpha_p)\\ (b_1,\beta_1),\ldots\ldots(b_m,\beta_m),(d_{m+1},\beta_{m+1})\ \ldots(d_q,\beta_q)\end{matrix}\right)$$

$$(2.11)\quad \prod_{i=1}^{n} I_{x\ 1/\alpha_i}^{-c_i,\, c_i-a_i}\ \prod_{i=n+1}^{p} K_{x\ 1/\alpha_i}^{a_i,\, c_i-a_i}\ \prod_{i=1}^{m} K_{x\ 1/\beta_i}^{d_i,\, b_i-d_i}$$

$$\times \prod_{i=m+1}^{q} I_{x\ 1/\beta_i}^{-b_i,\, b_i-d_i} \times H_{p,q}^{m,n}\left(ax \middle| \begin{matrix}(a_p,\alpha_p)\\ (b_q,\beta_q)\end{matrix}\right)$$

$$= H_{p,q}^{m,n}\left(ax \middle| \begin{matrix}(c_p,\alpha_p)\\ (d_q,\beta_q)\end{matrix}\right)$$

2.11 The Reduction of Dual Integral Equations.

consider the dual integral equation (2.1)(2.2). If some of the parameters in the Kernels of two equations are different they can be made equal by applying fractional integral operators. As an example, by applying (2.7) and (2.8), One can change all $(a_1,\alpha_1),\ldots(a_p,\alpha_p)$ in (2.1) into $(c_1,\alpha_1),\ldots\ldots(c_p,\alpha_p)$ and by applying (2.9) and (2.10), the parameters $(d_1,\beta_1),\ldots\ldots(d_q,\beta_q)$ in (2.2) can be changed in-to $(b_1,\beta_1),\ldots\ldots\ldots(b_q,\beta_q)$, making the Kernels in the two equations the same. Thus we may write (2.1) and (2.2) in the form of single integral equation.

(2.12)

$$\int_0^{\infty} H^{m,n}_{p,q}\left(ax \;\middle|\; \begin{matrix}(c_1,\alpha_1),\ldots(c_p,\alpha_p)\\(b_1,\beta_1),\ldots(b_q,\beta_q)\end{matrix}\right) f(y)\,dy = F(x),\ 0<x<\infty.$$

Where

$$F(x)=\begin{cases}\prod_{i=1}^{n} I_{x^{1/\alpha_i}}^{-c_i,\,c_i-a_i}\ \prod_{i=n+1}^{p} K_{x^{1/\alpha_i}}^{a_i,\,c_i-a_i}\ g(x),\ 0<x<1\,.\\[2ex] \prod_{i=1}^{m} K_{x^{1/\beta_i}}^{b_i,\,d_i-b_i}\ \prod_{i=m+1}^{q} I_{x^{1/\beta_i}}^{-d_i,\,d_i-b_i}\ h(x),\ X>1\,.\end{cases}$$

The problem of solving a single integral equation of the form (2.12) has been discussed by Fox [40]

2.2 An Application of Fractional Integral Operators of two Variables.

We consider the dual integral equations :

$$(2.14)\quad \int_0^\infty \int_0^\infty H^{0,n_1:\, m_2,n_2;\, m_2,n_2}_{p_1,q_1:\, p_2,q_2;\, p_2,q_2}\left[\begin{matrix} ax \\ by \end{matrix}\left|\begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});\ (e_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});\ (f_{q_2},F_{q_2}) \end{matrix}\right.\right] f(u,v)\,du\,dv$$

$$= g(x,y);\quad 0<x<1,\ 0<y<1.$$

$$(2.15)\quad \int_0^\infty \int_0^\infty H^{0,n_1:\, m_2,n_2;\, m_2,n_2}_{p_1,q_1:\, p_2,q_2;\, p_2,q_2}\left[\begin{matrix} ax \\ by \end{matrix}\left|\begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c'_{p_2},\gamma_{p_2});\ (e'_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d'_{q_2},\delta_{q_2});\ (f'_{q_2},F_{q_2}) \end{matrix}\right.\right] f(u,v)\,du\,dv$$

$$= h(x,y)\quad x>1,\ y>1.$$

where Kernels used in above, are H - functions of two variables (1.89). The importance of the pair(2.14), (2.15) is due to the very general yet simpleform of the Kernels which include as perticular cases many special functions used as Kernels of dual integral equations in earlier studies.

We use fractional integral operators and thir properties to transform (2.14), (2.15) in to equations with common Kernel, so that the problem is reduced to problem of solving a single integral equation in two variables. using (1.115) and (1.119) in the convolution theorem(1.114) we get

(2.16)

$$M\left[I^{\eta,\alpha,A} \; I^{\tau,\beta,B} \; ** \; H^{0,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle| \begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2}) \end{matrix}\right)\right]$$

$$= M\left[I^{\eta,\alpha,A} \; I^{\tau,\beta,B}(x,y)\right] M\left[H^{0,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle| \begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2}) \end{matrix}\right)\right]$$

On using (1.109) the L.H.S of the above equation becomes,

$$M\left[I^{\eta,\alpha}_{x\,A} \; I^{\tau,\beta}_{y\,B} \; H^{0,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle| \begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2}) \end{matrix}\right)\right]$$

and for R.H.S using (1.115) and (1.119) we get

$$\text{R.H.S} = \frac{\Gamma(1+\eta-s/A)\,\Gamma(1+\tau-t/B)}{\Gamma(1+\eta+\alpha-s/A)\,\Gamma(1+\tau+\beta-t/B)} \times \frac{a^{s} b^{-t} \prod_{j=1}^{n_1}\Gamma(1-a_j-\alpha_j s-A_j t)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j+\alpha_j s+A_j t)\prod_{j=1}^{q_1}\Gamma(1-b_j-\beta_j s-B_j t)} \times \frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j-\delta_j s)}$$

$$\times \frac{\prod_{j=1}^{n_2}\Gamma(1-c_j-\gamma_j s)}{\prod_{j=n_2+1}^{p_2}\Gamma(c_j+\gamma_j s)} \frac{\prod_{j=1}^{n_2}\Gamma(1-e_j-E_j t)\prod_{j=1}^{m_2}\Gamma(f_j+F_j t)}{\prod_{j=n_2+1}^{p_2}\Gamma(e_j+E_j t)\prod_{j=m_2+1}^{q_2}\Gamma(1-f_j-F_j t)}$$

By putting $\eta = -c_1'$, $\alpha = -c_1' - c_1$, $\eta+\alpha = -c_1$, $A = 1/\gamma_1$, $\tau = -e_1'$, $\beta = e_1' - e_1$, $B = 1/E_1$

in (2.16) and using Mellin -inverse formula, we get

$$I_{x}^{-c_1', c_1'-c_1}{}_{1/\gamma_1}\; I_{y}^{-e_1', e_1'-e_1}{}_{1/E_1}\; H^{0,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle| \begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2}) \end{matrix}\right)$$

(2.17)

$$= H^{0,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle| \begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_1',\gamma_1),(c_2,\gamma_2),\ldots(c_{p_2},\gamma_{p_2}); \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2}); \end{matrix}\right.$$

$$\left.\begin{matrix} (e_1',E_1),(e_2,E_2),\ldots,(e_{p_2},E_{p_2}) \\ (f_{q_2},F_{q_2}) \end{matrix}\right)$$

provided that Re $(c_1'-c_1) > 0$, Re $(e_1'-e_1) > 0$, and

the conditions (1.120) are satisfied.

By using same technique as above, we obtain the following formulae :

$$(2.18)\quad \mathbb{K}_{x^{1/\gamma_{p_2}}}^{c_{p_2},\, c'_{p_2}-c_{p_2}} \; \mathbb{K}_{y^{1/E_{p_2}}}^{e_{p_2},\, e'_{p_2}-e_{p_2}} \; H^{0,\, n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle|\begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2}) \end{matrix}\right)$$

$$= H^{0,\, n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle|\begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_1,\gamma_1),\ldots(c_{p_2-1},\gamma_{p_2-1}),(c'_{p_2},\gamma_{p_2});(e_1,E_1),\ldots(e_{p_2-1},E_{p_2-1}),(e'_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2}) \end{matrix}\right)$$

$$(2.19)\quad \mathbb{K}_{x^{1/\delta_1}}^{d'_1,\, d_2-d'_1} \; \mathbb{K}_{y^{1/F_1}}^{f'_1,\, f_1-f'_1} \; H^{0,\, n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle|\begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2}) \end{matrix}\right)$$

$$= H^{0,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle|\begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d'_1,\delta_1),(d_2,\delta_2),\ldots(d_{q_2},\delta_{q_2});(f'_1,F_1),(f_2,F_2)\ldots(f_{q_2},F_{q_2}) \end{matrix}\right)$$

$$(2.20)\quad I_{\frac{1}{x}/\delta_{q_2}}^{-d_{q_2},\, d_{q_2}-d'_{q_2}}\; I_{\frac{1}{y}/F_{q_2}}^{-f_{q_2},\, f_{q_2}-f'_{q_2}}\; H^{o,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle|\begin{matrix}(a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2})\\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2})\end{matrix}\right)$$

$$= H^{o,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle|\begin{matrix}(a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});\quad (e_{p_2},E_{p_2})\\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_1,\delta_1)\ldots(d'_{q_2},\delta_{q_2});(f_1,F_1),\ldots(f'_{q_2},F_{q_2})\end{matrix}\right)$$

Various conditions of validity for (2.18), (2.19) (2.20) can be given easily as in the case of (2.17).

The effect of these operators as is clear from these formulae is to change the para meters in H-function. By applying these operators step by step, we can obtain formulae of the following forms.

$$(2.21)\quad \prod_{i=1}^{n_2} I_{x^{1/\gamma_i}}^{-c'_i,\, c'_i-c_i}\; I_{y^{1/E_i}}^{-e'_i,\, e'_i-e_i}\; H^{o,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle|\begin{matrix}(a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2})\\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2})\end{matrix}\right)$$

$$= H^{o,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle|\begin{matrix}(a_{p_1},\alpha_{p_1},A_{p_1}):(c'_1,\gamma_1),\ldots(c'_{n_2},\gamma_{n_2}),(c_{n_2+1},\gamma_{n_2+1})\ldots(c_{p_2},\gamma_{p_2});\quad (e'_1,E_1),\ldots(e'_{n_2},E_{n_2}),(e_{n_2+1},E_{n_2+1}),\ldots(e_{p_2},E_{p_2})\\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});\quad (f_{q_2},F_{q_2})\end{matrix}\right)$$

(2.22)

$$\prod_{i=n_2+1}^{p_2} K_{x^{1/\gamma_i}}^{c_i, c_i'-c_i} \; K_{y^{1/E_i}}^{e_i, e_i'-e_i} \; H^{o,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle| \begin{matrix} (a_{p_1}, \alpha_{p_1}, A_{p_1}) : (c_{p_2}, \gamma_{p_2}) ; (e_{p_2}, E_{p_2}) \\ (b_{q_1}, \beta_{q_1}, B_{q_1}) : (d_{q_2}, \delta_{q_2}) ; (f_{q_2}, F_{q_2}) \end{matrix}\right)$$

$$= H^{o,n_1:m_2,m_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle| \begin{matrix} (a_{p_1}, \alpha_{p_1}, A_{p_1}) : (c_1, \gamma_1), \ldots (c_{n_2}, \gamma_{n_2}), (c'_{n_2+1}, \gamma_{n_2+1}) \cdots (c'_{p_2}, \gamma_{p_2}) ; (e_1, E_1), \cdots (e_{n_2}, E_{n_2}), (e'_{n_2+1}, E_{n_2+1}) \cdots (e'_{p_2}, E_{p_2}) \\ (b_{q_1}, \beta_{q_1}, B_{q_1}) : (d_{q_2}, \delta_{q_2}) ; (f_{q_2}, F_{q_2}) \end{matrix}\right)$$

(2.23)

$$\prod_{i=1}^{m_2} K_{x^{1/\delta_i}}^{d_i', d_i-d_i'} \; K_{y^{1/F_i}}^{f_i', f_i-f_i'} \; H^{0,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle| \begin{matrix} (a_{p_1}, \alpha_{p_1}, A_{p_1}) : (c_{p_2}, \gamma_{p_2}) ; (e_{p_2}, E_{p_2}) \\ (b_{q_1}, \beta_{q_1}, B_{q_1}) : (d_{q_2}, \delta_{q_2}) ; (f_{q_2}, F_{q_2}) \end{matrix}\right)$$

$$= H^{0,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\middle| \begin{matrix} (a_{p_1}, \alpha_{p_1}, A_{p_1}) : (c_{p_2}, \gamma_{p_2}) ; (e_{p_2}, E_{p_2}) \\ (b_{q_1}, \beta_{q_1}, B_{q_1}) ; (d_1', \delta_1) \cdots (d'_{m_2}, \delta_{m_2}), (d_{m_2+1}, \delta_{m_2+1}) \cdots (d_{q_2}, \delta_{q_2}) ; (f_1', F_1), \cdots (f'_{m_2}, F_{m_2}), (f_{m_2+1}, F_{m_2+1}), \cdots (f_{q_2}, F_{q_2}) \end{matrix}\right)$$

(2.24)

$$\prod_{i=m_2+1}^{q_2} I_{x^{1/\delta_i}}^{-d_i,\, d_i-d_i'}\; I_{y^{1/F_i}}^{-f_i,\, f_i-f_i'}\; H^{o,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\left|\begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2}) \end{matrix}\right.\right)$$

$$= H^{o,n_1;m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\left|\begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}) : (c_{p_2},\gamma_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_1,\delta_1),\dots(d_{m_2},\delta_{m_2}),(d'_{m_2+1},\delta_{m_2+1}),\dots(d'_{q_2},\delta_{q_2}); \end{matrix}\right.\right. \quad ;$$

$$\left.\left.\begin{matrix} (e_{p_2},E_{p_2}) \\ (f_1,F_1),\dots(f_{m_2},F_{m_2});(f'_{m_2+1},F_{m_2+1}),\dots(f'_{q_2},F_{q_2}) \end{matrix}\right.\right)$$

(2.25)

$$\prod_{i=1}^{n_2} I_{x^{1/\gamma_i}}^{-c_i',\, c_i'-c_i}\; I_{y^{1/E_i}}^{-e_i',\, e_i'-e_i}\; \prod_{i=n_2+1}^{p_2} K_{x^{1/\gamma_i}}^{c_i,\, c_i'-c_i}\; K_{y^{1/E_i}}^{e_i,\, e_i'-e_i}\; \prod_{i=1}^{m_2} K_{x^{1/\delta_i}}^{d_i',\, d_i-d_i'}\; K_{y^{1/F_i}}^{f_i',\, f_i-f_i'}$$

$$\times \prod_{i=m_2+1}^{q_2} I_{x^{1/\delta_i}}^{-d_i,\, d_i-d_i'}\; I_{y^{1/F_i}}^{-f_i,\, f_i-f_i'}\; H^{o,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\left|\begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c_{p_2},\gamma_{p_2});(e_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2}) \end{matrix}\right.\right)$$

$$= H^{o,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{matrix} ax \\ by \end{matrix}\left|\begin{matrix} (a_{p_1},\alpha_{p_1},A_{p_1}):(c'_{p_2},\gamma_{p_2});(e'_{p_2},E_{p_2}) \\ (b_{q_1},\beta_{q_1},B_{q_1}):(d'_{q_2},\delta_{q_2});(f'_{q_2},F_{q_2}) \end{matrix}\right.\right)$$

Various conditions of validity for (2.21), (2.22), (2.23) and (2.24) can be given easily as in the case of (2.17)

2.21 The Reduction of Dual Integral Equations. :

Consider the dual integral equations (2.14) and (2.15). If some of the the parameters in the Kernels of two equations are different they can be made equal by applying fractional integral operators. As an example, by applying (2.21) and (2.22), one can change all $(c_1,\gamma_1),(c_2,\gamma_2),\ldots\ldots(c_{p_2},\gamma_{p_2})$ and (e_1,E_1), $(e_2,E_2),\ldots\ldots(e_{p_2},E_{p_2})$ into $(c_1',\gamma_1'),(c_2',\gamma_2),\ldots\ldots(c_{p_2}',\gamma_{p_2})$ and $(e_1',E_1),(e_2',E_2),\ldots\ldots(e_{p_2}',E_{p_2})$, and by applying (2.23) and (2.24), the parameters $(d_1',\delta_1),(d_2',\delta_2),\ldots\ldots(d_{q_2}',\delta_{q_2})$ and $(f_1',F_1),(f_2',F_2),\ldots\ldots(f_{q_2}',F_{q_2})$ can be change into $(d_1,\delta_1),(d_2,\delta_2),\ldots(d_{q_2},\delta_{q_2})$ and (f_1,F_1), (f_2,F_2), $\ldots\ldots(f_{q_2},F_{q_2})$, making the Kernels in the two equations the same. Thus we may write (2.14) and (2.15) in the form of a single integral equation.

(2.26)

$$\int_0^\infty \int_0^\infty H^{o,n_1:m_2,n_2;m_2,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left(\begin{array}{c|c} ax & (a_{p_1},\alpha_{p_1},A_{p_1}):(c'_{p_2},\gamma_{p_2});(e'_{p_2},E_{p_2}) \\ by & (b_{q_1},\beta_{q_1},B_{q_1}):(d_{q_2},\delta_{q_2});(f_{q_2},F_{q_2}) \end{array} \right) f(u,v)\, du\, dv$$

$$= G(x,y);\ 0 < x < \infty,\ 0 < y < \infty.$$

Where

$$G(x,y) = \begin{cases} \prod_{i=1}^{n_2} I_{x^{1/\gamma_i}}^{-c'_i,\,c'_i-c_i}\ I_{y^{1/E_i}}^{-e'_i,\,e'_i-e_i} \prod_{i=n_2+1}^{p_2} K_{x^{1/\gamma_i}}^{c_i,\,c'_i-c_i}\ K_{y^{1/E_i}}^{e_i,\,e'_i-e_i}\ g(x,y), & 0<x<1,\ 0<y<1. \\ \prod_{i=1}^{m_2} K_{x^{1/\delta_i}}^{d_i,\,d'_i-d_i}\ K_{y^{1/F_i}}^{f_i,\,f'_i-f_i} \prod_{i=m_2+1}^{q_2} I_{x^{1/\delta_i}}^{-d'_i,\,d'_i-d_i}\ I_{y^{1/F_i}}^{-f'_i,\,f'_i-f_i}\ h(x,y), & x>1,\ y>1. \end{cases}$$

The problem of solving a single integral equation of the form (2.26) has been discussed by Srivastava and Panda [60].