CHAPTER - III

THE CLASS OF EXACT SOLUTIONS FOR FERROFLUID SPACE-TIME

1. INTRODUCTION

The imposition of the constraint of spherical symmetry in solving Einstein's field equations or Einstein-Maxwell field equations associated with relativistically moving objects, play a vital role in developing cosmological models. Moreover to obtain the particular types of models for given relativistic fluid distributions, some extra geometrical restrictions like isometry or self similarity are used by Eardley (1974), Cahill and Taub (1971), Wilson (1986).

In case of anisotropic distributions of matter involving viscous effects, the vibal geometrical restrictions like conformal symmetry are utilized in obtaining the appropriate mathematical models by the researchers like Penrose (1965), Garfinkle and Tlan (1987), Eardley et al (1986), Surve and Asgekar (1987) and Aherkar and Asgekar (1990). It is proved that conformal symmetry fits to static spherically symmetric distributions of matter (Herrera et al (1983)). Further the spherically symmetric distributions of matter including viscous effects is considered by Carot and Mas (1986). If a rotational fluid space-time admits a time-like conformal vector then the vortex lines are material curves in the fluid (Ehlers et al (1986)). It is shown by Dugal (1990) that the study of relativistic fluids under metric symmetries is relevant to the radiation like viscous fluid friedman-Robertson-Walker model with the conformal collineation symmetry vector parallel to its tilted velocity vector.

Our aim is to obtain a class of spherically symmetric space-time models compatible with the ferrofluid distribution admitting conformal symmetry.

The Einstein's field equations for spherically symmetric space-time filled with the ferrofluid admitting co-moving frame are formulated in Section 2. In Section 3 Maxwell equations are solved under the spherically symmetric background and the values of the magnetic permeability μ and the magnitude of the magnetic field are obtained as the functions of r. Sections 4 and 5 deal with the space-time admitting conformal symmetry. The system of Einstein's field equations is integrated to find the values of conformal potential ψ. Thus a class of models consistent with the ferrofluid space-time admitting conformal symmetry is developed. Some particular cases like $p = 0$ and $\rho = 3p$ are discussed in Section 6. Also the values of kinematical parameters have worked out with reference to the derived model.

2. SPHERICALLY SYMMETRIC LINE ELEMENT

We start with a spherically symmetric metric written in terms of Schwarzschild co-ordinates as

$$ds^2 = A^2(r)\,dt^2 - B^2(r)\,dr^2 - r^2(d\,\theta^2 + \sin^2\theta\; d\,\phi^2). \qquad \ldots (2.1)$$

The metric potentials are given by

$$g_{ab} = \text{diag.}\ (-B^2(r)\,,\ -r^2\,,\ -\ r^2\sin^2\theta\ ,\ A^2(r)\ ). \qquad \ldots (2.2)$$

We prefer to choose co-moving system so that the time-like flow vector is taken as

$$u^a = u^4\ \delta^a_4\ . \qquad \ldots (2.3)$$

The spherical symmetry with this choice of flow vector under the orthogonality relation $u^a H_a = 0$, demands that

$$H^a \equiv (H^1,\ 0\ ,0\ ,0\ ). \qquad \ldots (2.4)$$

Hence the unitary character of the flow vector u^a yields with the help of equation (2.1)

$$u^4 = \frac{1}{A}\ . \qquad \ldots (2.5)$$

THE COMPONENTS OF EINSTEIN'S TENSOR :

We can easily workout the following components of Einstein's tensor for the spherically symmetric line element (2.1).

$$G_1^1 \equiv R_1^1 - \frac{1}{2} Rg_1^1 = \frac{1}{B^2} \left(\frac{2A'}{Ar} + \frac{1}{r^2} \right) - \frac{1}{r^2} , \quad \ldots (2.6)$$

$$G_2^2 \equiv R_2^2 - \frac{1}{2} Rg_2^2 = G_3^3 = \frac{1}{B^2} \left[\frac{A''}{A} - \frac{A'B'}{AB} + \right.$$

$$\left. + \frac{1}{r} \left(\frac{A'}{A} - \frac{B'}{B} \right) \right] , \quad \ldots (2.7)$$

$$G_4^4 \equiv R_4^4 - \frac{1}{2} Rg_4^4 = - \frac{1}{B^2} \left[\frac{2B'}{Br} - \frac{1}{r^2} \right] - \frac{1}{r^2} . \quad \ldots (2.8)$$

(Here prime denote differentiation with respect to r)

The components of the stress energy tensor T_{ab} given by equation (I. (4.1)) under the choice of equations (2.3) and (2.4)

$$\left.\begin{aligned} T_1^1 &= - p + \frac{1}{2} \mu H^2 \equiv - P , \\ T_2^2 = T_3^3 &= - p - \frac{1}{2} \mu H^2 \equiv \bar{P} , \\ T_4^4 &= \rho + \frac{1}{2} \mu H^2 \equiv \bar{\rho} . \end{aligned}\right) \quad \ldots (2.9)$$

Now by utilizing these values (2.6) to (2.8) of the components in the Einstein's field equations (I.(5.1)) we get the following three differential equations pertaining to the ferrofluid ,

$$KP = \frac{1}{B^2}\left(\frac{2A'}{Ar} + \frac{1}{r^2}\right) - \frac{1}{r^2} \quad , \qquad \ldots (2.10)$$

$$K\bar{P} = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right] , \qquad \ldots (2.11)$$

$$K\bar{\rho} = \frac{1}{B^2}\left(\frac{2B'}{Br} - \frac{1}{r^2}\right) + \frac{1}{r^2} . \qquad \ldots (2.12)$$

3. DEDUCTIONS FROM MAXWELL EQUATIONS

We recall the set of Maxwell equations (II, (2.9)and) (2.11)) as in the form

$$[\mu(u^a H^b - u^b H^a)]_{;b} = 0 \quad ,$$

i.e., $\mu_{;b}(u^a H^b - u^b H^a) + \mu(u^a H^b - u^b H^a)_{;b} = 0. \quad \ldots (3.1)$

One of these equations for the value a = 1 is given by

$$\mu_{;b}(u^1 H^b - u^b H^1) + \mu[u^1 H^b_{;b} + u^1_{;b} H^b - u^b_{;b} H^1 - u^b H^1{}_{;b}] = 0. \qquad \ldots (3.2)$$

But by the choice we have

$$u^1 = u^2 = u^3 = H^2 = H^3 = H^4 = 0.$$

Hence the equation (3.2) reduces to

$$-\mu_{;b}u^b H^1 + \mu u^1{}_{;b}H^b - \mu u^b{}_{;b}H^1 - H^1{}_{;b}u^b = 0 \quad ,$$

i.e., $-\mu_{;4}u^4H^1 + \mu u^1{}_{;1}H^1 - \mu u^b{}_{;b}H^1 - H^1{}_{;4}u^4 = 0$,

i.e., $-\mu_{;4}u^4H^1 + \mu\,[\,u^1{}_{,1} + u^4\,\Gamma_{41}^{\;1}]\,H^1 - \mu u^6{}_{;b}H^1 -$

$$- \mu\,[\,H^1{}_{,4} + H^1\,\Gamma_{14}^{\;1}\,]\,u^4 = 0 ,$$

i.e., $-\mu_{;4}\,u^4H^1 + 0 - \mu H^1u^b{}_{;b} - \mu H^1{}_{,4}\,u^4 = 0$,

i.e., $-\mu_{,4}u^4H^1 - \mu H^1u^b{}_{;b} - \mu H^1_{,4}u^4 = 0$,

i.e., $$\frac{\mu_{,4}}{\mu} + \frac{u^b{}_{;b}}{u^4} + \frac{H^1{}_{,4}}{H^1} = 0. \qquad \ldots (3.3)$$

Now by using the standard formula

$$u^b{}_{;b} = \frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x^b}\left(\sqrt{-g}\;u^b\right) , \qquad \ldots (3.4)$$

and imposing the constraint

$$\mu = \mu(r) , \qquad \ldots (3.5)$$

We can write the equation (3.3) as

$$\frac{\frac{\partial}{\partial t}\left(u^4\sqrt{-g}\right)}{\sqrt{-g}\,u^4} + \frac{H^1{}_{,4}}{H^1} = 0 . \qquad \ldots (3.6)$$

An immediate integral of (3.6) provides

$$\log\,(u^4\sqrt{-g}\,) + \log H^1 = \log f^2 ,$$

where f is a function of r.

This implies that

$$H^1 = \frac{f^2}{u^4\sqrt{-g}} . \qquad \ldots (3.7)$$

Similarly by putting a = 4 in equation (3.1) and remembering that the only non-zero components are u^4 and H^1 we get

$$\mu_{;1}u^4H^1 + \mu u^4{}_{;1}H^1 + \mu u^4H^b{}_{;b} - \mu H^4{}_{;4}\, u^4 = 0 , \ldots (3.8)$$

i.e.,

$$\mu_{,1}u^4H^1 + \mu\, [u^4{}_{,1}H^1 + u^4\Gamma^4_{41}\,]\, H^1 + \mu u^4H^b{}_{;b} +$$

$$- \mu\, [\, H^4{}_{,4} + H^1\, \Gamma^4_{14}\,]u^4 = 0.$$

Since $H^4 = 0$ and $\Gamma^4_{41} = 0$, the above equation reduces to

$$\mu_{,1}u^4H^1 + \mu u^4{}_{,1}H^1 + \mu u^4H^b{}_{;b} = 0.$$

This after simplification provides

$$\frac{\mu_{,1}}{\mu} + \frac{u^4{}_{,1}}{u^4} + \frac{H^b{}_{;b}}{H^1} = 0.$$

This gives an easy solution in the form

$$\mu u^4H^1\sqrt{-g} = \phi^2 \; , \; \phi \text{ is arbitrary constant.}$$

So that we get the value of μ ,

$$\mu = \frac{\phi^2}{\sqrt{-g}\, u^4H^1}$$

i.e., $\mu = \dfrac{\phi^2}{f^2}$, ... (3.9)

[vide equation (3.7)]

Further we have

$$H_1 = g_{11} H^1 ,$$

i.e.,

$$H_1 = -B^2 \frac{f^2}{u^4 \sqrt{-g}} \quad . \left(\because g_{11} = -B^2 \text{ and } H^1 = \frac{f^2}{u^4 \sqrt{-g}} \right)$$

i.e., $H_1 = \dfrac{-Bf^2}{r^2}$ ... (3.10)

$$\left(\because u^4 \sqrt{-g} = Br^2 \right)$$

So we can find out the magnitude of the magnetic field by equations (3.7) and (3.10) as

$$H = -H_1 H^1 ,$$

i.e., $H^2 = \dfrac{f^4}{r^4}$ . ... (3.11)

Further we can calculate the value of μH^2 by using the equations (3.9) and (3.11)

$$\mu H^2 = \frac{\phi^2 f^2}{r^4} . \qquad \text{... (3.12)}$$

Thus the spherical symmetry described by the line element (2.1) under the selection of co-moving system has provided the values of the variable magnetic permeability μ given by equation (3.9) and the magnitude of the magnetic field $H^2(r)$ given by equation (3.11) via Maxwell field equations.

4. CONFORMAL SYMMETRY

The space-time is said to admit a conformal symmetry group if the metric potentials satisfy the conditions

$$\underset{X}{L}\, g_{ab} = \Psi g_{ab} \quad , \qquad \ldots (4.1)$$

where Ψ is a scalar function of co-ordinates.

If the space-time metric (2.1) satisfies the conditions (4.1) then we try to solve the system for getting the metric potentials in terms of the unknown scalar function Ψ . In doing so we start with a conformal killing equations (4.1) which can be written in explicit form

$$g_{ij,k}\; X^k + g_{kj}\, X^k{}_{,i} + g_{ik}\; X^k{}_{,j} = \Psi g_{ij}. \qquad \ldots (4.2)$$

Now if we choose the arbitrary vector field $\bar{X}$ in the form

$$\bar{X} = \lambda\,(r)\,\frac{\partial}{\partial r} \quad , \qquad \ldots (4.3)$$

then we write equation (4.2) as

$$g_{11,1}\, X^1 + 2g_{11}\, X^1{}_{,1} = \Psi g_{11}. \qquad \ldots (4.4)$$

This with the metric potential value given by (2.2) yields a relation

$$\frac{B'}{B} + \frac{\lambda'}{\lambda} = \frac{\psi}{2\lambda} . \qquad \ldots (4.5)$$

Also from equations (2.2), (4.2) and (4.3) we derive

$$2\lambda = r\psi . \qquad \ldots (4.6)$$

Hence the equations (4.5) and (4.6) then provides a differential equation

$$\frac{B'}{B} + \frac{\lambda'}{\lambda} = \frac{1}{r} . \qquad \ldots (4.7)$$

This gives a solution of the form

$$B\lambda = rn , \qquad \ldots (4.8)$$

where n is the arbitrary constant of integration.

Consequently from equation (4.6) we write the value of B in terms of ψ as follows:

$$B = \frac{2n}{\psi} . \qquad \ldots (4.9)$$

Further the equation (4.2) for the values $i = j = 4$ by utilizing equations (2.2), (4.3) implies

$$2A' \lambda = \psi A ,$$

i.e., $$\frac{2A'}{A} = \frac{\psi}{\lambda} = \frac{2}{r} . \qquad \ldots (4.10)$$

[vide equation (4.6)]

Integrating this system gives a solution with ℓ as the arbitrary constant

$$A = \ell r. \qquad \ldots (4.11)$$

Thus we have obtained the values of metric potential (2.1) as given by equations (4.9) and (4.11)

$$B = \frac{2n}{\psi} \quad , \qquad A = \lambda r. \qquad \ldots (4.12)$$

Now by making use of these values we rewrite the field equations (2.12) to (2.14) interms of potential ψ as follows :

$$KP = \frac{3\,\psi^2}{2n^2} \left(\frac{1}{r^2} \right) - \frac{1}{r^2} \quad , \qquad \ldots (4.13)$$

$$K\bar{P} = \frac{2}{4n^2} \left[\frac{2\,\psi'}{r\,\psi} + \frac{1}{r^2} \right] , \qquad \ldots (4.14)$$

$$K\,\bar{\rho} = \frac{1}{r^2} - \frac{\psi^2}{4n^2} \left[\frac{2\psi'}{r\psi} + \frac{1}{r^2} \right] . \qquad \ldots (4.15)$$

These equations can be put in explicit form by introducing the values $P = p - \frac{1}{2}\,\mu H^2$, $\bar{P} = p + \frac{1}{2}\,\mu H^2$ and $\bar{\rho} = \rho\,\frac{1}{2}\,\mu H^2$,

$$K\,(p - \frac{1}{2}\,\mu H^2\,) = \frac{3\,\psi^2}{4n^2} \left(\frac{1}{r2} \right) - \frac{1}{r^2} \quad , \qquad \ldots (4.16)$$

$$K(p + \frac{1}{2}\,\mu H^2) = \frac{\psi^2}{4n^2} \left[\frac{2\,\psi'}{r} + \frac{1}{r^2} \right] , \qquad \ldots (4.17)$$

$$K(\,\rho + \frac{1}{2}\,\mu H^2\,) = \frac{1}{r^2} - \frac{\psi^2}{4n^2} \left[\frac{2\,\psi'}{r\psi} + \frac{1}{r^2} \right] \ldots . \quad (4.18)$$

The addition and the substraction of equations (4.16) and (4.17) gives the values of p and μH^2 in terms of potential as follows :

$$2K p = \frac{\psi^2}{n^2 r^2} - \frac{1}{r^2} + \frac{\psi\psi'}{2n^2 r} , \qquad \ldots(4.19)$$

$$2K\mu H^2 = - \frac{\psi^2}{n^2 r^2} + \frac{2}{r^2} + \frac{\psi\psi'}{n^2 r} . \qquad \ldots(4.20)$$

Also by using the value of equation (4.20) in equation (4.18) we get

$$2K \rho = \frac{1}{r} \left[\frac{1}{r} - \frac{3\psi\psi'}{2n^2} \right]. \qquad \ldots(4.21)$$

Thus we have obtained a system of differential equations (4.19), (4.20) and (4.21) for the space-time of the ferromagnetofluid admitting conformal isometry.

5. SYSTEM INTEGRALS

In solving the above system of differential equations (4.19), (4.20) and (4.21) we make use of the equation (4.20) with the value of μH^2 given by the equation (3.12) and write

$$\frac{\phi^2 f^2}{r^4} = \frac{\psi\psi'}{2n^2 r} - \frac{2\psi^2}{4n^2 r^2} + \frac{1}{r^2} . \qquad \ldots(5.1)$$

This can be simplified and written as

$$2\psi\psi' - \frac{2\psi^2}{r} = \frac{4n^2\phi^2 f^2}{r^3} - \frac{4n^2}{r}. \qquad \ldots(5.2)$$

By making use of a substitution

$$\psi^2 = Y , \quad 2\psi\psi' = \frac{dY}{dr} , \qquad \ldots(5.3)$$

in equation (5.2) yields a linear equation in the form

$$\frac{dY}{dr} - \frac{2}{r} Y = \frac{4n^2\phi^2 f^2}{r^3} - \frac{4n^2}{r} . \qquad \ldots (5.4)$$

By solving this linear equation we get the integral providing the value of ψ^2 through the relation

$$Y \equiv \psi^2 = 4n^2\phi^2 r^2 \int r^{-5} f^2 \, dr + 2n^2 + c_1 r^2 \quad \ldots (5.5)$$

Further if we restrict the value of f^2 with the constraint equation

$$f^2 = r^5 \lambda' , \ (\lambda' \text{ is function of } r) \qquad \ldots (5.6)$$

then the integral (5.5) gets reduced to a simpler version

$$\psi^2 = 4n^2\phi^2 r^2 \lambda + 2n^2 + c_1 r^2 . \qquad \ldots (5.7)$$

Thus with this value of ψ^2 a class of spherically symmetric mathematical models describing the space-time of ferrofluid admitting a conformal group of motion is exhibited by the metric form

$$ds^2 = (l^2 r^2) \, dt^2 - \left(\frac{4n^2}{\psi^2} \right) dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\,\theta^2) .$$
$$\ldots (5.8)$$

<u>NOTE 1</u> : By substituting the value of ψ^2 (5.7) in the equations (4.19) to (4.21) we write the values of ρ, p and μH^2 in context of the model (5.8) as follows :

<u>To Find The Value of p</u> : We recall the equation (4.19)

$$2K\text{þ} = \frac{\psi^2}{n^2r^2} - \frac{1}{r^2} + \frac{\psi\psi'}{2n^2r} . \qquad \ldots (5.9)$$

Also from the value of ψ^2 (5.7) we get

$$\psi\psi' = 2n^2\phi^2 r\,\lambda' + 4n^2\phi^2 r\lambda + c_1 r . \qquad \ldots (5.10)$$

Hence equations (5.9) and (5.10) will give the value of þ in the form

$$\text{þ} = 3\phi^2\,\lambda + \frac{1}{2r^2} + \frac{3c_1}{4n^2} + \frac{\phi^2 r\,\lambda'}{2} . \qquad \ldots (5.11)$$

<u>To Find ρ</u> : We recall the equation (4.21)

$$2K\,\rho = \frac{1}{2}\left[\ \frac{1}{r} - \frac{3\psi\psi'}{2n^2}\ \right].$$

This by the substitution of the value from equation (5.10) generates the result

$$\rho = -\ 3\phi^2\,\lambda - \frac{3}{2}\,\phi^2 r\,\lambda' - \frac{3c_1}{4n^2} + \frac{1}{2r^2} . \qquad \ldots (5.12)$$

Further the value of μH^2 is provided by equations (4.20), (5.7) and (5.10) in the form

$$\mu H^2 = \phi^2\, r\lambda' . \qquad \ldots (5.13)$$

6. SOME PARTICULAR CASES

We include in this section two particular cases :

(a) Ferromagneto dust : This matter distribution is characterized by a dynamical condition

$$p = 0. \qquad \ldots (6.1)$$

(b) RADIATION DOMINATED MODEL : We take the characterizing feature of the ferrofluid space-time which is radiation dominating, through the dynamical restriction

$$\rho = 3p . \qquad \ldots (6.2)$$

THEOREM 1 : For static ferrofluid spheres admitting conformal motiohs

$$p = 0 \implies \psi^2 = n^2 + \frac{c_1}{r^4} .$$

PROOF : We recall the equation (4.19)

$$p = \frac{\psi^2}{2n^2r^2} + \frac{\psi\psi'}{4n^2r} - \frac{1}{2r^2} . \qquad \ldots (6.3)$$

Hence if $p = 0$ we get

$$\frac{\psi^2}{2n^2r^2} + \frac{\psi\psi'}{4n^2r} - \frac{1}{2r^2} = 0 ,$$

i.e.,

$$2\psi\psi' + 4\frac{\psi^2}{r} = \frac{4n^2}{r} . \qquad \ldots (6.4)$$

This differential equation has an immediate integral

$$\psi^2 = n^2 + \frac{c_1}{r^4} , \qquad \ldots (6.5)$$

where c_1 is constant of integration.

Here the proof is complete.

NOTE 2 : If we put $\psi^2 = n^2 + \dfrac{c_1}{r^4}$ in equations (4.20) and (4.21) then we get

$$\mu H^2 = \frac{1}{2r^2} - \frac{3c_1}{2n^2r^6} , \qquad \ldots (6.6)$$

$$\rho = \frac{1}{2r^2} + \frac{3c_1}{2n^2r^6} . \qquad \ldots (6.7)$$

THEOREM 2 : If the ferrofluid admitting conformal motions, satisfies a equation of state $\rho = 3p$, then

$$\psi^2 = \frac{4}{3} n^2 + \frac{c_1}{r^2} .$$

PROOF : If we use the equation of state $\rho = 3p$ then the equations (4.19) and (4.21) give rise to a differential equation

$$\frac{1}{2r^2} - \frac{3\psi\psi'}{4n^2r} = \frac{3\psi^2}{2n^2r^2} + \frac{3\psi\psi'}{4n^2r} - \frac{3}{2r^2} . \qquad \ldots (6.8)$$

This after simplification becomes

$$2\psi\psi' + \frac{2}{r}\psi^2 - \frac{8n^2}{3r} = 0. \qquad \ldots (6.9)$$

The immediate integral of (6.9) yields

$$\psi^2 = \frac{4}{3} n^2 + \frac{c_1}{r^2} . \qquad \ldots (6.10)$$

This is the required necessary condition of the theorem.

NOTE 3 : If we put this value of ψ^2 given by (6.10) in the equations (4.19) to (4.21) then we get

$$p = \frac{1}{6r^2} + \frac{c_1}{4n^2r^4} \,, \qquad \ldots (6.11)$$

$$\rho = \frac{1}{2r^2} + \frac{3c_1}{4n^2r^4} \,, \qquad \ldots (6.12)$$

$$\mu H^2 = \frac{1}{3r^2} - \frac{c_1}{n^2r^4} \,. \qquad \ldots (6.13)$$

CONCLUSIONS : We have studied the space-time of ferrofluid under the geometrical restrictions (i) static and spherically symmetric metric form, (ii) A group of conformal motions.

The set of Maxwell equations under the condition of co-moving system has helped in obtaining the values of variable magnetic permeability and scalar magnitude of magnetic field. The conditions of conformal symmetry are used to derive the metric potential values and thereby providing the values of unknown potential ψ^2 through Einstein field equations. The physically meaningful constraints are put on the spacetime structure and respective system of equation is solved in Section (6).

Thus we have succeeded in providing a class of space-time model associated with the ferrofluid distribution under the geometrical restriction known as the conformal symmetry group.

NOTE : The values of Kinematical parameters :

1. **Expansion Scalar (1)** :

$$\theta = u^a{}_{;a} \,, \qquad (\because \text{ by definition}) \qquad \ldots (6.14)$$

i.e., $$\theta = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} (u^a \sqrt{-g}) \,,$$

i.e., $$\theta = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial t} (u^4 \sqrt{-g}) \,,$$

($\because$ only u^4 is existing)

i.e., $$\theta = \frac{1}{ABr^2} \frac{\partial}{\partial t} \left(\frac{1}{A} ABr^2 \right) , \qquad (\because \sqrt{-g} = ABr^2)$$

i.e., $$\theta = \frac{1}{ABr^2} \frac{\partial}{\partial t} (Br^2) \,,$$

i.e., $$\theta = 0 \,. \qquad \ldots (6.15)$$

($\because B = \frac{2n}{\psi}$ where n is constant and ψ is function of r)

2. **ACCELERATION** :

$$\dot{u}^a = u^a{}_{;b} u^b \,, \qquad \text{(by definition)} \qquad \ldots (6.16)$$

$$\dot{u}^a = (u^a{}_{,b} + u^c{}_{cb}{}^a) u^b \,, \quad (\because u^a{}_{;b} u^b = (u^a{}_{,b} + u^c \Gamma_{cb}^a) u^b)$$

i.e., $$\dot{u}^a = u^a{}_{,b} u^b + u^c u^b \Gamma_{cb}^a \,, \qquad \ldots (6.17)$$

For a = 1

$$\dot{u}^1 = u^1{}_{;b}\, u^b + \Gamma^1_{cb}\; u^c u^b \quad ,$$

i.e., $\dot{u}^1 = 0 + \Gamma^1_{44}\; u^4 u^4$, ($\because$ only u^4 is existing)

i.e., $\dot{u}^1 = \Gamma^1_{44}\; (u^4)^2$,

i.e., $\dot{u}^1 = \dfrac{1}{2B^2}\; \dfrac{\partial}{\partial r}\,(A^2)\; \left(\dfrac{1}{A}\right)^2$,

($\because$ $u^4 = \dfrac{1}{A}$ and $\Gamma^1_{44} = \dfrac{1}{2B^2}\; \dfrac{\partial}{\partial r}\,(A^2)$

i.e., $\dot{u}^1 = \dfrac{2AA'}{2B^2} \cdot \dfrac{1}{A^2}$, by Tolman page 254)

Hence $\dot{u}^1 = \dfrac{A^1}{B^2 A}$ . ... (6.18)

Also we get

$$\dot{u}^2 = \dot{u}^3 = \dot{u}^4 = 0. \qquad \text{... (6.19)}$$

($\because$ only u^4 is existing and

$\dot{u}^4 = u^4{}_{,4}\; u^4 + \Gamma^4_{44}\; u^4 u^4$

$= 0 + 0$).

Hence $\dot{u}^a = \dot{u}^1 = \dfrac{A'}{B^2 A}$ . ... (6.20)

Therefore we get

$$\dot{u}^a \dot{u}_a = \dot{u}^1 \dot{u}_1 = g_{11} (\dot{u}^1)^2 \quad ,$$

i.e., $\dot{u}^a \dot{u}_a = - \left(\frac{A'}{AB} \right)^2$,

i.e., $\dot{u}^a \dot{u}_a = - \left(\frac{1}{lr \frac{2n}{\psi}} \right)^2$, ($\because$ $A = lr$ and $B = \frac{2n}{\psi}$)

i.e., $\dot{u}^a \dot{u}_a = - \frac{\psi^2}{4n^2r^2}$ . ... (6.21)

3. <u>SHEAR TENSOR COMPONENTS</u> :

$$\sigma_{ab} = \frac{1}{2} [u_{a;b} + u_{b;a}] - \frac{1}{2} [u_{a;c}u^c u_b + u_a u_{b;c} u^c] + + \frac{1}{3} \theta h_{ab} . \quad \text{(by definition)} \qquad \ldots (6.22)$$

Since $\theta = 0$ and only u_4 is existing and $u^a u_a = 1$, $u_{a,4} = 0$ we find

$$\sigma_{ab} = \frac{1}{2} [u_{a;b} + u_{b;a}] - u_4 \Gamma_{ab}^4 + \frac{1}{2} [\Gamma_{a4}^4 u_b + + u_a \Gamma_{b4}^4] . \qquad \ldots (6.23)$$

Clearly

$$\sigma_a^a = 0 \quad ,$$

and

$\sigma_{12} = 0$, ($\because$ $u_1 = 0 = u_2$ and $\Gamma_{12}^4 = 0$

Ref. Tolman p. 254)

$\sigma_{13} = 0$, $\quad (\because u_1 = 0 = u_3$ and $\Gamma_{13}^{\ 4} = 0)$

$\sigma_{23} = 0$, $\quad (\because u_2 = 0 = u_3$ and $\Gamma_{23}^{\ 4} = 0)$

$\sigma_{24} = 0$, $\quad (\because u_2 = 0 , u_{4,2} = 0$ and $\Gamma_{24}^{\ 4} = 0)$.

$\sigma_{43} = 0$, $\quad (\because u_{4,3} = 0 , u_3 = 0 , \Gamma_{43}^{\ 4} = 0 = \Gamma_{34}^{\ 4})$

$\sigma_{14} = \frac{1}{2} [u_4 - u_4 \Gamma_{14}^{\ 4}] \quad (\because u_1 = 0)$

i.e., $\sigma_{14} = \frac{1}{2} [A' - A. \frac{1}{2A^2} (2AA)]$, $(\because u^4 = \frac{1}{A})$

i.e., $\sigma_{14} = \frac{1}{2} (A' - A')$,

$\sigma_{14} = 0$,

Hence

$$\sigma^2 = \sigma^{ab} \sigma_{ab} = 0. \qquad \dots (6.24)$$

4. <u>ROTATION TENSOR COMPONENTS</u> :

$$w_{ab} = \frac{1}{2} [u_{a;b} - u_{b;a}] - \frac{1}{2} [u_{a;c} u^c u_b - u_a u_{b;c} u^c]. \qquad \dots (6.25)$$

(by definition)

By using $\Gamma_{ab}^{\ 4} = \Gamma_{ba}^{\ 4}$, $u_{a,4} = 0$, $u^a u_a = 1$ and

u^4 is existing, we get

$$W_{ab} = \frac{1}{2}[\, u_{a,b} - u_{b,a} \,] - \frac{1}{2}[\, u_a \Gamma_{b4}^{4} - \Gamma_{a4}^{4} u_b \,] , \qquad \ldots (6.26)$$

$W_{11} = 0$,

$W_{12} = 0$, ($\because u_1 = 0 = u_2$)

$W_{13} = 0$, ($\because u_1 = 0 = u_3$)

$W_{23} = 0$, ($\because u_2 = 0 = u_3$)

$W_{24} = 0$, ($\because u_2 = 0$, $u_{4,2} = 0$, $\Gamma_{24}^{4} = 0$)

$W_{43} = 0$, ($\because u_{4,3} = 0$, $u_3 = 0$, $\Gamma_{34}^{4} = 0$)

$W_{41} = \frac{1}{2}\left[u_{4,1} - u_4 \Gamma_{14}^{4} \right]$, ($\because u_{1,4} = 0$)

i.e.,

$W_{41} = \frac{1}{2}[\, A' - A.\frac{1}{2A^2}(2AA') \,]$,

i.e., $W_{41} = \frac{1}{2}(A' - A')$,

i.e., $W_{41} = 0$,

Hence

$$W^2 = W^{ab}W_{ab} = 0. \qquad \ldots (6.27)$$

Thus relative anisotropy $\frac{\sigma^2}{\rho} = \frac{0}{\rho} = 0.$ ... (6.28)

This proves that the flow of the ferrofluid admitting conformal motion is essentially accelerating.