C H A P T E R - I

B A S I C   E N T I T I E S

I. BASIC ENTITIES

We consider the space-time as 4-dimensional differentiable manifold M with a Laurentzian metric g of signature (-, -, -, +). The definitions of the terms given below are taken from the book, "The Large Scale Structure of The Space-time" by Hawking and Ellis (1973).

TYPES OF CURVES ON M :

At a point P on the curve of space-time M one can draw a tangent X_p. Then these vectors can be categorized in three types according to the condition (i) if $g(\bar{X}, \bar{X}) > 0$, then $\bar{X}$ is time-like, (ii) if $g(\bar{X}, \bar{X}) = 0$, then $\bar{X}$ is null, (iii) if $g(\bar{X}, \bar{X}) < 0$, then $\bar{X}$ is space-like.

1. CONGRUENCES IN SPACE-TIME M :

It is the family of curves on M. It is only in relativity, there exists three types of congruences. viz.: Time-like, null and space-like congruences due to the Laurentz's metric structure.

DEFINITION : If the curves of a congruence have always time-like (space-like) tangent vectors then the congruence is called time-like (space-like).

ILLUSTRATIONS : (1) The histories of the test particles

or the paths of Tardyons provide a time-like congruence.
(2) The path of the photons for a null congruence. The photons move with the velocity of light c and they have zero rest mass. This congruence is also referred as a ray congruence.
(3) The paths of Tachyons (which are hypothetical and supposed to travel with a velocity greater than that of the velocity of light) constitute a space-like congruence.

THE EQUATION FOR TIME-LIKE CONGRUENCE :

The parametric equation of a time-like congruence is given by

$$x^a = x^a\,(m^i,\ S),\ \ i = 1, 2, 3 \text{ and } a = 1, 2, 3, 4.$$

where m^i are the Lagrangian co-ordinates and S is the parameter along one of the curvesof the congruence.Let u^a be the unit 4-velocity vector tangent to the curve of the time-like congruence which is defined by

$$u^a = \frac{dx^a}{dS}\ ,\ (m^i \text{ is constant})$$

with normalizing condition

$$u^a u_a = 1.$$

The following relation holds trivially

$$u_{a;b}u^{a} = 0 \quad ,$$

where a semicolon denotes a covariant differentiation.

THE EQUATION OF SPACE-LIKE CONGRUENCE :

The parametric equation of the space-like congruence is described by the relation

$$X^{a} = X^{a}\ (m^{i}, s) \quad ,$$

where s is some parameter along the curve of the space-like congruence.

Let K^{a} represent unit tangent vector to the curve of the congruence. Then K^{a} is given by the expression

$$K^{a} = \frac{dX^{a}}{ds} \quad , \ (m^{i} \text{ is fixed})$$

where $K^{a}K_{a} = -1.$

obviously, we have

$$K_{a;b}K^{a} = 0$$

2. KINEMATICAL SCALARS

The kinematical parameters associated with the time-like congruence u^{a} of the fluid flow according to Greenberg (1970) are ,

(i) The expansion parameter

$$\theta = u^{a}{}_{;a} \quad , \qquad \dots (2.1)$$

(ii) The symmetric shear tensor

$$\sigma_{ab} = u_{(a;b)} - \dot{u}_{(a}u_{b)} + \frac{1}{3}\,\theta\, h_{ab} \quad , \qquad \dots (2.2)$$

(iii) The antisymmetric rotation tensor

$$w_{ab} = u_{[a;b]} - \dot{u}_{[a}u_{b]} \quad , \qquad \dots (2.3)$$

where the term ,

$$\dot{u}_{a} = u_{a;b}\, u^{b} \quad , \qquad \dots (2.4)$$

is known as the acceleration.

We have the projection operator

$$h_{ab} = g_{ab} - u_{a}u_{b} \quad , \qquad \dots (2.5)$$

with the properties

$$h_{ab}u^{b} = 0 \ , \quad h^{a}_{a} = 3. \qquad \dots (2.6)$$

The shear tensor and the rotation tensor are trace free ,

$$\text{i.e.,}\ \sigma^{a}_{a} = 0 = w^{a}_{a} \ . \qquad \dots (2.7)$$

Also by expressions (2) and (3) we have

$$\sigma_{ab}\, U^{b} = w_{ab}U^{b} = 0. \qquad \dots (2.8)$$

On the same line, it follows from the unitary character of the flow vector

$$\dot{U}_a u^a = 0. \qquad \ldots (2.9)$$

The invariants of these tensors are defined as

$$\sigma_{ab}\sigma^{ab} = 2\sigma^2,$$

$$w_{ab}w^{ab} = 2w^6$$

The definitions (2.1) to (2.4) help us to express the flow gradiant $u_{a;b}$ as

$$u_{a;b} = \sigma_{ab} + w_{ab} + \frac{1}{3}\theta h_{ab} + \dot{u}_a u_b . \qquad \ldots (2.10)$$

3. <u>GEOMETRICAL SYMMETRIES</u>

The particular types of geometrical symmetries utilized in the working of the dissertation are introduced below :

(1) <u>CONFORMAL SYMMETRY (YANO 1965)</u>:

If there exists a map M —> M such that the metric g transforms under the rule

$$g \rightarrow g' = e^{2\phi} g \ , \ \phi = \phi(x^a) \ , \qquad \ldots (3.1)$$

then M is said to have conformal symmetry.

There exist two subcases as selfsimilarity (ϕ = non zero constant) and isometry ($\phi = 0$). Equation (3.1) implies the existence of a one-parameter group of conformal motions generated by a conformal killing vector field $\bar{m}$ such that

$$\underset{m}{L}\, g_{ab} = m_{a;b} + m_{b;a} = 2\phi\, g_{ab} \quad , \qquad \ldots (3.2)$$

where L is the Lie-derivative operator.

(2) THE SPECIAL CONFORMAL MOTION :

The special types of conformal motions are defined through the conditions (Katzin et al., 1969).

$$\underset{m}{L}\, g_{ab} = \Psi g_{ab} \text{ and } \Psi_{;ab} = 0 \; . \qquad \ldots (3.3)$$

(3) The Ricci Collineation along the vector field $\bar{m}$ is defined as

$$\underset{m}{L}\, R_{ab} = R_{ab;c}\, m^c + R_{ac} m^c{}_{;b} + R_{cb} m^c{}_{;a} = 0. \qquad \ldots (3.4)$$

This provides a natural conservation law generator in the form

$$(R^a_b\, m^b)_{;a} = 0. \qquad \ldots (3.5)$$

4. THE STRESS ENERGY TENSOR FOR FERROFLUID :

The infinitely conducting charged relativistic fluid with variable magnetic permeability is called here as ferrofluid. The form of the stress-energy tensor characterizing ferrofluid Cissoko (1978) is given by

$$T_{ab} = (\rho + p + \mu H^2) u_a u_b - (p + \frac{1}{2}\mu H^2) g_{ab} - \mu H_a H_b . \quad \dots (4.1)$$

Here ρ is the matter energy density p is the isotropic pressure, μ is the variable magnetic permeability. The time-like flow vector $\bar{u}$ and space-like magnetic field vector $\bar{H}$ satisfy the conditions

$$u^a u_a = 1 \; , \; H^a H_a = - H^2 \text{ and}$$

$$u^a H_a = 0.$$

THE EIGEN VALUES OF T_{ab} :

The various contractions of (4.1) provide the following results :

$$T_{ab} u^a = (\rho + \frac{1}{2}\mu H^2) u_b \; , \quad \dots (4.2)$$

$$T_{ab} u^a u^b = \rho + \frac{1}{2}\mu H^2 \; , \quad \dots (4.3)$$

$$T_{ab} H^a = - (p + \frac{1}{2}\mu H^2) H_b + \mu H^2 H_b \; , \quad \dots (4.4)$$

$$T_{ab} H^a H^b = (p - \frac{1}{2}\mu H^2) H^2 \; , \quad \dots (4.5)$$

$$T_{ab} g^{ab} = T = \rho - 3p. \quad \dots (4.6)$$

It reveals from equation (4.2) to (4.5) that $\bar{u}$ is a time-like eigen vector of T_{ab} with eiven value $e_1 = \rho + \frac{1}{2}\mu H^2$

and $\bar{H}$ is the space-like eigen vector of T_{ab} with the eigen value $e_2 = (p - \frac{1}{2}\mu H^2) H^2$.

The equation (4.6) provides the trace value of T_{ab}.

The strong-energy condition to be satisfied by T_{ab} is given by (Hawking and Ellis, 1968).

$$T_{ab}u^a u^b - \frac{1}{2} T \geqslant 0. \qquad \ldots (4.7)$$

This for ferrofluid described by the expression (4.1) yields

$$\rho + 3p + \mu H^2 \geqslant 0. \qquad \ldots (4.8)$$

This shows that the stress-energy tensor (4.1) describing the ferrofluid is physically transparent.

5. A SYSTEM OF FIELD EQUATIONS FOR FERROFLUID

This system consists of

(1) Einstein field equations for gravitation.

(2) Maxwell field equations for electromagnetism.

(1) The well known Einstein field equations governing the geometrical and dynamical features of the space-time are written with the help of symmetric Ricci tensor R_{ab}, the scalar curvature R and the symmetric stress energy tensor T_{ab}. These are

given by ten independent non-linear differential equations as

$$R_{ab} - \frac{1}{2}Rg_{ab} = - KT_{ab} \; . \qquad \ldots (5.1)$$

For empty space ($T_{ab} = 0$) these are reduced to the form

$$R_{ab} = 0.$$

Hence these are the field equations for empty space-time.

<u>REMARK</u> : It is known that a flat space-time is empty but not the converse.

(2) Under the restriction of infinite conductivity, the only valid set of Maxwell equations applicable to ferro-fluid is given by (Ray and Banergy, 1979)

$$[\mu(H^a u^b - H^b u^a)]_{;b} = 0. \qquad \ldots (5.2)$$

By utilizing the values (4.1) and (4.6) in (5.1) we derive the expression for Ricci tensor for ferrofluid in the form

$$R_{ab} = - K \left[(\rho + p + \mu H^2) u_a u_b - \frac{1}{2} (\rho - p + \mu H^2) g_{ab} - - \mu H_a H_b \right] \qquad \ldots (5.3)$$

<u>REMARK</u> : Because of non-existence of another set of Maxwell's equations, the current becomes indeterminate.