C H A P T E R - II

THE CONSEQUENCES OF CONFORMAL SYMMETRY ON FERROFLUID SPACE-TIME

1. INTRODUCTION

A brief account of restrictions on geometry of space-time is presented below :

(1) ISOMETRY (YANO, 1965) :

Let (V_4, g) be a 4-dimensional Riemannian space with the fundamental metric form

$$ds^2 = g_{ab} dx^a dx^b . \quad \ldots (1.1)$$

In order that the infinitisimal point transformation

$$\bar{m}^a = m^a + X^a(M)\, dt \quad , \quad m^a m_a = M^2 \quad , \quad \ldots (1.2)$$

be a motion in V_4, it is necessary and sufficient that the Lie derivative of g_{ab} with respect to (1.2) vanishes i.e.,

$$\underset{X}{L}\, g_{ab} \doteq X_{a;b} + X_{b;a} = 0. \quad \ldots (1.3)$$

Thus (V_4, g) is said to admit a one parameter group of motions (infinitisimal isometry) generated by the vector field $\bar{X}$ if the conditions (1.3) are satisfied.

(2) SELF SIMILARITY (EARDLEY, 1974) :

DEFINITION : If there exist a smooth map $V_4 \longrightarrow V_4$ such that the metric g transforms under a constant scale factor,

i.e.,

$$g \longrightarrow g' = e^{2k} g \quad , \qquad \ldots (1.4)$$

then (V_4, g) is called a self similar space-time. The conditions (1·2) can be put in terms of Lie derivative as

$$\underset{X}{L}\, g_{ab} = C\, g_{ab} \quad , \qquad \ldots (1.5)$$

where C is any scalar.

It follows from such transformations that the geometry and physics at different points of a region ,(where self similarity holds) of a space-time, differ only by a change in the overall length scale.

REMARK : If in equation (1.5) C is constant then these conditions characterize the groups of homothetic motions.

(3) CONFORMAL SYMMETRY (YANO, 1955) :

In this case there exists a mapping $V_4 \longrightarrow V_4$ such that the metric g transforms under the rule

$$\underset{X}{L}\, g_{ab} = \Psi\, g_{ab} \quad , \qquad \ldots (1.6)$$

where Ψ is arbitrary function of co-ordinates.

Here the arbitrary vector $\bar{X}$ is the generator of conformal symmetry group.

(4) A SPECIAL CONFORMAL MOTION (Katcin et al., 1970) :

The arbitrary vector field $\bar{X}$ is said to generate a special conformal symmetry if

$$\underset{X}{L} g_{ab} = \psi g_{ab} \quad , \quad \psi_{;ab} = 0. \qquad \ldots (1.7)$$

In this chapter we have studied the space-time of the ferrofluid constrained under the conditions of the group of conformal symmetry and the special conformal symmetry.

The local conservation laws providing the equation of continuity and stream-lines are the contents of Section 2. It is proved that

(i) for expansion free flow of ferrofluid the matter energy density is conserved along the flow if and only if the magnetic permeability is conserved along the flow.

(ii) If the 4-acceleration of the ferrofluid is normal to magnetic lines then the conservation of isotropic pressure along the magnetic lines is the direct consequence of the conservation of magnetic permeability along magnetic lines. Section 3 deals with conformal symmetry group generated by flow vector $\bar{u}$ and magnetic field vector $\bar{H}$ involved in ferrofluid system. We have shown here (1) the flow of the ferrofluid admitting conformal motions must be expanding , (2) the 4-acceleration is normal to magnetic lines if and only if M is preserved along these lines. Moreover it is proved that the variation of isotropic pressure along magnetic lines and the divergence of magnetic lines depend explicitly on conformal potential ψ . Ricci Identities for conformal symmetry group are examined in Section 4. The expression for Lie derivative of Einstein's field equations along the arbitrary vector field $\bar{X}$ generating the conformal symmetry group is evaluated in Section 5. The next section six comprises a special case of conformal motions. It is proved that if the space-time of ferrofluid admits the special conformal group of motion then

$$\underset{X}{L}\, \rho + p = -\psi(\rho + p) - \mu H^2 .$$

The corresponding conservation law generators provide the results like $\underset{H}{L}\, H^2 = \rho \iff \rho - p - \mu H^2 = 0.$

2. SYSTEM OF DIFFERENTIAL EQUATIONS FOR FERROFLUID

The space-time of ferrofluid characterized by the stress energy tensor (I.(4.1)) has to satisfy the following set of differential equations.

(1) $T^{ab}{}_{;b} = 0.$ (Local conservation laws) , ... (2.1)

(2) $[\mu\,(u^b H^a - u^a H^b)\,]_{;b} = 0.$ (Maxwell equations)...(2.2)

We discuss below the implications of above differential equations (2.1) and (2.2).

THE LOCAL CONSERVATION LAWS :

The equation (2.1) with the expression of T_{ab} (I. (4.1)) generates

$$(\rho + p + \mu H^2)_{;b} u^a u^b + (\rho + p + \mu H^2) u^a_{;b} u^b +$$
$$+ (\rho + p + \mu H^2) u^a u^b_{;b} - (p + \frac{1}{2} \mu H^2)_{;b} g^{ab} -$$
$$- \mu_{;b} H^a H^b - \mu H^a_{;b} H^b - \mu H^a H^b_{;b} = 0. \qquad \ldots (2.3)$$

The time component of this equation ($T^{ab}_{;b} u_a$) with the use of the results

$$u^a u_a = 1 , \quad H^a u_a = 0 , \quad u^a_{;b} u_a = 0. \qquad \ldots (2.4)$$

is given by

$$(\rho + \frac{1}{2} \mu H^2)_{;b} u^b + (\rho + p + \mu H^2)\theta - \mu H^a_{;b} H^b u_a = 0 ,$$
$$\ldots (2.5)$$

where $u^a_{;a} = \theta$.

This after simplification under the notation $\dot{\rho} = \rho_{;a} u^a$ reduces to

$$\dot{\rho} + (\rho + p)\,\theta - \frac{1}{2}\,\dot{\mu}H^2 + \mu\left(\frac{1}{2}\,\dot{(H^2)} + H^2\theta + u^a{}_{;b}H_aH^b\right)$$

$$+ \dot{\mu}\,H^2 = 0. \qquad \ldots (2.6)$$

Further contracting the equation (2.3) with H_a and using the notations

$$\dot{u}^a = u^a{}_{;b}u^b \;,\quad H^aH_a = -H^2 \;, \qquad \ldots (2.7)$$

we get

$$(\rho + p)\dot{u}^aH_a - p_{;b}\,H^b + H^2\left[\mu(\dot{u}^aH_a + H^b{}_{;b}) + \right.$$

$$\left. + \frac{1}{2}\,\mu_{;b}\,H^b\right] = 0. \qquad \ldots (2.8)$$

<u>MAXWELL EQUATIONS</u> :

The only valid set of Maxwell equations to be used for ferrofluid is given by equation (2.2). This can be written as

$$\mu(H^a{}_{;b}u^b + H^au^b{}_{;b} - u^a{}_{;b}H^b - u^aH^b{}_{;b}) +$$

$$+ \mu_{;b}\,(H^au^b - u^aH^b) = 0. \qquad \ldots (2.9)$$

By transvecting this equation (2.9) with H_a and noting that $H^au_a = 0$, $H^aH_a = -H^2$ and $u^a{}_{;a} = \theta$, we get after simplification

$$\mu\left[\frac{1}{2}\,\dot{(H^2)} + H^2\,\theta + H_a\,u^a{}_{;b}H^b\right] + \dot{\mu}H^2 = 0. \qquad \ldots (2.10)$$

Further on contracting equation (2.9) with u_a we obtain after simplification

$$\mu(u_a H^a{}_{;b}u^b - u_a u^a{}_{;b} H^b - H^b{}_{;b}) - \mu_{;b}H^b = 0 ,$$

i.e.,

$$\mu(u_a H^a{}_{;b}u^b - H^b{}_{;b}) - \mu_{;b}H^b = 0 , (\because u^a{}_{;b}u_a = 0)$$

i.e.,

$$\mu(- u^a{}_{;b} H^a u^b - H^b{}_{;b}) - \mu_{;b} H^b = 0.$$

$$(\because H^a{}_{;b}u_a = - u_{a;b} H^a)$$

This finally can be written as

$$\mu \dot{u}_a H^a + (\mu H^b)_{;b} = 0. \qquad \ldots (2.11)$$

Now if we use the consequence (2.10) in the equation (2.6) then we obtain

$$\dot{\rho} + (\rho + p)\theta - \frac{1}{2} \dot{\mu} H^2 = 0. \qquad \ldots (2.12)$$

This is the equation of continuity for ferrofluid which explains the effect of variable permeability μ ,

On the variation of mass energy density ρ .

So also by virtue of equation (2.8) and (2.11) we deduce an important result

$$(\rho + p) \dot{u}^a H_a = (p_{;b} + \frac{1}{2} \mu_{;b}H^2) H^b . \qquad \ldots (2.13)$$

It is observed from this result that the 4-acceleration is orthogonal to magnetic lines if and only if the isotropic pressure p and the magnetic permeability μ are preserved along these lines.

To write the equation of streamlines we utilize the contindity equation (2.12) in the conservation equation (2.3). So that after simplification we get

$$(\rho + p + \mu H^2) \dot{u}^a + (p + \frac{1}{2} \mu H^2)_{;b} h^{ab} -$$

$$- (\mu H^b)_{;b} H^a = 0. \qquad \ldots (2.14)$$

This equation exhibits the factors causing the deviation of path lines from geodesic path.

3. THE GROUP OF CONFORMAL SYMMETRY

We recall the conditions for the conformal symmetry group generated by vector field $\bar{m}$ in the form

$$\underset{m}{L} g_{ab} = \Psi g_{ab} \ , \quad \Psi \text{ is a scalar function.}$$

i.e.,

$$m_{a;b} + m_{b;a} = \Psi g_{ab} . \qquad \ldots (3.1)$$

Our aim here is to study the effects of this conformal symmetry group generated by special vector fields given by

<u>CASE (A)</u> : $m_a = M u_a \quad , \quad m_a m^a = M^2$

Here the vector m_a is parallel to time-like vector u_a.

CASE (B) : $m_a = H_a$, $m_a m^a = H_a H^a = - H^2$

Here H_a is the space-like magnetic field vector.

CASE A. Let $m_a = Mu_a$: For this case the condition (3.1) becomes

$$(Mu_a)_{;b} + (Mu_b)_{;a} = \Psi g_{ab} . \qquad \text{... (3.2)}$$

The contraction of this result (3.2) with g^{ab} and $u^a u^b$ produce respectively the results

$$\dot{M} + M\,\theta = 2\Psi \qquad \text{... (3.3)}$$

and

$$\dot{M} = \frac{\Psi}{2} . \qquad \text{... (3.4)}$$

The above two equations yield in common

$$M\,\theta = \frac{3}{2}\Psi . \qquad \text{... (3.5)}$$

Further the contraction of equation (3.2) with u^a yield

$$\dot{M}\,u_b + M_{;b} + M\,\dot{u}_b = \Psi u_b . \qquad \text{... (3.6)}$$

From equation (3.4) this reduces to

$$M_{;b} + M\,\dot{u}_b = \frac{\Psi}{2} u_b . \qquad \text{... (3.7)}$$

This after contraction with H^b generates

$$M_{,b} H^b + M\,\dot{u}_b H^b = 0 . \qquad \text{... (3.8)}$$

Also the contractions of equation (3.2) with H^a and H^aH^b gives

$$M_{;a}H^a u_b + M(H^a u_{a;b} + u_{b;a} H^a) = \psi H_b . \qquad \ldots (3.9)$$

and

$$2M\, u_{a;b} H^a H^b = -\psi H^2. \qquad \ldots (3.10)$$

INTERPRETATIONS :

(1) The flow of the ferrofluid admitting conformal motions along $\bar{m}$ must be essentially expanding ($\theta \neq 0$).

[vide equation (3.5)]

(2) The 4-acceleration is normal to magnetic lines if and only if M is preserved along these lines.

[vide equation (3.8)]

CASE (B) : Let $m_a = H_a$, $m_a m^a = -H^2$

For this choice the conditions (3.2) become

$$H_{a;b} + H_{b;a} = \psi g_{ab} . \qquad \ldots (3.11)$$

These conditions when contracted with u^a, $u^a u^b$, $u^a H^b$, H^a, $H^a H^b$ and g^{ab} respectively give rise to following equations.

$$H_{a;b}\, u^a = \frac{\psi}{2} u_b , \qquad \ldots (3.12)$$

$$\dot{H}_a u^a = \frac{\psi}{2} , \qquad \ldots (3.13)$$

$$H_{a;b}u^aH^b = 0 \quad , \qquad \ldots (3.14)$$

$$-\frac{1}{2} H^2{}_{;b} + H_{b;a}H^a = \psi H_b \quad , \qquad \ldots (3.15)$$

$$\psi H^2 = H^2{}_{;b}H^b \quad , \qquad \ldots (3.16)$$

$$H^a{}_{;a} = 2\psi \; . \qquad \ldots (3.17)$$

It follows from equations (3.13), (3.16) and (3.17)

$$\dot{u}_aH^a \neq 0 \; , \; H^2{}_{;b}H^b \neq 0 \;\text{ and } H^a_{;a} \neq 0$$

as $\psi \neq 0$. ... (3.18)

Further by using equations (3.13) and (3.17) in Maxwell equation (2.11) we get

$$\mu_{;b}H^b = -\frac{3}{2}\psi\mu \quad ,$$

i.e.

$$\psi = \left(-\frac{2}{3\mu}\right)\left(\mu_{;b}H^b\right). \qquad \ldots (3.19)$$

This shows the explicit dependence of the conformal potential ψ on the variable magnetic permeability μ. Further by using equations (3.13) and (3.19) in equation (2.13) we derive

$$p_{;b}H^b = \psi\left(\frac{-2\rho - 2p + 3\mu H^2}{4}\right). \qquad \ldots (3.20)$$

This exhibits the explicit dependence of isotropic pressure p on ψ .

<u>REMARK</u> : If the equation (3.20) implies

$$p_{;b}H^b = 0 \Longleftrightarrow 2\rho + 2p - 3\mu H^2 = 0. \qquad \ldots (3.21)$$

We infer from this that the isotropic pressure of ferrofluid remains constant along magnetic lines if the ferrofluid satisfies the equation of state $2\rho + 2p - 3\mu H^2 = 0$. Further we obtain from equations (2.10) and (3.14)

$$\frac{\mu}{2} \, (\dot{H^2}) + \mu H^2 \theta + \dot{\mu} H^2 = 0. \qquad \ldots (3.22)$$

If $\theta = 0$, then this produces the result

$$(\dot{H^2}) = 0 \Longleftrightarrow \dot{\mu} = 0. \qquad \ldots (3.23)$$

We conclude from this that for the expansion free ferrofluid admitting the conformal motions along the magnetic lines, the magnitude of the magnetic lines is preserved along the flow if and only if the magnetic permeability is also preserved along the flow.

4. RICCI IDENTITIES FOR CONFORMAL SYMMETRY GROUP :

Let $\bar{X}$ be the conformal killing vector satisfying the conditions

$$X_{a;b} + X_{b;a} = \psi g_{ab} . \qquad \ldots (4.1)$$

CLAIM : For conformal killing vector $\bar{X}$ the Ricci Identities are

$$X_{m;np} = X_k R^k_{pnm} + \frac{1}{2} [\Psi_{;n} g_{mp} + \Psi_{;p} g_{mn} - $$

$$- \Psi_{;m} g_{np}] . \qquad \ldots (4.2)$$

<u>PROOF</u> : We have Ricci Identities

$$X_{m;np} - X_{m;pn} = X_k R^k{}_{mnp} . \qquad \ldots (4.3)$$

Similarly we write

$$X_{n;pm} - X_{n;mp} = X_k R^k{}_{npm} , \qquad \ldots (4.4)$$

$$X_{p;mn} - X_{p;nm} = X_k R^k{}_{pmn} . \qquad \ldots (4.5)$$

By additing equations (4.3), (4.4) and (4.5) we get

$$X_{m;np} + X_{n;pm} + X_{p;mn} - X_{m;pn} - X_{n;mp} -$$

$$- X_{p;nm} = X_k (R^k_{mnp} + R^k{}_{npm} + R^k{}_{pmn}).$$

i.e.,

$$X_{m;np} + X_{n;pm} + X_{p;mn} - X_{m;pn} - X_{n;mp} -$$

$$- X_{p;nm} = 0. \qquad \ldots (4.6)$$

(By cyclic property of curvature tensor)

From equation (4.1)

$$X_{m;p} + X_{p;m} = \Psi g_{mp} . \qquad \ldots (4.7)$$

This equation implies

$$X_{m;pn} + X_{p;mn} = \Psi_{;n}\, g_{mp} ,$$

i.e.

$$-X_{m;pn} = X_{p;mn} - \Psi_{;n}\, g_{mp} . \qquad \ldots (4.8)$$

Similarly we derive

$$-X_{n;mp} = X_{m;np} - \Psi_{;p}\, g_{mn} , \qquad \ldots (4.9)$$

and

$$-X_{p;nm} = X_{n;pm} - \Psi_{;m}\, g_{np} . \qquad \ldots (4.10)$$

Hence equation (4.6) with the use of equations (4.8), (4.9) and (4.10) reduces to

$$X_{m;np} + X_{n;pm} + X_{p;mn} + X_{p;mn} - \Psi_{;n}\, g_{mp} + {} + X_{m;np} - \Psi_{;p}\, g_{mn} + X_{n;pm} - \Psi_{;m}\, g_{np} = 0 ,$$

i.e.,

$$X_{m;np} + X_{n;pm} + X_{p;mn} = \tfrac{1}{2} [\Psi_{;n} g_{mp} + \Psi_{;p}\, g_{mn} + {} + \Psi_{;m}\, g_{np}]. \qquad \ldots (4.11)$$

Also by adjusting the terms in equation (4.11) we get

$$X_{m;np} - (X_{p;nm} - \Psi_{;m}\, g_{np} - X_{p;mn}) = \tfrac{1}{2} [\Psi_{;n}\, g_{mp} + \Psi_{;p}\, g_{mn} + \Psi_{;m}\, g_{np}] ,$$

i.e. $X_{m;np} - (X_{p;nm} - X_{p;mn})$

$$= \frac{1}{2} [\Psi_{;n} g_{mp} + \Psi_{;p} g_{mn} + \Psi_{;m} g_{np}] - - \Psi_{;m} g_{np} . \quad \ldots (4.12)$$

Thus due to the Ricci identities (4.3), the equation (4.12) produces

$$X_{m;np} - X_k R^k{}_{pnm} = \frac{1}{2} [\Psi_{;n} g_{mp} + \Psi_{;p} g_{mn} - -\Psi_{;m} g_{np}].$$

i.e.

$$X_{m;np} = X_k R^k{}_{pnm} + \frac{1}{2} [\Psi_{;n} g_{mp} + \Psi_{;p} g_{mn} - -\Psi_{;m} g_{np}]. \quad \ldots (4.13)$$

This is the required result.

5. THE GROUP OF CONFORMAL MOTIONS IN FERROFLUID :

(a) DEDUCTIONS FROM THE GROUP OF CONFORMAL MOTIONS :

We have the defining conditions for the conformal symmetry group

$$\underset{X}{L} g_{ab} = \Psi g_{ab} , \quad \ldots (5.1)$$

where Ψ is a scalar function of co-ordinates. For these transformations following the same procedure of Herrera et al (1984) we can write the transformation formulae for flow vector $\bar{u}$ and magnetic field vector $\bar{H}$ as follows.

$$\underset{X}{L} u_a = \frac{\Psi}{2} u_a \quad , \qquad \ldots (5.2)$$

and

$$\underset{X}{L} H_a = \frac{\Psi}{2} H_a \quad . \qquad \ldots (5.3)$$

We write by using the definition of Lie derivative

$$\underset{H}{L} u_a = H^b u_{a;b} + u_b H^b{}_{;a} \qquad \ldots (5.4)$$

But from the fact that $\bar{H}$ and $\bar{u}$ are orthogonal this yields

$$\underset{H}{L} u_a = H^b (u_{a;b} - u_{b;a}). \qquad \ldots (5.5)$$

By using the expression (I.2.10) of the gradient of the flow vector in terms of kinematical parameters we write the above equation after simplification as

$$\underset{H}{L} u_a = - H^b \dot{u}_b u_a + 2H^b w_{ab} \quad ,$$

i.e.,

$$\frac{\Psi}{2} u_a = - H^b \dot{u}_b u_a + 2H^b w_{ab} \; . \qquad \ldots (5.6)$$

[vide equation (5.2)]

This provides two results

$$\Psi = - 2\dot{u}_b H^b \quad , \qquad \ldots (5.7)$$

and

$$w_{ab} H^b = 0. \qquad \ldots (5.8)$$

CONCLUSION : We infer from equations (5.7) and (5.8)

i) As $\psi \neq 0$, $\dot{u}_b H^b \neq 0$.

This shows that the 4-acceleration can not be normal to magnetic field lines.

ii) The magnetic lines are always normal to the plane of rotation.

(b) TO FIND $\underset{X}{L} R_{ab} - \frac{1}{2} g_{ab} = - K \underset{X}{L} T_{ab}$:

From the expression (I.(4.1)) of the stress energy tensor of ferrofluid we write

$$\underset{X}{L} T_{ab} = \underset{X}{L} [(\rho + p + \mu H^2) u_a u_b - (p + \frac{1}{2} \mu H^2) g_{ab} - - \mu H_a H_b] \quad ,$$

i.e.,

$$\underset{X}{L} T_{ab} = [\underset{X}{L} \rho + \underset{X}{L} p + (\underset{X}{L} \mu) H^2 + \mu (\underset{X}{L} H^2)] u_a u_b +$$

$$+ (\rho + p + H^2) [u_a (\underset{X}{L} u_b) + u_b (\underset{X}{L} u_a)] -$$

$$- [\underset{X}{L} p + \frac{1}{2} \mu (\underset{X}{L} H^2) + \frac{1}{2} H^2 (\underset{X}{L} \mu)] g_{ab} -$$

$$- (p + \frac{1}{2} \mu H^2) (\underset{X}{L} g_{ab}) - (\underset{X}{L} \mu) H_a H_b -$$

$$- \mu H_b \left(\underset{X}{L} H_a \right) - \mu H_a \left(\underset{X}{L} H_b \right) . \qquad \ldots (5.9)$$

By using the conditions (i) $\underset{X}{L} u_a = \frac{\Psi}{2} u_a$,

(ii) $\underset{X}{L} g_{ab} = \Psi g_{ab}$ and (iii) $\underset{X}{L} H_a = \frac{\Psi}{2} H_a$. We can

write the above equation as

$$\underset{X}{L} T_{ab} = \Big[\underset{X}{L} \rho + \underset{X}{L} p + H^2 (\underset{X}{L} \mu) + \mu (\underset{X}{L} H^2) +$$

$$+ \Psi (\rho + p + \mu H^2) \Big] u_a u_b - \Big[\underset{X}{L} p +$$

$$+ \frac{1}{2} \mu (\underset{X}{L} H^2) + \frac{1}{2} H^2 (\underset{X}{L} \mu) +$$

$$+ \Psi (p + \frac{1}{2} \mu H^2) \Big] g_{ab} - \Big[\underset{X}{L} \mu + \Psi \mu \Big] H_a H_b .$$

$$\ldots (5.10)$$

Now to find the expression for $\underset{X}{L} R_{ab} - \frac{1}{2} R g_{ab} = - k \underset{X}{L} T_{ab}$

we separately evaluate the terms as follows.

From the definition of Ricci tensor, the expression for $\underset{X}{L} R_{ab}$ is given by (Herrera 1984).

$$\underset{X}{L} R_{ab} = \frac{1}{2} g^{cd} \Big[(\underset{X}{L} g_{cd})_{;ab} - (\underset{X}{L} g_{bd})_{;ca} -$$

$$- (\underset{X}{L} g_{ad})_{;cb} - (\underset{X}{L} g_{ab})_{;cd} \Big] , \qquad \ldots (5.11)$$

i.e.,

$$\frac{L}{X} R_{ab} = \frac{1}{2} g^{cd} [(\Psi g_{cd})_{;ab} - (\Psi g_{bd})_{;ca} -$$

$$- (\Psi g_{ad})_{;cb} - (\Psi g_{ab})_{;cd}],$$

(vide equation (5.1))

i.e., $\frac{L}{X} R_{ab} = \Psi_{;ab} + \frac{1}{2} g_{ab} \phi\Psi$,

where $\phi\Psi = g^{ab} \Psi_{;ab}$.

Hence

$$\frac{L}{X} R_{ab} = \Psi_{;ab} + \frac{1}{2} g_{ab} \phi\Psi . \qquad \ldots (5.12)$$

Also we write

$$\frac{L}{X} R = \frac{L}{X} g^{ab} R_{ab} ,$$

i.e.

$$\frac{L}{X} R = R_{ab} \frac{L}{X} g^{ab} + g^{ab} \frac{L}{X} R_{ab} . \qquad \ldots (5.13)$$

This with the result $\frac{L}{X} g^{cd} = - g^{ca} g^{db} \frac{L}{X} g_{ab}$

and equation (5.12) implies

$$\frac{L}{X} R = R_{ab} [- g^{ac} g^{db} \frac{L}{X} g_{cd} + g^{ab} [\Psi_{;ab} +$$

$$+ \frac{1}{2} g_{ab} \phi\Psi] ,$$

i.e.,

$$\underset{X}{L} R = - R^{cd} (\psi g_{cd}) + 3 \phi\psi \quad \text{,(vide equation (5.1))}$$

i.e., $\underset{X}{L} R = 3 \phi\psi - R\psi$ . ... (5.14)

Hence we find

$$\underset{X}{L} R_{ab} - \frac{1}{2} R g_{ab} = \underset{X}{L} R_{ab} - \frac{1}{2} g_{ab} \underset{X}{L} R - \frac{1}{2} R \underset{X}{L} g_{ab}$$

... (5.15)

From the equations (5.1), (5.12) and (5.15) we finally write

$$\underset{X}{L} R_{ab} - \frac{1}{2} Rg_{ab} = \psi_{;ab} - g_{ab} \phi\psi \quad . \qquad \text{... (5.16)}$$

Thus from the two expressions (5.10) and (5.16) the transformed version of Einstein field Equations $R_{ab} - \frac{1}{2} Rg_{ab} = - K T_{ab}$ under the conformal transformation (5.1) is given by

$$\begin{aligned} \psi_{;ab} - g_{ab} \phi\psi = - K \Big[& \underset{X}{L} \rho + \underset{X}{L} p + H^2 (\underset{X}{L} \mu) + \mu(\underset{X}{L} H^2) + \\ & + \psi(\rho + p + \mu H^2) \Big] u_a u_b - \\ & - \Big[\underset{X}{L} p + \frac{1}{2} \mu (\underset{X}{L} H^2) + \frac{1}{2} H^2(L \mu) + \\ & + \psi(p + \frac{1}{2} \mu H^2) \Big] g_{ab} - \\ & - [L \mu + \psi \mu] H_a H_b \quad . \end{aligned} \qquad \text{... (5.17)}$$

On transvecting this equation (both sides) with $u^a u^b$, $H^a H^b$, g^{ab} and $u^a H^b$ respectively we obtain the following equations:

$$\Psi_{;ab} u^a u^b - \phi\Psi = -K\left[\underset{X}{L} \rho + \frac{1}{2} H^2 (\underset{X}{L} \mu) + + \frac{1}{2} \mu(\underset{X}{L} H^2) + \Psi\rho + \frac{1}{2} \Psi\mu H^2\right] , \quad \ldots (5.18)$$

$$\Psi_{;ab} H^a H^b + \phi\Psi H^2 = -K\left[\underset{X}{L} p + \Psi p + \frac{1}{2} \mu(L H^2) - - \frac{1}{2} (\underset{H}{L} \mu + \Psi\mu)H^2 \right] , \quad \ldots (5.19)$$

$$-3 \phi\Psi = -K\left[\underset{X}{L} \rho - 3 \underset{X}{L} p + \Psi\rho - 3 \Psi p - \mu(\underset{X}{L} H^2)\right] , \quad \ldots (5.20)$$

$$\Psi_{;ab} u^a H^b = 0. \quad \ldots (5.21)$$

6. THE CASE OF SPECIAL CONFORMAL MOTIONS

These special types of conformal motions are described by the condition (5.1) with extra condition on the conformal potential as $\Psi_{;ab} = 0$. (Katzin et al., 1969)
For this choice the final equation (5.17) gets reduced to the form

$$\left[\underset{X}{L} \rho + \underset{X}{L} p + H^2 (\underset{X}{L} \mu) + \mu(\underset{X}{L} H^2) + \Psi (\rho + p + \mu H^2)\right]u_a u_b -$$

$$- \left[\underset{X}{L} p + \frac{1}{2} \mu (\underset{X}{L} H^2) + \frac{1}{2} H^2 (\underset{X}{L} \mu) + \psi (p + \frac{1}{2} \mu H^2) \right] g_{ab} -$$

$$- \left[\underset{X}{L} \mu + \psi \mu \right] H_a H_b = 0. \qquad \ldots (6.1)$$

THEOREM : If the space-time of ferrofluid admits the special conformal group of motion then

$$\underset{X}{L} (\rho + p) = - \psi (\rho + p) - \mu H^2.$$

PROOF : On substracting the equation (5.18) from equation (5.20) we get

$$\underset{X}{L} p + \psi p + \frac{1}{6} H^2 (\underset{X}{L} \mu + \psi \mu) + \frac{\mu}{2} (\underset{X}{L} H^2) = 0. \qquad \ldots (6.2)$$

If we add this in equation (5.19) we get

$$\underset{X}{L} \mu + \psi \mu = 0. \qquad \ldots (6.3)$$

Hence the equations (5.18) and (5.19) get reduced in the form

$$\underset{X}{L} \rho + \psi \rho + \frac{1}{2} \mu (\underset{X}{L} H^2) = 0 , \qquad \ldots (6.4)$$

$$\underset{X}{L} p + \psi p + \frac{1}{2} \mu (\underset{X}{L} H^2) = 0. \qquad \ldots (6.5)$$

Consequently these two equations provide the required result

$$\frac{L}{X}(\rho + p) = -\psi(\rho + p) - \mu H^2. \quad \ldots (6.6)$$

Hence the proof is complete.

7. THE CONSERVATION LAW GENERATORS :

If the space-time of ferrofluid admits the special conformal symmetry group then it leads to Ricci Collineation. The corresponding conservation law implied by Ricci Collineation is

$$(R^a_b \; X^b)_{;a} = 0. \quad \ldots (7.1)$$

Our aim in this article is to investigate this conservation law for selected vector fields i) $\bar{X} = N\bar{u}$, ii) $\bar{X} = \bar{H}$.

CASE (i) : For the choice $\bar{X} = N\bar{u}$.

The conservation law (7.1) for this case provides

$$(R^a_b \, N \, u^b)_{;a} = 0 \quad ,$$

i.e. $$\left[\left[(\rho + p + \mu H^2)u^a u_b - \frac{1}{2}(\rho - p + \mu H^2)\, g^a_b - \right.\right.$$
$$\left.\left. - \mu H^a H_b \right] N u^b\right]_{;a} = 0$$

[vide (I.5.3)]

i.e., $$[(\rho + 3p + \mu H^2) N u^a]_{;a} = 0,$$

i.e., $$N u^a{}_{;a}(\rho + 3p + \mu H^2) + N_{;a} u^a (\rho + 3p + \mu H^2) +$$
$$+ N u^a (\rho + 3p + \mu H^2)_{;a} = 0 \quad ,$$

i.e.,

$$(N\theta + \dot{N})(\rho + 3p + \mu H^2) + \underset{X}{L}\rho + 3\underset{X}{L}p + \underset{X}{L}(\mu H^2) = 0 \qquad \ldots (7.2)$$

$$(\because \; u^a{}_{;a} = \theta, \; N_{;a}u^a = \dot{N}, \; \underset{X}{L}\rho = \rho_{;a}(Nu^2)).$$

By using the result (3.3) this reduces to

$$\psi\rho + 3\psi p + 2\psi\mu H^2 + [\underset{X}{L}\rho + \psi\rho + \tfrac{1}{2}\mu(\underset{X}{L}H^2)] +$$

$$+ 3[\underset{X}{L}p + \psi p + \tfrac{1}{2}\mu(\underset{X}{L}H^2)] -$$

$$- \mu(\underset{X}{L}H^2) + H^2(\underset{X}{L}\mu) = 0. \qquad \ldots (7.3)$$

This equation can further be simplified by using the equations (6.4) and (6.5) as

$$\psi\rho + 3\psi p + 2\psi\mu H^2 - \mu(L H^2) + H^2(L\mu) = 0.$$

By adjusting the terms and using the earlier derived condition $\underset{X}{L}\mu + \psi\mu = 0$ we get

$$\psi\rho + 3\psi p + \psi\mu H^2 - \mu(\underset{X}{L}H^2) = 0,$$

i.e.,

$$\underset{X}{L}H^2 = -\frac{\psi}{\mu}(\rho + 3p + \mu H^2). \qquad \ldots (7.4)$$

We infer from this result that the conservation of H^2 along

$\overline{X}$ explicitly depends on the scalar potential ψ and the variable magnetic permeability μ.

CASE (ii) $\overline{X} = \overline{H}$: For this choice the conservation law (7.1) becomes

$$(R^a_b H^b)_{;a} = [(\mu H^2 + p - \rho)H^a]_{;a} = 0 \quad ,$$

i.e.,

$$(R^a_b H^b)_{;a} = (\mu H^2 + p - \rho)H^a_{;a} + H^a[(\mu_{;a})H^2 +$$

$$+ \mu(H^2_{;a}) + p_{;a} - \rho_{;a}] = 0 \quad ,$$

i.e.,

$$(R^a_b H^b)_{;a} = (\mu H^2 + p - \rho)2\psi + (\underset{\overline{H}}{L}\mu)H^2 +$$

$$+ \mu(\underset{\overline{H}}{L} H^2) + \underset{\overline{H}}{L} p - \underset{\overline{H}}{L}\rho = 0. \quad \ldots (7.5)$$

($\because$ vide equation (3.17))

By using equations (6.3), (6.4) and (6.5) and simplifying equation (7.5) becomes

$$(\mu H^2 + p - \rho) + \mu(\underset{\overline{H}}{L} H^2) = 0.$$

Hence we can put a claim

$$\underset{\overline{H}}{L} H^2 = 0 \iff \rho - p - \mu H^2 = 0.$$