CHAPTER ONE

Introduction

"In the intellectual life of man there can be clearly two domains, the domain of action, of shaping and construction to which active artists, scientists, technicians and statesmen devote themselves, and a domain of reflection of which the fulfilment lies in insight, and which since we struggle in it to find the meaning of our activity, is to be regarded as the proper domain of the philosopher."

... H.WEYL

CHAPTER - I

INTRODUCTION

1.1 Fractional Calculus

The two most basic operations of Mathematics are differentiation and integration. One can expect much from the natural extension of these operations. The fractional calculus has its origin in the question of the extension of meaning. In generalized integration and differentiation the question of extension of meaning is : Can the meaning of derivatives of integral order $d^n y/ dx^n$ be extended to have meaning where n is any number .... irrational, fractional, or complex ? The concept of differentiation and integration to noninteger order is by no means new. Interest in this subject was evident almost as soon as the ideas of the classical calculus were known. The subject is old, dating back atleast to Leibnitz in its theory and to Heaviside in its application. But the application of these ideas has not yet been fully exposed.

Leibnitz [19] in 1695 replied, 'It will lead to a paradox', adding prophetically 'from which one day useful consequences will be drawn', to L'Hospital's letter asking about 'What if n be 1/2 ?' In 1697, Leibnitz, referring to Wallis's infinite product for $\pi/2$, used the notation $d^{1/2}y$ and stated that differential calculus might have used to achieve the same result.

In 1819 the first mention of a derivative of arbitrary order appears in a text. The French mathematician, S.F. Lacroix [37], in his 700 page text on differential and integral calculus, has devoted less than two pages to fractional calculus. He develops a formula for fractional differentiation for the nth order derivative of x^m by induction. Then, he formally replaces n with the fraction $\frac{1}{2}$, and together with the fact that $\Gamma(1/2) = \sqrt{\pi}$, he obtains

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(x) = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

The systematic studies seem to have been made in the beginning and middle of the 19th century by Liouville [40], Riemann [54] and Holmgren [31] although Euler [20], Lagrange [38], and others made contributions even earlier.

Abel [1] was probably the first to give an application of fractional calculus. He used derivatives of arbitrary order to solve the tautochrone problem. This problem, sometimes called the isochrone problem, is that of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire slides to the lowest point of the wire in the same time regardless of where the bead is placed. The brachistochrone problem deals with the shortest time of slide. The integral he worked with

$$\int_0^x (x - t)^{-\frac{1}{2}} f(t)dt$$

is precisely of the same form that Riemann used to define fractional operations.

It was Liouvile [40] who expanded functions in series of exponentials and defined the qth derivative of such a series by operating term-by-term as though q were a positive integer. Riemann [54] proposed a different definition that involved a definite integral and was applicable to power series with noninteger exponents. Grunwald [24] first unified the results of Liouville and Riemann. Krug [35], working through Cauchy's integral formula for ordinary derivatives, showed that Riemann's definite integral had to be interpreted as having a finite lower limit while Liouville's definition, in which no distinguishable lower limit appeared, correspond to a lower limit $-\infty$.

Notable contributions, in the present century, have been made to both the theory and application of the fractional calculus. Weyl [68], Hardy [26], Hardy and Littlewood [27,28], Kober [33], and Kuttner [36] examined some rather special, but natural, properties of fractional operators of functions belonging to Lebesgue and Lipschitz classes. Erdelyi [13, 14, 15] and Osler [51] have given definitions of fractional operators with respect to arbitrary functions, and Post [52] used quotients to define generalized differentiation. Riesz [55] has developed a theory of fractional integration for functions of more than one variable. Erdelyi [18, 19] has applied the fractional calculus to integral equations and Higgins [30] has used fractional integral operators to solve differential equations. Prabhakar [53] studied some integral equations containing hypergeometric functions in two variables with the help of fractional integration.

1.11 A fractional integral is a straightforward generalization of the elementary concept of a repeated integral. If the function f(x) is integrable in any interval (o,a) where $a > 0$ we define the first integral $F_1(x)$ of f(x) by the formula

$$F_1(x) = \int_0^x f(t)\, dt$$

and the subsequent integrals by the recursion formula

$$F_{r+1}(x) = \int_0^x F_r(t)dt, \; r = 1, 2, \ldots..$$

It can easily be proved by induction that for any positive integer n

$$(1.1) \qquad F_{n+1}(x) = \frac{1}{n!} \int_0^x (x - t)^n f(t)\, dt.$$

Similarly we could define an indefinite integral $G_r(x)$ by the formulae

$$G_1(x) = - \int_x^\infty f(t)dt, \quad G_{r+1}(x) = - \int_x^\infty G_r(t)dt$$

$r = 1, 2, \ldots.$ and show by induction that for any positive integer n

$$(1.2) \qquad G_{n+1}(x) = \frac{1}{n!} \int_x^\infty (t - x)^n f(t)\, dt,$$

provided that $f(x)$ is of such a nature that the integral exists.

The earliest generalization of the integral on the right-hand side of equation (1.1) would appear to be the Riemann-Liouville fractional integral of order α defined for $R(\alpha) > 0$ by

$$(1.3) \qquad R_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)\,(x-t)^{\alpha-1}\, dt.$$

The upper limit of integration x may be real or complex; in the latter case the path of integration is the straight segment $t = xs$, $0 \leq s \leq 1$. Integrals of this type arise in the theory of linear ordinary differential equations where they are called Euler transforms of the first kind.

Hardy and Littlewood [28] consider the fractional integral

$$(1.4) \qquad f_\alpha(x) = \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}\, dt, \quad (0 < R(\alpha) < 1),$$

while Love and Young [41] consider the integral

$$(1.5) \qquad f_\alpha^{+}(a,x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} f(t)(x-t)^{\alpha-1}\, dt, \quad (a \leq x \leq b,\ R(\alpha) > 0),$$

$f(x)$ being integrable in (a,b); Zygmund [71] discusses the same integral but denotes it by $F_\alpha(x)$.

The Weyl fractional integral is a generalization of the integral on the right hand side of equation (1.2); it is defined by the equation (Weyl [68])

$$(1.6) \qquad W_\alpha\left\{f(t);x\right\} = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1} f(t)\,dt, \quad R(\alpha) > 0.$$

In general x and α are complex, the path of integration being one of the rays $t = xs$ $(s > 0)$, or $t = x + s$ $(s > 0)$. When they occur in the theory of linear ordinary differential equations, fractional integrals of this kind are called Euler transforms of the second kind.

A fractional integral

$$(1.7) \qquad f_\alpha^{-}(x,b) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} f(t)(t-x)^{\alpha-1}\, dt, \quad R(\alpha) > 0$$

closely related to Weyl's has been introduced by Love and Young [41].

The fundamental theorems on fractional integrals and derivatives as given by Weyl and Hardy and Littlewood were extended by Kober [33] over a wider range. He studied the applications of these operators for Mellin and Hankel transforms. By introducing complex parameter η he dealt with the operators

$$(1.8) \qquad I^{+}_{\eta,\alpha} f = \frac{1}{\Gamma(\alpha)} z^{-\eta-\alpha} \int_0^z (z-t)^{\alpha-1}\, t^{\eta}\, f(t)\, dt,$$

$$(1.9) \qquad K^{-}_{\eta,\alpha} f = \frac{1}{\Gamma(\alpha)} z^{\eta} \int_z^{\infty} (t-z)^{\alpha-1}\, t^{-\eta-\alpha}\, f(t)\, dt,$$

$$(1.10) \qquad I^{-}_{\eta,\alpha} f = \frac{z^{-\eta-\alpha}}{\Gamma(\alpha)} \int_z^{\infty} (t-z)^{\alpha-1}\, t^{\eta}\, f(t)\, dt,$$

$$(1.11) \qquad K^{+}_{\eta,\alpha} f = \frac{1}{\Gamma(\alpha)} z^{\eta} \int_0^z (z-t)^{\alpha-1}\, t^{-\eta-\alpha}\, f(t)\, dt.$$

Erdelyi [14, 15] further generalized these operators and discussed in detail the importance of these in the theory of Mellin and Hankel transformations. For $A > 0$, the fractional integral operators with respect to x^A are defined by

$$(1.12) \qquad I^{\eta,\alpha}_{x^A} f(x) = \frac{1}{\Gamma(\alpha)} x^{-A\eta-A\alpha} \int_0^x (x^A - t^A)^{\alpha-1}\, t^{A\eta} f(t)\, d(t^A),$$

$$(1.13) \qquad K^{\eta,\alpha}_{x^A} f(x) = \frac{1}{\Gamma(\alpha)} x^{A\eta} \int_x^{\infty} (t^A - x^A)^{\alpha-1}\, t^{-A\eta-A\alpha}\, f(t)\, d(t^A).$$

We know [66, Th. 44, P. 60] that if $f \in L(0,\infty)$, $g \in L(0,\infty)$, then $(f*g)(x) \in L(0,\infty)$, where

$$(1.14) \qquad (f*g)(x) = \int_0^\infty f(x/u)\, g(u)\, du/u.$$

Hence the set $L(0,\infty)$ of complex valued functions forms an algebra over the field of complex numbers with the usual definitions of addition and scalar multiplication and the convolution (1.14) as the product. It is easy to see that the algebra is commutative and associative.

Buschman [12, P.101] has introduced

$$(1.15) \qquad I^{\eta,\alpha,A}(x) = \frac{A}{\Gamma(\alpha)} (x^A-1)^{\alpha-1}\, x^{-A\eta-A\alpha}\, U(x-1),$$

and

$$(1.16) \qquad K^{\eta,\alpha,A}(x) = \frac{A}{\Gamma(\alpha)} (1-x^A)^{\alpha-1}\, x^{A\eta}\, U(1-x)$$

where $U(x)$ is the Heaviside unit function

$$U(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x > 0 \end{cases}.$$

He developed some additional identities by showing the connection between these operators and algebra of functions which has Mellin convolution as the product. He proves

$$(1.17) \qquad I^{\eta,\alpha}_{x^A} f(x) = (I^{\eta,\alpha,A} * f)(x),$$

$$(1.18) \qquad K^{\eta,\alpha}_{x^A} f(x) = (K^{\eta,\alpha,A} * f)(x),$$

Where $I^{\eta,\alpha,A}(x)$ belongs to $L(0,\infty)$ for $R(\alpha) > 0$, $R(\eta) > 1/A - 1$ and $K^{\eta,\alpha,A}(x)$ belongs to $L(0,\infty)$ for $R(\alpha) > 0$ and $R(\eta) > -1/A$. He has further pointed out that

$$(1.19) \qquad M\left[I^{\eta,\alpha,A}(x)\right] = \frac{\Gamma(1+\eta - s/A)}{\Gamma(1+\eta+\alpha - s/A)}$$

$$R(\alpha) > 0, \quad R(s/A) < 1 + R(\eta)$$

and

$$(1.20) \qquad M\left[K^{\eta,\alpha,A}(x)\right] = \frac{\Gamma(\eta + s/A)}{\Gamma(\eta+\alpha+s/A)}$$

$$R(\alpha) > 0, \; R(s/A) > -R(\eta).$$

Finally, he used these facts to reduce a pair of integral equations to a single integral equation.

Kesarwani [32] extended the earlier work of Bushman to solve certain dual integral equations. Sueddon [63] modified the Erdelyi-Kober operators and applied them to solve certain dual integral equations. Mourya [46] has developed fractional integrals for the functions of two variables on the lines of Erdelyi and Kober and discussed some of their fundamental properties and simple identities. The algebra of these operators has been developed by Koranne [34] and used in the solutions of certain dual integral equations of functions of two variables, Lowndes [42] introduced the generalization of Erdelyi-Kober operators (1.8) to (1.11) by using Bessel functions. Saxena and Kumbhat [58] also gave the generalizations of these operators by utilizing generalized hypergeometric functions.

1.12 Since an indefinite integral is probably the simplest integral transformation of all, no apology is needed for using this as the starting point for a theory of fractional calculus, another topic which has sprung to life in recent years with the publications of Oldham and Spanier [50] and Ross [56]. We are well accustomed to the use of the notation $\frac{d^n f}{dx^n}$ for the nth derivative of a function f with respect to x when n is a nonnegative integer. Because integration and differentiation are inverse operations it is natural to associate the symbol $\frac{d^{-1} f}{[dx]^{-1}}$ with indefinite integration of f with respect to x. Hence we may define

$$(1.21) \qquad \frac{d^{-1} f}{[dx]^{-1}} = \int_0^x f(y)dy \text{ or } \frac{d^{-1} f}{[d(x-a)]^{-1}} \equiv \int_a^x f(y)dy.$$

In general the multiple integration with lower limit a is the natural extension

$$(1.22) \qquad \frac{d^{-n} f}{[d(x-a)]^{-n}} \equiv \int_a^x dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \cdots\cdots$$

$$\cdots\cdot \int_a^{x_2} dx_1 \int_a^{x_1} f(x_o)\, dx_o \,.$$

This n-fold integration of f with respect to x may be symbolized as

$$(1.23) \qquad f^{(-n)} \equiv \int_{a_n}^{x} dx_{n-1} \int_{a_{n-1}}^{x_{n-1}} dx_{n-2} \cdots\cdots \int_{a_2}^{x_2} dx_1 \int_{a_1}^{x_1} f(x_o)\, dx_o,$$

where $a_1, a_2, \ldots, a_n$ are completely arbitrary.

Using the definition of the first derivative in terms of a backward difference and repeating the process we can have

$$(1.24) \qquad \frac{d^n f}{[dx]^n} \equiv \lim_{N\to\infty} \left\{ \left[\frac{x-a}{N}\right]^{-n} \sum_{j=0}^{N-1} (-1)^j \binom{n}{j} f\left(x - j\left(\frac{x-a}{N}\right)\right) \right\}$$

Also using the definition of an integral as a limit of a Riemann Sum, we get

$$(1.25) \qquad \frac{d^{-n} f}{[d(x-a)]^{-n}} \equiv \lim_{N\to\infty} \left\{ \left[\frac{x-a}{N}\right]^{n} \sum_{j=0}^{N} \binom{j+n-1}{j} f\left(x - j\left(\frac{x-a}{N}\right)\right) \right\}.$$

Now comparing the above two formulas and recalling

$$(-1)^j \binom{n}{j} = \binom{j-n-1}{j} = \frac{\Gamma(j-n)}{\Gamma(-n)\,\Gamma(j+1)},$$

we have

$$(1.26)\qquad \frac{d^q f}{\left[d(x-a)\right]^q} \equiv \lim_{N\to\infty}\left[\frac{\left[\frac{x-a}{N}\right]^q}{\Gamma(-q)} \sum_{r=0}^{N-1} \frac{\Gamma(r-q)}{\Gamma(r+1)} f\left(x - r\left[\frac{x-a}{N}\right]\right)\right]$$

where q is any integer of either sign. A new term 'differintegrals' has been coined by Oldham and Spanier [50] to avoid the cumbersome alternate 'derivatives or integrals to arbitrary order'. They have defined the differenintegral of order q by the formula

$$(1.27)\qquad \frac{d^q f}{\left[d(x-a)\right]^q} = \lim_{N\to\infty}\left[\frac{\left[\frac{x-a}{N}\right]^{-q}}{\Gamma(-q)} \sum_{r=0}^{N-1} \frac{\Gamma(r-q)}{\Gamma(r+1)} f\left(x - r\left[\frac{x-a}{N}\right]\right)\right]$$

where q is arbitrary, which was first given by Grünwald [24]. Note that this definition involves only evaluations of the function itself; no explicit use is made of derivatives or integrals of f. They have further established that

$$(1.28)\qquad \frac{d^n}{dx^n}\left[\frac{d^q f}{\left[d(x-a)\right]^{n+q}}\right] = \frac{d^{n+q} f}{\left[d(x-a)\right]^{n+q}}$$

for all positive integers n and all q.

Consider the formula

$$(1.29) \quad \frac{d^{-1}f}{[d(x-a)]^{-1}} = \int_a^x f(y)dy = \frac{1}{n!}\frac{d^n}{dx^n}\int_a^x (x-y)^n f(y)dy,$$

$$n = 0, 1, 2, \ldots.$$

A single integration of (1.29) for n = 1 produces

$$\frac{d^{-2}f}{[d(x-a)]^{-2}} = \int_a^x dx_1 \int_a^{x_1} f(x_0)dx_0 = \int_a^x (x-y)\, f(y)\, dy,$$

and an (n-1) - fold integration produces Cauchy's formula for repeated integration.

$$(1.30) \quad \frac{d^{-n}f}{[d(x-a)]^{-n}} = \int_a^x dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \cdots \int_a^{x_1} f(x_0)dx_0 =$$

$$= \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1}\, f(y)dy.$$

Thus an iterated integral may be expressed as a weighted single integral with a very simple weight function, a fact that provides an important clue for generalizations involving noninteger orders. By replacing -n by q we get the Riemann-Liouville fractional integral

$$(1.31) \quad \left[\frac{d^q f}{d(x-a)^q}\right]_{R-L} = \frac{1}{\Gamma(-q)}\int_a^x (x-y)^{-q-1}\, f(y)dy, q < 0.$$

By establishing the identities

$$(1.32) \quad \left[\frac{d^q f}{d(x-a)^q}\right]_{R-L} = \sum_{K=0}^{n-1} \frac{(x-a)^{-q+K}\, f^{(K)}(a)}{\Gamma(-q+K+1)}$$

...

$$+ \left[\frac{d^{q-n} f^{(n)}}{d(x-a)^{q-n}} \right]_{R-L}$$

which provides an analytic continuation of the formula (1.31) for $R(q) < n$ and

$$(1.33) \quad \left[\frac{d^q f}{d(x-a)^q} \right]_{R-L} \equiv \frac{d^n}{dx^n} \left[\frac{d^{q-n} f}{d(x-a)^{q-n}} \right]_{R-L}$$

where d^n/dx^n effects ordinary n-fold differentiation and n is an integer chosen so large that $q-n < 0$, Oldham and Spanier [50] proved that the operators (1.27) and (1.31) coincides for all functions f.

Riemann [54] considered power series with noninteger exponents to be extensions of Taylor's series and built up a generalized derivative for such functions by use of the formula

$$(1.34) \qquad \frac{d^q x^p}{dx^q} = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} x^{p-q},$$

this being an obvious generalization of the formula

$$\frac{d^n x^p}{dx^n} = P(P-1)(P-2) \ldots (P-n+1)x^{P-n}$$
$$= \frac{\Gamma(P+1)}{\Gamma(P-n+1)} x^{p-n}$$

for n a nonnegative integer. This is similar to the approach taken by Scott Blair [60], Heaviside [29], and others.

Liouville [40] defined a generalized derivative for functions expansible as a series of exponentials,

$$f = \sum c_j \exp (b_j x), \text{ by}$$

$$(1.35) \qquad \frac{d^q f}{dx^q} \equiv \sum_{j=0}^{\infty} c_j b_j^q \exp (b_j x),$$ which leads to a different operator than those of (1.27).

Erdelyi [18] defined a qth order differintegral of a function f(z) with respect to the function z^n by

$$(1.36) \qquad \frac{d^q f}{\left[d(z^n - a^n)\right]^q} \equiv \frac{1}{\Gamma(-q)} \int_a^z \frac{f(t) n t^{n-1}\, dt}{\left[z^n - t^n\right]^{1+q}}$$

Oslar [51] has extended Erdelyi's work by defining differintegral of a function f(z) with respect to an arbitrary function g(z) by considering the Riemann-Liouville integral

$$(1.37) \qquad \frac{d^q f}{\left[d(g(z) - g(a))\right]^q} \equiv \frac{1}{\Gamma(-q)} \int_a^z \frac{f(t) g^{(1)}(t)\, dt}{\left[g(z) - g(t)\right]^{q+1}},$$

where a is chosen to give $g(a) = 0$, that is $a = g^{-1}(0)$. Upon setting $g(z) = z-a$, one obtains the Riemann-Liouville Integral once again. Certain choices of g have been shown by Erdelyi and Osler which lead to a number of formulas of interest in classical analysis.

1.13 The mathematical problem of defining fractional integration and differentiation :

The symbols

$$ {}_{c}D_{x}^{-v} \quad f(x), \quad v \geqslant 0, $$

invented by Harold T. Davis, denote integration of arbitrary order along x-axis. The subscripts c and x denote the limits of integration of a definite integral which defines fractional integration.

For every function $f(z)$, $z = x + iy$, of a sufficiently wide class, and every number v, irrational, fractional, or complex, a function ${}_{c}D_{z}^{v} f(z) = g(z)$, or ${}_{c}D_{x}^{v} f(x) = g(x)$ when z is purely real, should be assigned subject to the following criteria :

(1) If $f(z)$ is an analytic function of the complex variable z, the derivative ${}_{c}D_{z}^{v}$ is an analytic function of v and z.

(2) The operation ${}_{c}D_{x}^{v} f(x)$ must produce the same result as ordinary differentiation when v is a positive integer. If v is a negative integer, say $v = -n$, then ${}_{c}D_{x}^{-n} f(x)$ must produce the same result as ordinary n-fold integration and ${}_{c}D_{x}^{-n}$ must vanish along with its n-1 derivatives at $x = c$.

(3) The operation of order zero leaves the function unchanged :

$$ {}_cD_x^0 \; f(x) = f(x) $$

(4) The fractional operators must be linear :

$$ {}_cD_x^{-v}\left[af(x) + bg(x)\right] = a\,{}_cD_x^{-v} \; f(x) + b\,{}_cD_x^{-v} \; g(x). $$

(5) The law of exponents for integration of arbitrary order holds :

$$ {}_cD_x^{-u} \; {}_cD_x^{-v} \; f(x) = {}_cD_x^{-u-v} \; f(x). $$

A definition which fulfills these criteria named in honour of Riemann and Liouville is

$$ (1.38) \qquad {}_cD_x^{-v} \; f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} \; f(t)\; dt. $$

This definition can be obtained in atleast four different ways. The definition (1.38) is for integration of arbitrary order. For differentiation of arbitrary order it cannot be used directly. However, by means of a simple trick, we can find a convergent expression. Let $v = m - p$ where for convenience m is the least integer greater than v, and $0 \leqslant p \leq 1$. Then for differentiation of arbitrary order we have

$$(1.39) \qquad {}_cD_x^v f(x) = {}_cD_x^m \; {}_cD_x^{-p} f(x)$$

$$= \frac{d^m}{dx^m} \cdot \frac{1}{\Gamma(p)} \int_c^x (x-t)^{p-1} f(t)dt,$$

where we take advantage of the knowledge that ${}_cD_x^m$ is an ordinary mth derivative operator d^m/dx^m. The question of extending the definition (1.38) for integration of arbitrary order to differentiation of arbitrary order is answered by letting v be real and greater than zero we have

$$(1.40) \qquad g(v, x) = {}_0D_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt$$

which is in general convergent for $v > 0$. For any v we can write

$$h(v, x) = {}_0D_x^{-v} f(x) = {}_0D_x^m \; {}_0D_x^{-p} f(x)$$

$$= \frac{d^m}{dx^m} \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} f(t)\, dt,$$

where $-v = m - p$, $m = 0, 1, 2 \ldots.$

When $V > 0$ choose $m = 0$, thus $v = p$ and $g = h$. Now (1.40) can be written

$$g(v, x) = \frac{d}{dx} \int_0^x \left[\frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt \right] dx.$$

By Dirichlet's formula, we have

$$g(v, x) = \frac{d}{dx} \frac{1}{\Gamma(v+1)} \int_0^x (x-t)^v f(t)\, dt$$

which is convergent for $v > -1$. We then have $g(v,x) =$
$= h(v,x)$ for $m = 1$. This process can be repeated for $v > -n$, n a positive integer. Now g is analytic in R_1 where $v > 0$ and h is analytic in R_2 for $v > -n$. Since $g = h$ on a set of points in $R_1 \cap R_2$ with a limit point in the right half plane, then h is the analytic continuation of g.

1.14 No claim can be made that the fractional calculus approach is better than some other approach. However, there is a succinctness of notation and simplicity of formulation in the fractional calculus that might suggest a solution to a complicated functional equation that is not readily obtained by other means.

Fractional calculus can be categorized as applicable mathematics. The properties and theory of these fractional operators are proper objects of study in their own right. Scientists and applied mathematicians, in the last decade, found the fractional calculus useful in various fields. Within mathematics, the subject makes a contact with a very large segment of classical analysis and provides a unifying theme for a great many known, and some new, results. Applications outside mathematics include such otherwise unrelated topics as : transmission line theory, quantitive biology, electro-chemistry, scattering theory, diffusion, and dissemination of atmospheric pollutants. Virtually no area of classical analysis has been left untouchaed by the fractional calculus.

1.2 Generalizations of the Hankel Transform

The Well-known Hankel transform denoted by H_ν , is defined by the integral equation

$$(1.41) \qquad g(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\nu(xy) f(y)dy.$$

where $0 < x < \infty$, ν is a real number and J_ν is the Bessel function of first kind of order ν. Functions which are their own Hankel transforms i.e., solutions of the integral equation

$$(1.42) \qquad f(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\nu(xy) f(y)dy$$

have been called self-reciprocal in Hankel transform of order ν.

According to Titchmarsh [66, P. 252] we shall say that $f(x)$ belongs to the class $A(\alpha,a)$, where $0 < \alpha \leq \pi$, $a < \frac{1}{2}$, if (i) $f(x)$ is an analytic function of $x = re^{i\theta}$ regular in the angle defined by $r > 0$, $|\theta| < \alpha$, and (ii) $f(x)$ is $O(|x|^{-a-\epsilon})$ for small x, and $O(|x|^{a-1+\epsilon})$ for large x, for every positive ϵ and uniformly in any angle $|\theta| \leq \alpha - \eta < \alpha$.

Generalizations of the Hankel transform (1.41), have been given from time to time by various mathematicians like Agrawal [2, P. 164], Bhatanagar [5], Mehra [43], R. Narain [47, 48], Bhise [7] and several others. Sharma [61] has established and studied symmetrical fourier kernel interms of Fox's H - function [23, P. 408] as

$$(1.43) \quad 2\beta\gamma\, x^{\gamma-1}\, H^{q,p}_{2p,2q}\left[\beta^2 x^{2\gamma}\left|\begin{matrix}(a_p, \alpha_p), (1-a_p-\alpha_p, \alpha_p)\\ (b_q, \beta_q), (1-b_q-\beta_q, \beta_q)\end{matrix}\right.\right]$$

where β and γ are real constants. With the help of (1.43) a new reciprocal transform may be introduced in the form

$$(1.44) \quad g(x) = 2\beta\gamma\, x$$

$$\times \int_0^\infty (xy)^{\gamma-\frac{1}{2}}\, H^{q,p}_{2p,2q}\left[\beta^2 (xy)^{2\gamma}\left|\begin{matrix}(a_p, \alpha_p), (1-a_p-\alpha_p, \alpha_p)\\ (b_q, \beta_q), (1-b_q-\beta_q, \beta_q)\end{matrix}\right.\right] f(y)dy,$$

which by applying known identity

$$(1.45) \quad x^k\, H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_p, \alpha_p)\\ (b_q, \beta_q)\end{matrix}\right.\right] \equiv H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_p + K.\alpha_p, \alpha_p)\\ (b_q + K.\beta_q, \beta_q)\end{matrix}\right.\right]$$

may be put in the form

$$(1.46) \quad g(x) = 2\,\gamma\,\beta^{1/2\gamma}\, x$$

$$\times \int_0^\infty H^{q,p}_{2p,2q}\left[\beta^2 (xy)^{2\gamma}\left|\begin{matrix}(a_p + \frac{2\gamma-1}{4\gamma}\alpha_p, \alpha_p), (1-a_p - \frac{2\gamma+1}{4\gamma}\alpha_p, \alpha_p)\\ (b_q + \frac{2\gamma-1}{4\gamma}\beta_q, \beta_q), (1-b_q - \frac{2\gamma+1}{4\gamma}\beta_q, \beta_q)\end{matrix}\right.\right] \times$$

$\times \; f(y)dy$.

We shall denote the generalized Hankel transform (1.44) or (1.46) symbolically by $g(x) = H_T\left[f(y);\ x;\ (a_p, \alpha_p);\ (b_q, \beta_q)\right]$. If the functions f(x) and g(x) satisfy the integral equation (1.44) or (1.46) we shall call g(x) to be $H_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$- transform of f(x) and f(x) as $H_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$ - transform of g(x). The functions f(x) and g(x) shall be called as a pair of $H_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$ - transforms. If $f(x) \equiv g(x)$, it shall be called as self-reciprocal in $H_T\left[(a_p,\alpha_p);\ (b_q,\beta_q)\right]$ transform and we shall say that f(x) is $R_T\left[(a_p, \alpha_p);\ (b_q, \beta_q)\right]$.

The Kernel in the transform defined in (1.46) is very general and yields as particular cases a number of known generalizations of Hankel transform (1.41) as follows :

(i) Taking $\alpha_p = \beta_q = 1$, the result (1.46) reduces to [61, P.3]

(1.47) $$g(x) = 2\,\gamma\beta^{\,1/2\gamma} \times$$

$$\times \int_0^\infty G^{q,p}_{2p,2q}\left[\beta^2\,(xy)^{2\gamma}\,\middle|\,\begin{matrix}\left(\dfrac{2\gamma-1}{4\gamma}+a_p\right),\ \left(\dfrac{2\gamma-1}{4\gamma}-a_p\right)\\ \left(\dfrac{2\gamma-1}{4\gamma}+b_q\right),\ \left(\dfrac{2\gamma-1}{4\gamma}-b_q\right)\end{matrix}\right] f(y)dy.$$

We shall denote the generalized Hankel transform (1.47) symbolically by

$$g(x) = G_T\left[f(y);\ x\ ;\ a_p, b_q\right]$$

(ii) With $\beta = \frac{1}{2}$, $\alpha = 1$, (1.47) reduces to a generalization given by R. Narain [48, P. 298]

$$(1.48)\quad g(x) = \int_0^\infty (xy)^{\frac{1}{2}}\, G^{q,p}_{2p,2q}\left[\left(\frac{xy}{2}\right)^2 \middle|\ \begin{matrix} a_p, & -a_p \\ b_q, & -b_q \end{matrix} \right] f(y)dy.$$

(iii) On having $\beta = \frac{1}{2}$, $\alpha = 1$, $p = 1$, $q = 2$,

$a_1 = k - m - \frac{\nu}{2} - \frac{1}{2}$, $b_1 = \frac{\nu}{2}$ and $b_2 = \frac{\nu}{2} + 2m$,

(1.47) yields another generalization by R.Narain [47,P.270]

$$(1.49)\ g(x) = 2^{-\nu} \int_0^\infty (xy)^{\frac{1}{2}+\nu}\, \chi_{\nu,k,m}\left(\frac{x^2 y^2}{4}\right) f(y)dy,$$

known as $\chi_{\nu,k,m}$ - transform, where

$$\chi_{\nu,k,m}(x) \equiv x^{-\nu}\ G^{2,1}_{2,4}\left[x \middle|\ \begin{matrix} k-m-\frac{1}{2}, & \nu-k+m+\frac{1}{2} \\ \nu, & \nu+2m, & -2m, & 0 \end{matrix} \right].$$

A function, self-reciprocal in (1.49) has been denoted by $R_\nu(K, m)$.

(iv) Taking $\alpha = 1$, $\beta = 2^{-n}$, where n is a positive integer and giving suitable values to the parameters (1.47) reduces to the generalizations due to Bhatanagar [5,P.176]

$$(1.50) \quad g(x) = \int_0^\infty \tilde{w}_{\mu_1,\ldots,\mu_n}(xy) f(y)\,dy,$$

$$(\mu_1, \ldots, \mu_n > -\tfrac{1}{2}),$$

where $\tilde{w}_{\mu_1, \ldots, \mu_n}(x)$ is defined as

$$\tilde{w}_{\mu_1,\ldots,\mu_n}(x) \equiv 2^{1-n} x^{\frac{1}{2}} \times$$

$$\times\, G^{n,0}_{0,2n}\left[\left(\frac{x}{2^n}\right)^2 \middle| \frac{\mu_1}{2}, \ldots, \frac{\mu_n}{2}, -\frac{\mu_1}{2}, \ldots, -\frac{\mu_n}{2} \right].$$

A function self-reciprocal in (1.50) will be represented by $R_{\mu_1, \ldots, \mu_n}$

(v) With n = 2, (1.50) reduces to

$$(1.51) \quad g(x) = \int_0^\infty \tilde{w}_{\mu,\nu}(xy) f(y)\,dy, \quad (\mu, \nu > -\tfrac{1}{2}),$$

where $\tilde{w}_{\mu,\nu}(x)$ is Watson's Kernel [67, P.298] defined

$$\tilde{w}_{\mu,\nu}(x) \equiv \tfrac{1}{2} x^{\frac{1}{2}} G^{2,0}_{0,4}\left[\left(\frac{x}{4}\right)^2 \middle| \frac{\mu}{2}, \frac{\nu}{2}, -\frac{\mu}{2}, -\frac{\nu}{2} \right].$$

A function self-reciprocal in (1.51) will be denoted by $R_{\mu,\nu}$.

(vi) On having $\beta = \frac{1}{2}$, $\alpha = 1$, p = 1, q = 2, $a_1 = k - m - \nu/2$, $b_1 = \nu/2$ and $b_2 = \nu/2 + 2m$, (1.47) yields a generalization by Mehra [43]

$$(1.59) \quad g(x) = 2^{-\nu} \int_0^\infty (xy)^{\frac{1}{2}+\nu} \chi_{\nu,k+\frac{1}{2},m}\left(\frac{x^2y^2}{4}\right) f(y)\,dy.$$

(vii) Putting $\gamma = \frac{1}{2}$, $p = 1$, $q = 2$, $\alpha_1 = \beta_1 = \beta_2 = 1$, $\beta = 1$, and $a_1 = k - m - \nu/2 - \frac{1}{2}$, $b_1 = \nu/2$, $b_2 = \nu/2 + 2m$,

(1.46) reduces to a modified form of $\chi_{\nu,k,m}$ transform introduced by Bhise [6, p. 198]

$$(1.53) \quad g(x) = \int_0^\infty G^{2,1}_{2,4}\left[xy \left| \begin{matrix} k-m-\nu/2-\frac{1}{2}, & -k+m+\nu/2+\frac{1}{2} \\ \nu/2, \; \nu/2+2m, & -\nu/2, \; -\nu/2-2m \end{matrix} \right. \right] \times$$

$$\times f(y)dy$$

(viii) Having $\gamma = 1$, $\beta = \frac{1}{2}$, $p = 0$, $q = 1$, $b_1 = \nu/2$ and using the identity [3, p. 216(3)]

$$(1.54) \quad G^{1,0}_{0,2}\left[x \left| \begin{matrix} - \\ a, b \end{matrix} \right. \right] \equiv x^{\frac{1}{2}(a+b)} J_{a-b}(2x^{\frac{1}{2}}),$$

(1,47) reduces to (1.41).

(ix) With $\gamma = 2$, and $\beta = 2^{-4}$, $p = 0$, $q = 2$, $\beta_1 = \beta_2 = 1$ and using the identity (1.45), we have

$$(1.55) \quad g(x) = \frac{1}{4} \int_0^\infty (xy)^{3/2} G^{2,0}_{0,4}\left[\left(\frac{xy}{4}\right)^4 \left| -\frac{3}{8}, -\frac{1}{8}, \frac{3}{8}, \frac{1}{8} \right. \right] f(y)dy,$$

and

$$(1.56) \quad g(x) = \frac{1}{4} \int_0^\infty (xy)^{3/2} G^{2,0}_{0,4}\left[\left(-\frac{xy}{4}\right)^4 \left| \frac{1}{8}, \frac{3}{8}, -\frac{1}{8}, -\frac{3}{8} \right. \right] f(y)dy$$

where the Kernels used are due to Guinand [25, p. 192]

(x) On putting $\beta = (\frac{1}{2}x^{-1})^{2k}$, $\gamma = k$ and choosing the parameters suitably in (1.47), we have

$$(1.57)\quad g(x) = \int_0^{\infty} (xy)^{\frac{1}{2}} J_{o,k}(xy) f(y)dy$$

and

$$(1.58)\quad g(x) = \int_0^{\infty} (xy)^{\frac{1}{2}} J_{\frac{1}{2},k}(xy) f(y)dy,$$

where the Kernels used are due to Everitt [21, p.271] and given in terms of Meijer's G - function as

$$x^{\frac{1}{2}} J_{o,k}(x) = (2k)^{\frac{1}{2}} (\tfrac{1}{2}x/k)^{k-\frac{1}{2}} \quad x$$

$$x\ G^{k,0}_{0,2k}\left[\left(\frac{x}{2k}\right)^{2k} \middle|\ \begin{matrix} - & - & - & - \\ 0,\ 1/2k,\ \ldots, & \frac{k-1}{2k}, 0, & -1/2k, \ldots & -\frac{k-1}{2k} \end{matrix}\right]$$

and

$$x^{\frac{1}{2}} J_{\frac{1}{2},k}(x) = (2k)^{\frac{1}{2}} (\tfrac{1}{2}x/k)^{k-1} \quad x$$

$$x\ G^{k,0}_{0,2k}\left[\left(\frac{x}{2k}\right)^{2k} \middle|\ \begin{matrix} - & - & - & - \\ 1/4k, 3/4k, \ldots & \frac{2k-1}{4k}, & -1/4k, \ldots, & -\frac{2k-1}{4k} \end{matrix}\right].$$

1.3 The H-function and its asymptotic expansions :

The H-function is applicable in a number of problems arising in physical sciences, engineering and statistics. The importance of this function lies in the fact that nearly all the special functions occurring in applied mathematics and statistics are its special cases. Besides, the function considered by Boresma [9] , Mittag-Leffer, generalized

Bessel function due to Wright [70], the generalization of hypergeometric function studied by Fox [22], and Wright [70] are all special cases of H-function. Except the function of Boresma, the aforesaid functions cannot be obtained as special cases of G-function of Meijer [44]; hence the study of H-function will cover wider range than G-function and gives deeper, more general and more useful results directly applicable in various problems of physical and biological sciences.

Of all the integrals which contain gamma functions in their integrands the most important ones are the so-called Mellin-Barnes integrals. Such integrals were first introduced by S. Pincherle in 1888, Barnes in 1908 and Mellin in 1910. They were used for a complete integration of the hypergeometric differential equation by E.W. Barnes in 1908. Dixon and Ferrar in 1936 have given the asymptotic expansion of general Mellin-Barnes type integrals, [3, Vol.I, p. 49]. Functions close to H-function occur in the study of the solutions of certain functional equations considered by Bochner in 1958 and Chandrasekharan and Narasimhan in 1962.

The H-function has been used by Fox [23] in the study of Fourier Kernels. This function has been studied in detail with reference to analytic continuations and asymptotic expansions by Braaksma [8]. Following Braaksma [8], it is defined in terms of Mellin-Barnes integrals as follows:

$$(1.59)\quad H^{m,n}_{p,q}\left[z \right] = H^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \right. \right]$$

$$= H^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_1,\alpha_1), \ldots\ldots, (a_p,\alpha_p) \\ (b_1,\beta_1), \ldots\ldots, (b_q,\beta_q) \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i} \int_L \chi(s) z^s ds,$$

where $i = (-1)^{\frac{1}{2}}$, $z \neq 0$ is a complex variable and

$$(1.60)\quad z^s = \exp\left[s\mathrm{Log}\,|z| + i.\ \arg z \right],$$

in which $\mathrm{Log}\,|z|$ represents the natural logirithm of $|z|$ and arg z is not necessarily the principal value. An empty product is interpreted as unity. Also

$$(1.61)\quad \chi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)}$$

where m,n, p,q are nonnegative integers satisfying $0 \leq n \leq p$, $1 \leq m \leq q$; α_j $(j = 1,2,\ldots, p)$ and $B_j (j = 1,2 \ldots, q)$ are assumed to be positive quantities. Also, $a_j (j = 1,2, \ldots,p)$ and b_j $(j = 1,2, \ldots, q)$ are complex numbers such that none of the points

(1.62) $s = \left[(b_h + \nu)/\beta_h \right]$, $h = 1,2, \ldots, m$; $\nu = 0,1,2, \ldots$
which are the poles of $\Gamma(b_h - \beta_h s)$, $h = 1,2, \ldots, m$;
and the points

(1.63) $s = \left[(a_i - \eta - 1) / \alpha_i \right]$, $i=1,2, \ldots, n$; $\eta = 0,1,2,\ldots$
which are the poles of $\Gamma(1 - a_i + \alpha_i s)$ coincide with one another, i.e.,

(1.64) $\alpha_i (b_h + \nu) \neq \beta_h (a_i - \eta - 1)$
for $\nu, \eta = 0, 1, 2, \ldots$; $h = 1, 2, \ldots, m$, $i = 1,2, \ldots n$.
Further, the contour L runs from $- i\infty$ to $+ i\infty$ such that the poles of $\Gamma(b_h - \beta_h s)$, $h = 1,2, \ldots, m$, lie to the right of L and the poles of $\Gamma(1 - a_j + \alpha_j s)$, $j = 1,2, \ldots, h$, lie to the left of L. Such a contour is possible on account of (1.64). These assumptions will be retained throughout.

The H-function is an analytic function of z and makes sense if the following existence conditions are satisfied:

Case I : For all $z \neq 0$ with $\mu > 0$,
Case II: For $0 < |z| < B^{-1}$ with $\mu = 0$, where

$$\mu = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

and

$$B = \prod_{j=1}^{p} (\alpha_j)^{\alpha_j} \prod_{j=1}^{q} (\beta_j)^{-\beta_j} .$$

It does not depend upon the choice of L due to the occurrence of the factor z^s in the integrand of (1.59), it is ingeneral, many-valued but one-valuedon Riemann surface of Logz and the result of the H-function are obtainable in a more compact form and without much difficulty. This is not the case with G-function.

The behaviour of the H-function for small and large values of the argument has been discussed by Braaksma [8] in detail. Here, we shall give some of his results. The following definitions will be used :

$$A = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j ;$$

$$B = \prod_{j=1}^{p} (\alpha_j)^{\alpha_j} \prod_{j=1}^{q} (\beta_j)^{-\beta_j} ;$$

$$C = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + p/2 - q/2;$$

$$D = \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \quad \text{and}$$

$$\mu = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j .$$

Then according to Braaksma [8, p. 278]

(1.65) $H^{m,n}_{p,q}\left[x\right] = O\left(|x|^{c}\right)$ for small x,

where $\mu \geqslant 0$, and $c = \min\limits_{1 \leq j \leq m} R(b_j/\beta_j)$;

(1.66) $H^{m,n}_{p,q}\left[x\right] = O(|x|^{d})$ for large x,

where $\mu \geqslant 0$, $A > 0$, $|\arg x| < A\pi/2$ and

$d = \max\limits_{1 \leq j \leq n} \left\{ (R\left[(a_j - 1)/\alpha_j\right]) \right\}$.

The two useful identities of the H-function are :

(1.67) $H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_p,1)\\(b_q,1)\end{matrix}\right] \equiv G^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}a_p\\b_q\end{matrix}\right]$.

(1.68) $H^{1,0}_{0,2}\left[x \,\middle|\, \begin{matrix}- \quad\quad -\\(0,1),\ (-\lambda,\mu)\end{matrix}\right] \equiv J^{\mu}_{\lambda}(x)$

where $J^{\mu}_{\lambda}(x) = \sum\limits_{r=0}^{\infty} \dfrac{(-x)^r}{r!\,\Gamma(1+\lambda+\mu r)}$ is the

Maitland's generalized Bessel function.

We now state some of the important properties of H-function

(1.69) The H-function is symmetrical in pairs of $(a_1,\alpha_1),\ldots,(a_n,\alpha_n)$ and in $(a_{n+1},\alpha_{n+1}),\ldots,(a_p,\alpha_p)$ likewise in $(b_1,\beta_1),\ldots,(b_m,\beta_m)$ and in $(b_{m+1},\beta_{m+1}),\ldots,(b_q,\beta_q)$.

(1.70) If one of the (a_j,α_j) $(j = 1,2, \ldots,n)$ is the same as the one of (b_h, β_h) $(h = m+1, \ldots,q)$ or one of (b_h, β_h) $(h = 1,2, \ldots,m)$ is the same as one of (a_j, α_j) $(j=n+1, \ldots,p)$, the H-function reduces to one of lower order.

$$(1.71)\quad H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] \equiv H_{p,q}^{m,n}\left[1/x \,\middle|\, \begin{matrix}(1-b_q,\beta_q)\\(1-a_p, \alpha_p)\end{matrix}\right]$$

$$(1.72)\quad H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] \equiv k\, H_{p,q}^{m,n}\left[x^k \,\middle|\, \begin{matrix}(a_p,k\alpha_p)\\(b_q, k\beta_q)\end{matrix}\right], \; k > 0 .$$

1.4 Fractional Integrations of H-function

In this section we evaluate some integrals which will be useful in the development of the work in the dissertation.

Riemann-Liouville integral

Let

$$(1.73)\quad f(x) = H_{p,q}^{m,n}\left[ax^b \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] \quad , \; b>0$$

then

$$R_\alpha\left[f(t);\, x\right] = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} H_{p,q}^{m,n}\left[at^b \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] dt$$

$$= \frac{1}{2\pi i} \cdot \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \int_L \chi(s) a^{-s} t^{-bs} \, ds dt$$

$$= \frac{1}{2\pi i} \int_L \chi(s) a^{-s} \, ds . \frac{1}{\Gamma(\alpha)} \int_0^x t^{-bs} (x-t)^{\alpha-1} \, dt$$

changing the order of integration and using

$$(1.74) \quad \chi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j+\beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-\beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j+\alpha_j s)}$$

Now evaluating the t - integral with the help of the known result [4, p. 185(7)] and then using the definition of H-function we get

$$(1.75) \quad R_\alpha \left[f(t);x \right] = x^\alpha \, H^{m,n+1}_{p+1,q+1} \left[ax^b \middle| \begin{matrix} (o,b), \, (a_p,\alpha_p) \\ (b_q,\beta_q), (-\alpha,b) \end{matrix} \right] ,$$

$p+q < 2(m+n)$, $R(\alpha) > 0$, $R(b_j/\beta_j) > -1$, $j = 1,2,\ldots,m.$

<u>Weyl Integral</u>

Using (1.73) we have

$$W_\alpha \left[f;x \right] = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} \, H^{m,n}_{p,q} \left[at^b \middle| \begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \right] dt$$

$$= \frac{1}{2\pi i} \cdot \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} \int_L \chi(s) a^{-s} t^{-bs} \, ds dt$$

$$= \frac{1}{2\pi i} \int_L \chi(s) a^{-s} ds. \frac{1}{\Gamma(\alpha)} \int_x^{\infty} t^{-bs} (t-x)^{\alpha-1} dt$$

changing the order of integrations and using the relation(1.74). Now evaluating the t-integral with the help of the known result [4, P.201 (6)] and then using the definition of H-function, we obtain

$$(1.76) \quad W_\alpha \left[f;x \right] = x^\alpha H^{m+1,n}_{p+1,q+1} \left[ax^b \middle| \begin{matrix} (a_p,\alpha_p), (o,b) \\ (-\alpha,b), (b_q,\beta_q) \end{matrix} \right],$$

$p+q < 2(m+n)$, $o < R(\alpha) < R\left(\frac{1-a_j}{\alpha_j}\right)$, $j = 1,2, \ldots, n.$

1.5. The Mellin transform

The Mellin transform F(s) of the function f(x) is defined by the equation

$$(1.77) \qquad F(s) = M(f(x)) = \int_0^\infty f(x) x^{s-1} dx, \; s = \sigma + i\tau .$$

Under certain conditions [66, P.46], f(x), the inverse Mellin transform of $F(s)$, may be represented as an integral

$$(1.78) \qquad M^{-1}\left[F(s)\right] \equiv f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s} ds.$$

Associated with this transform is the following convolution theorem [66, Th. 44, P.60]. If $s = c + it$, $x^c f(x)$ and $x^c g(x)$ belong to $L(0, \infty)$, then

$F(s)G(s) = M\left[(f*g)(x)\right]$ and $x^c(f*g)(x)$ belongs to $L(0,\infty)$ where

$$(1.79)\quad (f*g)(x) = \int_0^\infty f(x/u)g(u)du/u.$$

The Mellin transform of the H-function follows from the definition (1.59) in view of (1.77) and (1.78). We have

$$(1.80)\quad \int_0^\infty x^{s-1} H^{m,n}_{p,q}\left[ax \,\middle|\, \begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix} \right] dx = a^{-s}\,\chi(s)$$

where $\chi(s)$ is given by (1.74) and the conditions

$$A > 0,\quad |\arg a| < A\pi/2 \quad \text{and} \quad \mu > 0,$$

$$-\min_{1\leq j\leq m} R(b_j/\beta_j) < R(s) < \min_{1\leq j\leq n} R((1-a_j)/\alpha_j)$$

are satisfied.

Motivation of the work done

Erdelyi-Kober [14, 33], Srivastava [65], Bhise [7] and Sharma [61] have used the operators of fractional integration to study the theory of Hankel transform and its generalizations. The generalized Hankel transform (1.46) yields almost all earlier known generalizations as special cases. This very situation motivated us to study the theory of the generalized Hankel transform with the help of fractional integral operaters. Thus the study of this generalized Hankel transform will not only help us to study some of the generalizations which were not studied earlier with the help of fractional integral operaters but will also unify the scattered results in this field.

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